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Approximation of Derivatives of Analytic Functions from One Hardy Class by Another Hardy Class

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Abstract—In the Hardy space $\mathcal{H}^p(D_\varrho)$, $1 \le p \le \infty$, of functions analytic in the disk $D_\varrho = \{z \in \mathbb{C} : |z| < \varrho\}$, we denote by $NH^p(D_\varrho)$, $N > 0$, the class of functions whose L^p-norm on the circle $\gamma_\varrho = \{z \in \mathbb{C} : |z| = \varrho\}$ does not exceed the number N and by $\partial H^p(D_\varrho)$ the class consisting of the derivatives of functions from $1H^p(D_\varrho)$. We consider the problem of the best approximation of the class $\partial H^p(D_\rho)$ by the class $NH^p(D_R)$, $N > 0$, with respect to the L^p-norm on the circle γ_r, $0 < r < \rho < R$. The order of the best approximation as $N \to +\infty$ is found:

$$\mathcal{E}\left(\partial H^p(D_\rho), NH^p(D_R)\right)_{L^p(\Gamma_r)} \asymp N^{-\beta/\alpha} \ln^{1/\alpha} N, \quad \alpha = \frac{\ln R - \ln \rho}{\ln R - \ln r}, \quad \beta = 1 - \alpha.$$

In the case where the parameter N belongs to some sequence of intervals, the exact value of the best approximation and a linear method implementing it are obtained. A similar problem is considered for classes of functions analytic in annuli.

Keywords: analytic functions, Hardy class, best approximation of a class by a class.

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Let $D_\varrho := \{z \in \mathbb{C} \colon |z| < \varrho\}$ be a disk of radius ϱ centered at zero, and let $\gamma_\tau := \{z \in \mathbb{C} \colon |z| = \tau\}$ be a circle of radius τ centered at zero. Denote by $\mathcal{A}(D_\varrho)$ the set of functions analytic in the disk D_ϱ. For a function $f \in \mathcal{A}(D_\varrho)$ and a number τ, $0 < \tau < \varrho$, we define the p-mean of f on the circle γ_τ:

$$\|f\|_{L^p(\gamma_\tau)} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\tau e^{it})|^p \, dt\right)^{1/p}, \quad 1 \le p < \infty.$$

Denote by $\mathcal{H}^p(D_\varrho)$, $1 \le p < \infty$, the Hardy space of functions $f \in \mathcal{A}(D_\varrho)$ such that

$$\sup\{\|f\|_{L^p(\gamma_\tau)} \colon 0 < \tau < \varrho\} < +\infty$$

and by $\mathcal{H}^\infty(D_\varrho)$ the Hardy space of functions analytic and bounded in D_ϱ. It is known that, for an arbitrary function $f \in \mathcal{H}^p(D_\varrho)$, there exist nontangent (angular) limit boundary values almost everywhere at the boundary γ_ϱ of the disk D_ϱ. These values compose a function, also denoted by f, from the space $L^p(\gamma_\varrho)$. The space $\mathcal{H}^p(D_\varrho)$ is equipped with the norm

$$\|f\|_{\mathcal{H}^p(D_\varrho)} := \begin{cases} \sup\{\|f\|_{L^p(\gamma_\tau)} \colon 0 < \tau < \varrho\}, & 1 \le p < \infty, \\ \sup\{|f(z)| \colon z \in D_\varrho\}, & p = \infty; \end{cases}$$

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it is known that $\|f\|_{\mathcal{H}^p(D_\varrho)} = \|f\|_{L^p(\gamma_\varrho)}$. Obviously, $\mathcal{H}^p(D_R) \subset \mathcal{H}^p(D_\rho)$ for $0 < \rho < R$.

In the space $\mathcal{H}^p(D_\varrho)$, we consider a Hardy class $NH^p(D_\varrho)$, $N > 0$, of functions satisfying the inequality $\|f\|_{\mathcal{H}^p(D_\varrho)} \le N$. We will use the notation $H^p(D_\varrho)$ for $N = 1$.

Introduce the class $\partial H^p(D_\varrho)$ consisting of derivatives of functions from the Hardy class $H^p(D_\varrho)$; i.e., $\partial H^p(G_\varrho) := \{g' \colon g \in H^p(D_\varrho)\}$. Note that the class $\partial H^p(D_\varrho)$ is not contained in the Hardy space $\mathcal{H}^p(D_\varrho)$; however, it is a subset of the weighted Bergman space (see details in [8, Subect. 10.1] and further references therein).

1. STATEMENT AND DISCUSSION OF THE PROBLEM

Suppose that Q_1 and Q_2 are classes in a linear space, B is a Banach space, and for any $q_1 \in Q_1$ there exists $q_2 \in Q_2$ such that $q_1 - q_2 \in B$. The *best approximation of the class Q_1 by the class Q_2 in the norm of the space B* is the value

$$\mathcal{E}\left(Q_1, Q_2\right)_B := \sup\left\{E\left(q_1, Q_2\right)_B : q_1 \in Q_1\right\},$$

where $E\left(q_1, Q_2\right)_B$ is the best approximation of q_1 by the class Q_2 defined by the equality

$$E\left(q_1, Q_2\right)_B := \inf\left\{\|q_1 - q_2\|_B \colon q_2 \in Q_2\right\}.$$

Let three numbers r, ρ, and R be related by the inequalities $0 < r < \rho < R$.

In the present paper, we consider the problem of the best approximation of the class $\partial H^p(D_\rho)$ by the class $NH^p(D_R)$ in the norm of the space $L^p(\gamma_r)$ (or, equivalently, in the norm of the space $\mathcal{H}^p(D_r)$), i.e., the problem of finding

$$\begin{aligned}\mathcal{E}\left(\partial H^p(D_\rho), NH^p(D_R)\right)_{L^p(\gamma_r)} &= \sup\Big\{E\left(q, NH^p(D_R)\right)_{L^p(\gamma_r)} : q \in \partial H^p(D_\rho)\Big\}\\ &= \sup\Big\{E\left(g', NH^p(D_R)\right)_{L^p(\gamma_r)} : g \in H^p(D_\rho)\Big\}. \qquad (1)\end{aligned}$$

We obtain the order of (1) as $N \to +\infty$; in the case where the parameter N belongs to some sequence of intervals, we obtain the best approximation of a class by a class and a linear method implementing the best approximation. We also consider a similar problem for classes of functions analytic in annuli.

The problem of approximating one class of functions by another is classical for approximation theory. The mutual connection between the problem on the modulus of continuity of an unbounded operator on a class and the corresponding problem of the best approximation of one class by another in conjugate spaces is known. There is also a connection between Stechkin's problem on the approximation of an unbounded operator by bounded operators on a class and the corresponding problem of the best linear approximation of one class by another (see [3]). The connection between Stechkin's problem on the best approximation of differentiation operators by bounded operators and the problem of the best approximation of one class of differentiable functions of a real variable by another class of smoother functions has been studied most thoroughly (see details in [4; 5; 6, Sect. 7]). In the present paper we investigate problem (1) in the case $1 < p < \infty$; it is related to problems (on the modulus of continuity of an operator and the best approximation of an operator by bounded operators) on the class $H^{p'}(D_{R'})$ for the operator taking the trace of a function on the circle $\gamma_{r'}$ to its derivative on the circle $\gamma_{\rho'}$. This operator is considered as an operator from $L^{p'}(\gamma_{r'})$ to $L^{p'}(\gamma_{\rho'})$ with parameters defined by the equalities $1/p + 1/p' = 1$, $r' = 1/R$, $\rho' = 1/\rho$, and $R' = 1/r$. These problems were studied in [2]. In further discussion, the connection between the problems will not be used explicitly. However, the construction of a linear method providing the best approximation (1) and the analysis of the properties of this method will essentially employ the ideas of construction of a best approximation operator from [2].

The closest to (1) is the problem of the best approximation of a Hardy class $H^p(D_\rho)$ by another Hardy class $NH^p(D_R)$; this problem was studied by Taikov in [7] and then by the author in [1]. For the comparison with the results of the present paper, we give an exact statement of this problem and the known results. The best approximation of a Hardy class $H^p(D_\rho)$ by a Hardy class $NH^p(D_R)$ in the norm of the space $L^p(\gamma_r)$ is the value

$$\mathcal{E}\left(H^p(D_\rho), NH^p(D_R)\right)_{L^p(\gamma_r)} := \sup\left\{E\left(g, NH^p(D_R)\right)_{L^p(\gamma_r)} : g \in H^p(D_\rho)\right\}. \tag{2}$$

Define

$$\alpha := \frac{\ln R - \ln \rho}{\ln R - \ln r}, \quad \beta := \frac{\ln \rho - \ln r}{\ln R - \ln r}, \tag{3}$$

$$\eta_* := \alpha + 2\sum_{k=1}^{\infty}(-1)^k \frac{(\rho/R)^k - (\rho/R)^{-k}}{(r/R)^k - (r/R)^{-k}}, \quad \eta^* := \beta + 2\sum_{k=1}^{\infty}(-1)^k \frac{(r/\rho)^k - (r/\rho)^{-k}}{(r/R)^k - (r/R)^{-k}}.$$

Theorem A [7, Theorems 1, 2; 1, Theorem 3]. *Let numbers r, ρ, and R satisfy the condition $0 < r < \rho < R$. Then the following statements hold for arbitrary p, $1 \le p \le \infty$.*

1. *The following order equality holds for the value* (2):

$$\mathcal{E}\left(H^p(D_\rho), NH^p(D_R)\right)_{L^p(\gamma_r)} \asymp N^{-\beta/\alpha} \quad \text{as } N \to +\infty.$$

2. *If a positive number N can be presented in the form $N = \rho^{-n} R^n(\beta - \eta)$, where n is a positive integer and $\eta \in [-\eta_*, \eta^*]$, then the following equalities holds for the value* (2):

$$\mathcal{E}\left(H^p(D_\rho), NH^p(D_R)\right)_{L^p(\gamma_r)} = \rho^{-n}\, r^n\,(\alpha + \eta) = (\alpha + \eta)\,(\beta - \eta)^{\beta/\alpha}\, N^{-\beta/\alpha}.$$

In the cases where the exact values of (2) were found, linear approximation methods implementing them were constructed in [1, 7].

2. CONSTRUCTION OF THE APPROXIMATION METHOD

For $0 < \rho_1 < \rho_2$, denote by $C_{\rho_1,\rho_2} := \{z \in \mathbb{C}\colon \rho_1 < |z| < \rho_2\}$ the annulus centered at zero with inner and outer radii ρ_1 and ρ_2, respectively. Let number r_0, r, ρ, and R be related by the inequalities $0 < r_0 < r < \rho < R$. For an integer n and an arbitrary function g analytic in the annulus $C_{r_0,\rho}$ and presentable in $C_{r_0,\rho}$ by the Laurent series

$$g(z) = \sum_{k=-\infty}^{+\infty} g_k z^k, \tag{4}$$

we define the function

$$f(z) := \sum_{k=-\infty}^{+\infty} v_{n+k}\, g_{n+k}\, z^{n+k-1}, \tag{5}$$

$$v_n = \frac{n\ln\rho - n\ln r - 1}{\ln R - \ln r}, \quad v_{n+k} = \frac{(n-k)\rho^{2k} - (n+k)r^{2k}}{R^{2k} - r^{2k}}, \quad k \ne 0.$$

It can be easily verified with the use of the Cauchy–Hadamard theorem that the function f is analytic in the annulus $C_{r_0,R^2/\rho}$ (which is larger than the original annulus $C_{r_0,\rho}$). It is convenient to interpret equality (5) as the definition $V_n g := f$ of a linear operator V_n from the space $\mathcal{A}(C_{r_0,\rho})$ to the space $\mathcal{A}(C_{r_0,R^2/\rho})$. Note that if a function g is analytic in the disk D_ρ, then the coefficients of the series (4) satisfy the property $g_k = 0$ for $k < 0$ and, hence, $f \in \mathcal{A}(D_{R^2/\rho}) \subset \mathcal{H}^\infty(D_R) \subset \mathcal{H}^p(D_R)$.

Let present the difference $g' - f$ in terms of the coefficients of the series (4):

$$g'(z) - f(z) = \sum_{k=-\infty}^{+\infty} (k - v_k)\, g_k\, z^{k-1} = \sum_{k=-\infty}^{+\infty} l_k\, g_k\, z^{k-1}, \quad l_k = k - v_k, \tag{6}$$

$$l_n = \frac{n \ln R - n \ln \rho + 1}{\ln R - \ln r}, \quad l_{n+k} = \frac{(n+k)R^{2k} - (n-k)\rho^{2k}}{R^{2k} - r^{2k}}, \quad k \neq 0.$$

Equality (6) can also be interpreted as the definition $T_n g := g' - f$ of a linear operator T_n in the space $\mathcal{A}(C_{r_0,\rho})$.

Now, using (5), we express the values of f on the circle γ_R in terms of the values of g on the circle γ_ρ. We obtain the representation

$$f(Re^{ti}) = R^{-1} e^{-it} \sum_{k=-\infty}^{+\infty} (R/\rho)^k\, v_k\; g_k\, \rho^k e^{ikt} = R^{-1} e^{-it}\, \frac{1}{2\pi} \int_0^{2\pi} \mathcal{V}_n(t-\tau) g(\rho e^{i\tau})\, d\tau, \tag{7}$$

whose kernel $\mathcal{V}_n$ can be expressed in terms of the sum μ_n of a cosine series

$$\mathcal{V}_n(t) = (R/\rho)^n\; e^{int}\, \mu_n(t), \quad \mu_n(t) = \mu_{n,0} + 2 \sum_{k=1}^{+\infty} \mu_{n,k}\, \cos kt.$$

The coefficients of the latter series are given by the formulas

$$\mu_{n,0} = v_n = \frac{n \ln \rho - n \ln r - 1}{\ln R - \ln r}, \tag{8}$$

$$\mu_{n,k} = \left(\frac{R}{\rho}\right)^k v_{n+k} = \left(\frac{\rho}{R}\right)^k v_{n-k} = \frac{(n-k)(\rho/r)^k - (n+k)(r/\rho)^k}{(R/r)^k - (r/R)^k}, \quad k \geq 1.$$

Similarly, using equality (6), we express the values of the function $g' - f$ on the circle γ_r in terms of the values of g on the circle γ_ρ. We have the representation

$$g'(re^{it}) - f(re^{it}) = r^{-1} e^{-it} \sum_{k=-\infty}^{+\infty} (r/\rho)^k\, l_k\; g_k\, \rho^k e^{ikt} = r^{-1} e^{-it}\, \frac{1}{2\pi} \int_0^{2\pi} \Lambda_n(t-\tau) g(\rho e^{i\tau})\, d\tau, \tag{9}$$

whose kernel Λ_n can be expressed in terms of the sum λ_n a cosine series

$$\Lambda_n(t) = (r/\rho)^n\; e^{int}\, \lambda_n(t), \quad \lambda_n(t) = \lambda_{n,0} + 2 \sum_{k=1}^{+\infty} \lambda_{n,k}\, \cos kt.$$

The coefficients of the latter series are given by the formulas

$$\lambda_{n,0} = l_n = \frac{n \ln R - n \ln \rho + 1}{\ln R - \ln r}, \tag{10}$$

$$\lambda_{n,k} = \left(\frac{r}{\rho}\right)^k l_{n+k} = \left(\frac{\rho}{r}\right)^k l_{n-k} = \frac{(n+k)(R/\rho)^k - (n-k)(\rho/R)^k}{(R/r)^k - (r/R)^k}, \quad k \geq 1.$$

In what follows, we will need a statement on the properties of the functions λ_n and μ_n.

Lemma 1. *For an integer n satisfying the inequality*

$$|n| \geq \frac{\pi}{\sin\alpha\pi}\,\frac{1}{\ln R - \ln r}, \tag{11}$$

the functions λ_n and μ_n have the same constant sign on the period; i.e., $\lambda_n(t)\mu_n(t) > 0$ for $t \in [0, 2\pi]$.

Proof. The proof follows the scheme from [2, Lemma 2]. Consider the functions

$$g_\pm(t,y) = \frac{e^{ny}\sin(\alpha\pi)}{\xi(t) \pm \cos(\alpha\pi)}, \quad \xi(t) = \cosh\frac{\pi t}{\ln R - \ln r}, \quad y = \ln R - \ln\rho.$$

For λ_n and μ_n, we have

$$\lambda_n(t) = \rho^{n-1}\frac{\partial}{\partial(1/\rho)}\left\{\frac{1}{\rho^n}\Lambda_+(t)\right\}, \quad \mu_n(t) = \rho^{n-1}\frac{\partial}{\partial(1/\rho)}\left\{\frac{1}{\rho^n}\Lambda_-(t)\right\},$$

where

$$\Lambda_+(t) = \alpha + 2\sum_{k=1}^{\infty}\frac{(R/\rho)^k - (\rho/R)^k}{(R/r)^k - (r/R)^k}\cos kt, \quad \Lambda_-(t) = \beta + 2\sum_{k=1}^{\infty}\frac{(\rho/r)^k - (r/\rho)^k}{(R/r)^k - (r/R)^k}\cos kt.$$

On the other hand [1, Lemma 1],

$$\Lambda_\pm(t) = \frac{\pi}{\ln R - \ln r}\,e^{-ny}\sum_{k=-\infty}^{+\infty} g_\pm(t + 2\pi k, y).$$

We obtain the representations

$$\lambda_n(t) = \frac{\pi}{\ln R - \ln r}\left(\frac{\rho}{R}\right)^n \sum_{k=-\infty}^{+\infty}\frac{\partial}{\partial y}g_+(t+2\pi k, y),$$

$$\mu_n(t) = \frac{\pi}{\ln R - \ln r}\left(\frac{\rho}{R}\right)^n \sum_{k=-\infty}^{+\infty}\frac{\partial}{\partial y}g_-(t+2\pi k, y).$$

Calculating the derivatives of $g_\pm$, we get

$$\frac{\partial g_\pm}{\partial y} = \frac{\pi e^{ny}\left[\xi(t)(nu + \cos\alpha\pi) \pm (nu\cos\alpha\pi + 1)\right]}{(\ln R - \ln r)(\xi(t) \pm \cos\alpha\pi)^2}, \quad u = \frac{\sin\alpha\pi(\ln R - \ln r)}{\pi}.$$

The signs of the derivatives coincide with the signs of the expressions in square brackets. Since $|nu| \geq 1$, we have

$$\left|\frac{nu + \cos\alpha\pi}{nu\cos\alpha\pi + 1}\right| \geq 1,$$

because the linear fractional function $h(z) = (z + \cos\alpha\pi)/(1 + z\cos\alpha\pi)$ maps the exterior of the unit disk to itself. For arbitrary $t \in \mathbb{R}$, we have $\xi(t) \geq 1$. Therefore, the condition $|nu| \geq 1$, which is equivalent to (11), implies the constancy of the sign of the expression $\xi(t)(nu+\cos\alpha\pi)\pm(nu\cos\alpha\pi+1)$. This fact completes the proof of Lemma 1.

Remark. The coefficients (10) and (8) are the mean values of the functions λ_n and μ_n on the period. From (10) and (8), we have $\mu_{n,0}+\lambda_{n,0}=n$. Therefore, if condition (11) is satisfied, then the functions λ_n and μ_n and the coefficients $\lambda_{n,0}$ and $\mu_{n,0}$ have the same sign coinciding with the sign of n.

As follows from Lemma 1, there exists an open interval I_n (of positive length) defined by equality $I_n=\{\eta\in\mathbb{R}\colon (\lambda_n(t)+\eta)(\mu_n(t)-\eta)>0,\ t\in[0,2\pi]\}$. The interval $I_n=(\eta_n^-,\eta_n^+)$ has boundary points

$$\eta_n^-=\max_{t\in[0,2\pi]}\min\{-\lambda_n(t),\mu_n(t)\},\quad \eta_n^+=\min_{t\in[0,2\pi]}\max\{-\lambda_n(t),\mu_n(t)\},$$

related by the inequality $\eta_n^-<0<\eta_n^+$. Denote by S_n the closed interval $[\eta_n^-,\eta_n^+]$.

For an integer n satisfying inequality (11) and a number η from the interval S_n, we define a linear operator $V_{n,\eta}$ from the space $\mathcal{A}(C_{r0,\rho})$ to the space $\mathcal{A}(C_{r0,R})$ by the equality

$$(V_{n,\eta}g)(z):=(V_ng)(z)-\eta g_nz^{n-1}=f(z)-\eta g_nz^{n-1}, \tag{12}$$

in which g_n is the coefficient with index n of the Laurent series (4) of the function g, and the operator V_n with values $V_ng=f$ is defined by (5).

3. MAIN RESULT

Theorem 1. *Let numbers r, ρ, and R satisfy the inequalities $0<r<\rho<R$. Then, for arbitrary p, $1\le p\le\infty$, the following statements hold.*

1. *The best approximation* (1) *satisfies the order equality*

$$\mathcal{E}\left(\partial H^p(D_\rho),NH^p(D_R)\right)_{L^p(\gamma_r)}\asymp N^{-\beta/\alpha}\ln^{1/\alpha}N \quad \text{as } N\to+\infty, \tag{13}$$

where the numbers α and β are defined in (3).

2. *If a positive number N can be presented in the form*

$$N=\rho^{-n}R^{n-1}(\mu_{n,0}-\eta), \tag{14}$$

where n is an arbitrary positive integer satisfying inequality (11) *and η is an arbitrary number from S_n, then the best approximation* (1) *satisfies the equality*

$$\mathcal{E}\left(\partial H^p(D_\rho),NH^p(D_R)\right)_{L^p(\gamma_r)}=\rho^{-n}r^{n-1}(\lambda_{n,0}+\eta). \tag{15}$$

Here $\mu_{n,0}$ and $\lambda_{n,0}$ are defined by formulas (8) *and* (10).

The linear method defined by (12) *yields the best approximation of a class by a class in* (1).

Proof. As follows from the remark after Lemma 1, the functions $\lambda_n+\eta$ and $\mu_n-\eta$ and the numbers $\lambda_{n,0}+\eta$ and $\mu_{n,0}-\eta$ are positive for $\eta\in S_n$.

Let us check that the function $(V_{n,\gamma}g)(z)=f(z)-\eta g_nz^{n-1}$ belongs to the class $NH^p(D_R)$ if $g\in H^p(D_\rho)$. As shown earlier, $f\in\mathcal{H}^p(D_R)$. Consider the p-norm on the circle γ_R. Using representation (7) and the positivity of $\mu_n-\eta$, we obtain

$$\|V_{n,\eta}g\|_{L^p(\gamma_R)}=\left\|f(Re^{it})-\eta g_nR^{n-1}e^{i(n-1)t}\right\|_{L^p(0,2\pi)}$$

$$= \left\| \frac{R^{n-1}}{2\pi\rho^n} \int_0^{2\pi} e^{in(t-\tau)}(\mu_n(t-\tau)-\eta)g(\rho e^{i\tau})\,d\tau \right\|_{L^p(0,2\pi)}$$

$$\le \frac{R^{n-1}}{2\pi\rho^n} \|g\|_{L^p(\gamma_\rho)} \int_0^{2\pi} |\mu_n(\tau)-\eta|\,d\tau \le \frac{R^{n-1}}{2\pi\rho^n} \int_0^{2\pi} (\mu_n(\tau)-\eta)\;d\tau = \rho^{-n}R^{n-1}(\mu_{n,0}-\eta).$$

Arguing similarly and using representation (9) and Lemma 1, we derive a bound for the approximation of the derivative of a function $g \in H^p(D_\rho)$ by the method $V_{n,\eta}$:

$$\|g' - V_{n,\eta}g\|_{L^p(\gamma_r)} = \left\| g'(re^{it}) - \left(f(re^{it}) - \eta g_n r^{n-1}e^{i(n-1)t}\right)\right\|_{L^p(0,2\pi)}$$

$$= \left\| \frac{r^{n-1}}{2\pi\rho^n} \int_0^{2\pi} e^{in(t-\tau)}(\lambda_n(t-\tau)+\eta)g(\rho e^{i\tau})\,d\tau \right\|_{L^p(0,2\pi)}$$

$$\le \frac{r^{n-1}}{2\pi\rho^n} \int_0^{2\pi} |\lambda_n(\tau)+\eta|\,d\tau\,\|g\|_{L^p(\gamma_\rho)} \le \frac{r^{n-1}}{2\pi\rho^n} \int_0^{2\pi} (\lambda_n(\tau)+\eta)\;d\tau = \rho^{-n}r^{n-1}(\lambda_{n,0}+\eta).$$

Consequently, for $N = \rho^{-n}R^{n-1}(\mu_{n,0}-\eta)$, the best approximation (1) satisfies the upper bound

$$\mathcal{E}\left(\partial H^p(D_\rho), NH^p(D_R)\right)_{L^p(\gamma_r)} \le \rho^{-n}r^{n-1}(\lambda_{n,0}+\eta). \tag{16}$$

Let us derive a lower bound for (1). The function $g_0(z) = \rho^{-n}z^n$ belongs to the class $H^p(D_\rho)$. The best approximation of its derivative $g_0'(z) = n\rho^{-n}z^{n-1}$ by the class $NH^p(D_R)$ under the condition

$$0 \le N \le n\rho^{-n}R^{n-1} \tag{17}$$

is implemented by the function $NR^{1-n}z^{n-1}$, and

$$E\left(g_0', NH^p(D_R)\right)_{L^p(\gamma_r)} = \left(n\rho^{-n} - NR^{1-n}\right)r^{n-1}.$$

The value $N = \rho^{-n}R^{n-1}(\mu_{n,0}-\eta)$ satisfies condition (17). In this case,

$$E\left(g_0', NH^p(D_R)\right)_{L^p(\gamma_r)} = \rho^{-n}r^{n-1}\left(n - (\mu_{n,0}-\eta)\right) = \rho^{-n}r^{n-1}(\lambda_{n,0}+\eta).$$

This implies the lower bound

$$\mathcal{E}\left(\partial H^p(D_\rho), NH^p(D_R)\right)_{L^p(\gamma_r)} \ge E\left(g_0', NH^p(D_R)\right)_{L^p(\gamma_r)} = \rho^{-n}r^{n-1}(\lambda_{n,0}+\eta). \tag{18}$$

Inequalities (16) and (18) give the assertion (15).

Finally, the order equality (13) for the best approximation follows from (15), the monotonicity of (1) in the parameter N, and the fact that the value (14), for example, for $\eta = 0$, tends to infinity as $n \to \infty$. The theorem is proved.

Comparing Theorem 1 with Theorem A, we see that the best approximation (1) tends to zero slower in order than (2); specifically, these values differ in order by the factor $\ln^{1/\alpha} N$. In both Theorem A and Theorem 1, there are countably many intervals on which the dependence of the best approximation on N is linear; however, the interval S_n in Theorem 1 depends on n in contrast to the interval $[-\eta_*, \eta^*]$, which plays the similar role in Theorem A.

4. THE CASE OF AN ANNULUS

In this section we consider the problem for the case of classes of functions analytic in an annulus. Let $\mathcal{H}^p(C_{\varrho_1,\varrho_2})$, $1 \le p \le \infty$, be the Hardy space of functions f analytic in an annulus C_{ϱ_1,ϱ_2} and such that $\sup\{\|f\|_{L^p(\gamma_\tau)}\colon \varrho_1 < \tau < \varrho_2\} < +\infty$. Introduce the classes of functions

$$NH^p(C_{\varrho_1,\varrho_2}) := \left\{f\colon f \in \mathcal{H}(C_{\varrho_1,\varrho_2}), \|g\|_{L^p(\gamma_{\varrho_2})} \le N\right\}, \quad N > 0;$$

$$\partial H^p(C_{\varrho_1,\varrho_2}) := \left\{g'\colon g \in H^p(C_{\varrho_1,\varrho_2})\right\}.$$

Using the properties of the operator (12) and the fact that $g_0(z) = n\rho^{-n}z^{n-1} \in \partial H^p(C_{r_0,\rho})$ for an arbitrary integer value of the parameter n and arguing similarly to the proof of Theorem 1, we obtain the following result.

Theorem 2. *Let numbers r_0, r, ρ, and R satisfy the condition $0 < r_0 < r < \rho < R$. For arbitrary p, $1 \le p \le \infty$, if a positive number N can be presented in the form*

$$N = \rho^{-n}R^{n-1}\left|\mu_{n,0} - \eta\right|,$$

where n is an arbitrary integer satisfying the inequality (11) *and η is an arbitrary number from the interval S_n, then*

$$\mathcal{E}\left(\partial H^p(C_{r_0,\rho}), NH^p(C_{r,R})\right)_{L^p(\gamma_r)} = \rho^{-n}r^{n-1}\left|\lambda_{n,0} + \eta\right|.$$

Here $\mu_{n,0}$ and $\lambda_{n,0}$ are defined by (8) *and* (10).

The linear method defined by equality (12) *yields the best approximation of a class by a class.*

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