Nonstationary Laminar Bénard-Marangoni Convection for Newton-Richmann Heat Exchange

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Abstract. The paper presents a mathematical modeling of nonstationary laminar Bénard-Marangoni convection of a viscous incompressible fluid moving in an infinite band. The main attention is paid to the study of the position and displacement of the stagnation point of the solution with time, the appearance and disappearance of counterflows. It is shown that the overdetermined initial boundary value problem within the here presented class of exact solutions of the Oberbeck-Boussinesq equations is reducible to the Sturm-Liouville problem. The hydrodynamic fields indicating the presence of counterflows in the fluid and their change during fluid acceleration are analyzed with the use of a computational experiment.

INTRODUCTION

The study of convective flows of a viscous incompressible fluid is one of the most common tasks in a variety of theoretical and applied scientific disciplines. In recent decades, the interest to the study of solutions describing natural convection has been continuously increasing. This is due to the fact that convection is the first example of the self-organization of dynamic systems that can be observed experimentally [1]. The first personalized self-organization structure also belongs to convection, namely Bénard cells, which are hexagonal prisms with fluid elevation in the center [2, 3, 4]. Even in his first hydrodynamic experiments, Bénard supposed that an important role in the occurrence of convection is played not as much by gravity as by the Marangoni thermocapillary effect. Note that the Marangoni effect clarifies not only thermocapillary convection, but also concentration convection and bioconvection [6].

In modeling the above-mentioned fluid motion processes, it is very important to have a certain reserve of exact solutions since complete integration of the Navier-Stokes equations in the Boussinesq approximation (the Oberbeck-Boussinesq system) remains a challenge. The first exact solution describing the thermocapillary motions of a viscous incompressible fluid was proposed by Birikh [5]; the first exact convective solution was discussed in [7].

An exhaustive list of exact convective motions can be found in [8–17] and in the references those studies.

Among the exact solutions there are solutions presented in the form of elementary functions and quadratures, solutions at which the original system of partial derivative equations is reducible to a system of ordinary differential equations, linear partial differential equations, or linear integral equations (as a rule, to one of these types) [19, 20].

Thus, the solutions of the Oberbeck-Boussinesq equations, obtained due to the use of numerical methods, can be treated as exact.

In this paper, we study the exact solution describing nonstationary thermocapillary convection when heat exchange at the boundaries of the fluid layer obeys the Newton-Richmann law. The study of such motions is extremely important for practical problems. Even in laboratory research, it is always difficult reach a thermally insulated boundary for closed vessels and impossible for thermocapillary flows.

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The characteristic feature of the obtained solution is that the velocities are one-dimensional in the coordinates, the temperature and pressure fields being three-dimensional in the coordinates. The chosen conditions correspond to experimental research on hydrodynamics. For the stable motion of a nonisothermal fluid, the unidimensionality of the velocities is practically reached in a rectangular layer; this is not observed for temperature and pressure even in the simplest cases.

PROBLEM STATEMENT

The problem of nonstationary convection in a laminar flow of a viscous incompressible fluid is considered. The General system describing convection in an incompressible Oberbeck-Boussinesq fluid [8, 11] has the form

$$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}=-\frac{\partial P}{\partial x}+\nu\Delta V_x\,,$$
$$\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}=-\frac{\partial P}{\partial y}+\nu\Delta V_y\,,$$
$$\frac{\partial P}{\partial z}=g\beta T\,,$$
$$\frac{\partial T}{\partial t}+V_x\frac{\partial T}{\partial x}+V_y\frac{\partial T}{\partial y}=\chi\Delta T\,,$$
$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0\,. \tag{1}$$

Here, V_x, V_y are the dimensionless components of the fluid velocity vector, where $V=g\beta\vartheta L^2/\nu$ is a characteristic velocity scale; the dimensionless horizontal coordinates x and y are determined by the characteristic length scale L and the transverse coordinate z is determined by the layer thickness h; $\delta=h/L$ is the length scale ration; $\mathrm{Gr}=g\beta\vartheta L^3/\nu^2$ is the Grashof number; β is the volume expansion coefficient of the fluid; g is the acceleration of gravity, ϑ is the difference between the maximum and minimum temperatures, ν is the kinematic (molecular) viscosity coefficient of the fluid; $\mathrm{Ra}=\mathrm{Pr}\,\mathrm{Gr}$ is the Rayleigh number; $\mathrm{Pr}=\nu/\chi$ is the Prandtl number; χ is the thermal diffusivity; $\mathrm{Ek}=Ro/\mathrm{Gr}=\nu/(L\omega)$ is the Ekman number; $\mathrm{Ro}=VL/(L^2\omega)$ is the Rossby number; ω is the Coriolis parameter; $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{\delta^2}\frac{\partial^2}{\partial z^2}$ is the Laplace operator.

The boundary conditions on the lower surface defined by the equation z=0, due to the no-slip condition, have the form

$$V_x(t,x,y,0)=0\,,\ V_y(t,x,y,0)=0\,,\ V_z(t,x,y,0)=0\,.$$

We assume that the fluid temperature on the plane z=0 is uniform and constant and take zero as the reference value,

$$T(t,x,y,0)=0\,. \tag{2}$$

On the free surface $z=h$, the heat transfer conditions are given by Newton's law as

$$\left.\frac{\partial T}{\partial n}\right|_{z=h}=-\vartheta\left(T-T_c\right). \tag{3}$$

$T_c(x,y,z)=Ax+By$ is the ambient temperature; A and B are constants; ϑ is the heat transfer coefficient, the parameter characterizing the ratio between the heat exchange coefficient α and the heat conductivity coefficient λ.

When the condition of a plane (rigid) free boundary is met [8], the surface tension is described by the formulas

$$\eta \left.\frac{\partial V_x}{\partial n}\right|_{z=h} = \sigma \left.\frac{\partial T}{\partial x}\right|_{z=h}, \ \eta \left.\frac{\partial V_y}{\partial n}\right|_{z=h} = \sigma \left.\frac{\partial T}{\partial y}\right|_{z=h}, \tag{4}$$

where σ and η are the temperature coefficient of surface tension and the dynamic viscosity factor, respectively; the pressure on the outer surface is constant and equal to the atmospheric pressure $P(t,h) = S/\rho_0$, where S is the atmospheric pressure; ρ_0 is fluid density.

The initial conditions are as follows:

$$V_x(0,x,y,z) = 0, \ V_y(0,x,y,z) = 0, \ T(0,x,y,z) = 0. \tag{5}$$

Let us proceed to the study of the laminar motion of a viscous incompressible fluid. In this case, the fluid flow is plane parallel [20], i.e. the vertical component of the velocity vector becomes zero. In the study of laminar flows, the topology of hydrodynamic fields is determined by the transverse coordinate z; thus, according to [14; 18; 21], the exact solution of system (1) will be computed as follows:

$$\begin{gathered} V_x = U(t,z), \ V_y = V(t,z), \\ T = T_0(t,z) + xT_1(t,z) + yT_2(t,z), \\ P = P_0(t,z) + xP_1(t,z) + yP_2(t,z). \end{gathered} \tag{6}$$

The structure of hydrodynamic fields (6) is such that the velocities are one-dimensional in the coordinates, whereas the temperature and the pressure are three-dimensional. By substituting expressions (6) into the Oberbeck-Boussinesq system (1), with the application of the method of undetermined coefficients, we obtain the following nonlinear system of partial differential equations:

$$\frac{\partial T_1}{\partial t} = \chi \frac{\partial^2 T_1}{\partial z^2}, \ \frac{\partial T_2}{\partial t} = \chi \frac{\partial^2 T_2}{\partial z^2} \tag{7}$$

$$\frac{\partial P_1}{\partial z} = g\beta T_1, \ \frac{\partial P_2}{\partial z} = g\beta T_2 \tag{8}$$

$$\frac{\partial U}{\partial t} = -P_1 + \nu \frac{\partial^2 U}{\partial z^2}, \ \frac{\partial V}{\partial t} = -P_2 + \nu \frac{\partial^2 V}{\partial z^2} \tag{9}$$

$$\frac{\partial T_0}{\partial t} + UT_1 + VT_2 = \chi \frac{\partial^2 T_0}{\partial z^2}, \ \frac{\partial P_0}{\partial z} = g\beta T_0 \tag{10}$$

System (7)–(10) consists of several systems of nonlinear heat-conduction-type equations. Subsystem (7) consists of homogeneous equations, and subsystems (9) and (10) consist of heterogeneous equations.

It is obvious that the continuity equation included in system (1) is satisfied identically.

CONSTRUCTION OF A NON-STATIONARY SOLUTION

System (7)–(10) can be divided into three subsystems with corresponding boundary and initial conditions. The boundary conditions (2), (3), by virtue of class (6), become as follows:

$$\begin{gathered} T_1(t,0) = 0, \ T_2(t,0) = 0, \\ \left[\frac{\partial T_1}{\partial z} - \vartheta\left(A - T_1\right)\right]_{z=h} = 0, \end{gathered} \tag{11}$$

$$\left[\frac{\partial T_2}{\partial z} - \vartheta\left(B - T_2\right)\right]_{z=h} = 0,$$

for subsystem (9),

$$\eta \frac{\partial U}{\partial z}\bigg|_{z=h} = \sigma T_1(t,h), \eta \frac{\partial V}{\partial z}\bigg|_{z=h} = \sigma T_2(t,h) \tag{12}$$

$$\mathrm{U}\left(\mathrm{t},0\right) = 0,\ \mathrm{V}\left(\mathrm{t},0\right) = 0$$

for subsystem (7) and

$$P_0(t,h) = S/\rho_0,\ P_1(t,h) = 0,\ P_2(t,h) = 0 \tag{13}$$

$$T_0(t,0) = 0,\ T_0(t,h) = 0. \tag{14}$$

for systems (8) and (10).

The initial conditions are

$$U\left(0,z\right) = 0,\ V\left(0,z\right) = 0,\ T_0\left(0,z\right) = 0,\ T_1\left(0,z\right) = 0\ T_2\left(0,z\right) = 0. \tag{15}$$

The solutions of the subsystems are sought in the form of the sum of the stationary solution and the Fourier series with respect to the eigenfunctions of the subsystems. The boundary conditions (11) are inhomogeneous. The substitution gives

$$T_1\left(t,z\right) = \frac{A\vartheta z}{1+\vartheta h} + u\left(t,z\right). \tag{16}$$

According to the variable separation method, the solution of equation (16) is represented in the multiplicative form as

$$u_k(t,z) = u_k^t(t)\varphi_{1k}(z),\ \ k \in N.$$

The original boundary value problem is reduced to the Sturm-Liouville problem for the equation

$$\frac{d^2\varphi_{1k}}{dz^2} = \omega_{1k}^2\varphi_{1k}$$

with the boundary conditions

$$\varphi_{1k}(0) = 0,\ \frac{d\varphi_{1k}}{dz}\bigg|_{z=h} = -\vartheta\varphi_{1k}(h), \tag{17}$$

which follow from the boundary conditions (12). Hereinafter, N is a set of natural numbers, ω_{1k} and φ_{1k} are the eigenvalues and eigenfunctions of the Sturm-Liouville problem, respectively. This results in the general solution of equation (17)

$$\varphi_{1k}(z) = C_1 \sin(\omega_{1k} z) + C_2 \cos(\omega_{1k} z),$$

where C_1 and C_2 are the integration constants found from the boundary conditions. It follows from the boundary condition (17) that $C_2 = 0$, and the equation for finding the eigenvalues is as follows:

$$\omega_{1k} \cos\left(\omega_{1k} h\right) + \vartheta \sin\left(\omega_{1k} h\right) = 0 \tag{18}$$

Equation (18) is transcendent, it has a denumerable number of roots. The auxiliary function $u(t,z)$ acquires the form

$$u\left(t,z\right) = -\frac{A\vartheta}{1+h\vartheta} \sum_{n=1}^{\infty} \alpha_n e^{-\chi\omega_{1n}^2 t} \sin\left(\omega_{1n} z\right).$$

Here, the coefficients α_n denote the expansion of the linear function z in a Fourier series with respect to the orthogonal system of functions $\sin\left(\omega_{1n} z\right)$ on the interval $\left(0,h\right)$. Thus, we obtain a solution to first equation (7),

$$T_1\left(t,z\right) = \frac{A\vartheta}{1+h\vartheta}\left(T_1^0 - \sum_{n=1}^{\infty} T_1^n(t) \sin\left(\omega_{1n} z\right)\right),$$

where $T_1^0 = z$ is the stationary solution, the functions $T_1^k(t) = \alpha_k e^{-\chi\omega_{1k}^2 t}$ describe the transition process. The solution of $T_2(t,z)$ is found similarly,

$$T_2(t,z) = \frac{B\vartheta}{1+h\vartheta}\left(z + \sum_{n=1}^{\infty} T_1^n(t) \sin(\omega_{1n} z)\right).$$

Solving system (8), we find the pressure components

$$P_1(t,z) = \frac{A\vartheta\beta g}{1+\vartheta h}\left\{P_1^0 - \sum_{n=1}^{\infty} \frac{T_1^n(t)}{\omega_{1n}}\left(\cos(\omega_{1n} h) - \vartheta\cos(\omega_{1n} z)\right)\right\},$$
$$P_2(t,z) = \frac{B\vartheta\beta g}{1+\vartheta h}\left\{P_1^0 - \sum_{n=1}^{\infty} \frac{T_1^n(t)}{\omega_{1n}}\left(\cos(\omega_{1n} h) - \vartheta\cos(\omega_{1n} z)\right)\right\},$$

where $P_1^0 = \dfrac{(z^2 - h^2)}{2}$ is the stationary solution.

Let us now integrate the equations of subsystem (9), which has the inhomogeneous boundary (12) and homogeneous initial conditions (15). Making the substitution

$$U(t,z) = U_1(t,z) + \frac{\sigma}{\eta} T_1(t,h) z\,,$$

we reduce the equation of subsystem (8) to the equation

$$\frac{\partial U_1(t,z)}{\partial t} = \nu \frac{\partial^2 U_1(t,z)}{\partial z^2} + \frac{\sigma}{\eta}\frac{A\vartheta\chi}{1+\vartheta h} z \sum_{m=1}^{\infty} \omega_{1m}^2 \alpha_m e^{-\chi\omega_{1m}^2 t} \sin(\omega_{1m} h) -$$
$$-\frac{A\vartheta\beta g}{1+\vartheta h}\left[\frac{z^2 - h^2}{2} - \sum_{n=1}^{\infty} e^{-\chi\omega_{1n}^2 t} \frac{\alpha_n}{\omega_{1n}}\left(\cos(\omega_{1n} h) - \vartheta\cos(\omega_{1n} z)\right)\right] \tag{19}$$

with the homogeneous boundary conditions

$$U_1(t,0)=0,\left(\frac{dU_1(t,z)}{dz}\right)_{z=h}=0 \tag{20}$$

and the initial conditions $U_1(0,z)=0$.

The eigenvalues of problem (19), (20) are found based on the boundary conditions (20), and they are

$$\omega_{2k}=\frac{\pi}{h}\left(k-\frac{1}{2}\right)=\frac{\pi(2k-1)}{2h},\ k\in N,$$

and the eigenfunctions have the form

$$\varphi_{2k}(z)=\sin(\omega_{2k}z),\ k\in N.$$

In the solution of the heterogeneous equation (19), we separate the stationary solution

$$U^0(z)=\frac{g\beta}{\nu}\left(\frac{z^4}{24}-\frac{h^2z^2}{4}+\frac{h^3z}{3}\right)+\frac{\sigma hz}{\eta}.$$

Note that, with the ideal thermal contact ($\vartheta\to\infty$), this solution is equal to that found in [21]. The non-stationary solution of subsystem (9), as for subsystem (7), is sought in the form of the sum of the stationary solution and the series with respect to the eigenfunctions,

$$U(t,z)=\frac{A\vartheta}{1+h\vartheta}\left[U^0+\sum_{k=1}^{\infty}(-U_1^k(t)+U_2^k(t))\sin(\omega_{2k}z)-zU_3(t)\right], \tag{21}$$

where

$$U_1^k(t)=\frac{g\beta}{\nu}\gamma_{1k}e^{-\nu\omega_{2k}^2t},$$

$$U_2^k(t)=\sum_{n=1}^{\infty}\alpha_n\left(\beta g\frac{\gamma_{3nk}}{\omega_{1n}}+\frac{\sigma\chi}{\eta}\omega_{1n}^2\gamma_{2k}\sin(\omega_{1n}h)\right)\frac{e^{-\chi\omega_{1n}^2t}-e^{-\nu\omega_{2k}^2t}}{\nu\omega_{2k}^2-\chi\omega_{2n}^2},$$

$$U_3(t)=\frac{\sigma}{\eta}\sum_{n=1}^{\infty}\alpha_n e^{-\chi\omega_{1n}^2t}\sin(\omega_{1n}h).$$

Here, γ_{1m}, γ_{2m}, γ_{3nm} are the coefficients of the expansion in the Fourier series with respect to $\sin(\omega_{2m}z)$ of the polynomial $z^4/24-h^2z^2/4+h^3z/3$, the linear function z and the function $\cos(\omega_{1n}z)-\cos(\omega_{1n}h)$, respectively.

The solution of subsystem (9) for the velocity $V(t,z)$ with the boundary conditions (16) has a similar form

$$V(t,z)=\frac{B\vartheta}{1+h\vartheta}\left[U^0+\sum_{k=1}^{\infty}(U_1^k(t)+U_2^k(t))\sin(\omega_{2k}z)+zU_3(t))\right].$$

It is obvious from expression (21), that the solution for velocity consists of three groups of terms changing with essentially different velocities. The first term is associated with friction forces. The characteristic time is large. The processes reach the steady state slowly. The rate of change of the second term, as well as the first one, is determined by friction. The third velocity term is determined by the processes of heat conduction, and these terms become steady-state within a short time. It is rather difficult to solve a problem of this type numerically. The time step must be chosen small enough so that an acceptable accuracy can be provided. However, at large *t*, this leads to a large calculation time and error accumulation. A larger step causes a loss of accuracy, particularly at the initial stage. The

analytical representation of the solution eliminates these problems. The time of the influence of the heat conduction terms and the beginning of the most essential influence of friction can be evaluated.

After the substitution of the previously calculated solutions, the equation of the third subsystem (10) becomes

$$\frac{\partial T_0}{\partial t} = \chi \frac{\partial^2 T_0}{\partial z^2} - \frac{(A^2+B^2)\vartheta^2}{(1+h\vartheta)^2}\left(z - \sum_{n=1}^{\infty} \alpha_n e^{-\chi\omega_{1n}^2 t} \sin\omega_{1n} z \right) \times$$
$$\times \left\{ \frac{g\beta}{\nu}\left(\frac{z^4}{24} - \frac{h^2 z^2}{4} + \frac{h^3 z}{3} \right) + \frac{\sigma h z}{\eta} - \frac{g\beta}{\nu} \sum_{k=1}^{\infty} \gamma_{1k} e^{-\nu\omega_{2k}^2 t} \sin(\omega_{2k} z) - \right.$$
$$\left. - \sum_{k,m=1}^{\infty} \frac{e^{-\chi\omega_{1m}^2 t} - e^{-\nu\omega_{2k}^2 t}}{\nu\omega_{2k}^2 - \chi\omega_{1m}^2} \left(\beta g \frac{\gamma_{3mk}}{\omega_{1m}} + \frac{\sigma\chi}{\eta} \alpha_m \omega_{1m}^2 \gamma_{2k} \sin(\omega_{1m} h) \right) \sin(\omega_{2k} z) - \frac{\sigma z}{\eta} \sum_{k=1}^{\infty} \alpha_k e^{-\chi\omega_{1k}^2 t} \sin(\omega_{1k} h) \right\}, \tag{22}$$

with the boundary conditions (19) and the initial conditions (20). From the boundary conditions (19), we find the eigenvalues of the boundary value problem $\omega_{3n} = \pi n / h$ and the eigenfunctions $\sin(\omega_{3n} z),\ n \in N$. As before, the solution is sought in the form of the sum of the stationary solution and the Fourier series with respect to the eigenfunctions.

$$T_0(t,z) = \frac{(A^2+B^2)\vartheta^2}{(1+h\vartheta)^2} \Big[T_0^0(z) +$$
$$+ \sum_{k=1}^{\infty} (T_0^{1k}(t) + T_0^{2k}(t) + T_0^{3k}(t) + T_0^{4k}(t) + T_0^{5k}(t) + T_0^{6k}(t) + T_0^{7k}(t)) \sin(\omega_{3k} z) \Big],$$

where T_0^0 is the stationary solution

$$T_0^0 = -\frac{(A^2+B^2)\vartheta^2}{(1+\vartheta h)^2 \chi} \left[\frac{\beta g}{\nu} \left(\frac{z^7}{1008} - \frac{h^2 z^5}{80} + \frac{h^3 z^4}{36} - \frac{41 h^6 z}{2520} \right) - \frac{\sigma h z (z^3 - h^3)}{12\eta} \right].$$

The functions T_0^{jk} have the form

$$T_0^{1k} = -\sum_{m=1}^{\infty} \frac{e^{-\chi\omega_{1m}^2 t} - e^{-\nu\omega_{3k}^2 t}}{\nu\omega_{3k}^2 - \chi\omega_{1m}^2} \alpha_m \gamma_{4mk},$$

$$T_0^{2k} = -\sum_{n,m=1}^{\infty} \left(\frac{e^{-\chi\omega_{1n}^2 t} - e^{-\chi\omega_{5k}^2 t}}{\chi(\omega_{3k}^2 - \omega_{1n}^2)} - \frac{e^{-\nu\omega_{2k}^2 t} - e^{-\chi\omega_{5k}^2 t}}{\chi\omega_{3k}^2 - \nu\omega_{2m}^2} \right) \frac{\alpha_n \gamma_{5km}}{\nu\omega_{2m}^2 - \chi\omega_{1n}^2} \left(g\beta \frac{\gamma_{3nm}}{\omega_{1n}} - \frac{\sigma\chi g\beta}{\eta\nu} \omega_{1n}^2 \gamma_{1m} \sin(\omega_{1n} h) \right),$$

$$T_0^{3k} = \gamma_{6k} \sum_{m=1}^{\infty} \alpha_m \sin(\omega_{1m} h) \frac{e^{-\chi\omega_{1m}^2 t} - e^{-\chi\omega_{3k}^2 t}}{\chi(\omega_{3k}^2 - \omega_{1m}^2)},$$

$$T_0^{4k} = \sum_{n,m,j=1}^{\infty} \alpha_n \alpha_j \left(g\beta \frac{\gamma_{3nm}}{\omega_{1n}} - \frac{\sigma\chi g\beta}{\eta\nu} \omega_{1n}^2 \gamma_{1m} \sin(\omega_{1n} h) \right) \frac{\gamma_{7mjk}}{\nu\omega_{2m}^2 - \chi\omega_{1n}^2} \times$$
$$\times \left(\frac{e^{-\chi(\omega_{1n}^2 + \omega_{1k}^2)t} - e^{-\chi\omega_{3j}^2 t}}{\chi(\omega_{3j}^2 - \omega_{1n}^2 + \omega_{1k}^2)} - \frac{e^{-\nu\omega_{2m}^2 t - \chi\omega_{1k}^2 t} - e^{-\chi\omega_{3j}^2 t}}{\chi(\omega_{3j}^2 - \omega_{1k}^2) - \nu\omega_{2m}^2} \right),$$

$$T_0^{5k} = \frac{g\beta}{\nu} \sum_{m,j=1}^{\infty} \alpha_j \gamma_{1m} \gamma_{7jmk} \frac{e^{-\chi\omega_{1n}^2 t - \nu\omega_{2m}^2 t} - e^{-\chi\omega_{5k}^2 t}}{\chi(\omega_{3k}^2 - \omega_{1j}^2) - \nu\omega_{2m}^2},$$

$$T_0^{6k} = \frac{\sigma}{\eta} \sum_{n,j=1}^{\infty} \alpha_n \alpha_j \gamma_{8nk} \sin(\omega_{1j} h) \frac{e^{-\chi(\omega_{1n}^2 + \omega_{1j}^2)t} - e^{-\chi\omega_{3k}^2 t}}{\chi(\omega_{3k}^2 - \omega_{1n}^2 - \omega_{1j}^2)},$$

$$T_0^{7k} = \frac{g\beta}{\nu}\sum_{k=1}^{\infty}\gamma_{1k}\gamma_{5km}\frac{e^{-\nu\omega_{2k}^2 t} - e^{-\chi\omega_{3m}^2 t}}{\chi\omega_{3m}^2 - \nu\omega_{2k}^2},$$

where γ_{4mk} are the coefficients of the expansion of $U^0(z)\sin(\omega_{1m}z)$ in a series with respect to $\sin(\omega_{3k}z)$; γ_{5km} are the coefficients of the expansion of the product $z\sin(\omega_{2k}z)$ in a series with respect to $\sin(\omega_{3m}z)$; γ_{6k} are the coefficients of the expansion of z^2 in a series with respect to $\sin(\omega_{3k}z)$; γ_{7mjk} are the coefficients of the expansion of $\sin(\omega_{2m}z)\sin(\omega_{1j}z)$ in a series with respect to $\sin(\omega_{3k}z)$, γ_{8nj} are the coefficients of the expansion of $z\sin(\omega_{1n}z)$ in a series with respect to $\sin(\omega_{3j}z)$.

The solution P_0 is obtained by integrating the corresponding terms of T_0 with respect to z :

$$P_0 = \frac{(A^2+B^2)\vartheta^2}{(1+h\vartheta)^2}\Big[P_0^0 +$$

$$+\sum_{k=1}^{\infty}(T_0^{1k}+T_0^{2k}+T_0^{3k}+T_0^{4k}+T_0^{5k}+T_0^{6k}+T_0^{7k})\frac{(\cos(\omega_{3k}h)-\cos(\omega_{3k}z))}{\omega_{3k}}\Big],$$

where P_0^0 is the stationary solution,

$$P_0^0 = -g\beta\left[\frac{\beta g}{\nu}\left(\frac{z^8}{8064}-\frac{h^2z^6}{480}+\frac{h^3z^5}{180}-\frac{41h^6z^2}{5040}+\frac{61h^8}{13440}\right)-\frac{\sigma}{\eta}\left(\frac{z^5}{60}-\frac{h^3z^2}{24}+\frac{h^5}{40}\right)\right].$$

NUMERICAL EXPERIMENT

A change in the parameter T_1 for water is computed in the example. The parameter A is supposed to be equal to ~1, with the layer thickness h=1/2.

The physical parameters are assumed to be as follows:

- density $\rho \approx 1000$ kg/m^3,
- thermal conductivity $\lambda = 59.9$ W/(mK),
- heat capacity $C_P = 4.183$ kJ/(kgK),
- thermal diffusivity $\chi = 14.3\cdot 10^{-8}$, m^2/s
- kinematic viscosity $\nu = 1.006\cdot 10^{-6}$ m^2/s,
- coefficient of volume expansion $\beta = 1.82\cdot 10^{-4}$ 1/K,
- coefficient of surface tension $\sigma = 726.4\cdot 10^{-4}$ N/m^2,
- coefficient of heat exchange with a metallic wall $\alpha = 350$ W/(m^{2}K).

The relation of the parameters is $\vartheta = \alpha/\lambda$, $\chi = \alpha/C_P$.

The frequency equation (18) is solved numerically. The Fourier series expansion coefficients are calculated beforehand to be stores in tables. In the computation of the flow parameters, the series are added together up to 50 terms with the use of the data from the tables. When the number of terms is smaller, e.g. 5, although the qualitative behavior of the solution is maintained, the error is much larger. The computation results are presented in the following figures, which show the evolution of the position of the stagnation point with a change in one of the temperature gradient component and a change in the counterflow zone.

In Figs. 5 and 6, the counterflow disappears, but the stagnation point is preserved in Fig. 5.

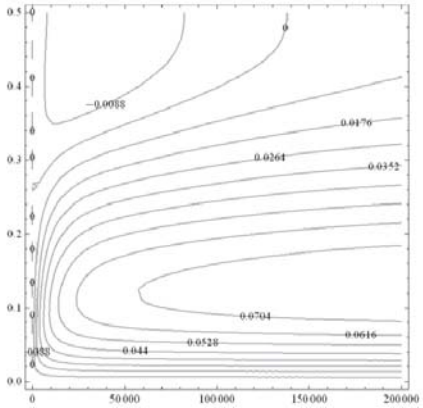

FIGURE 1. The level lines of the function $U(t,z)$ with $A=-0.08, B=0.049$

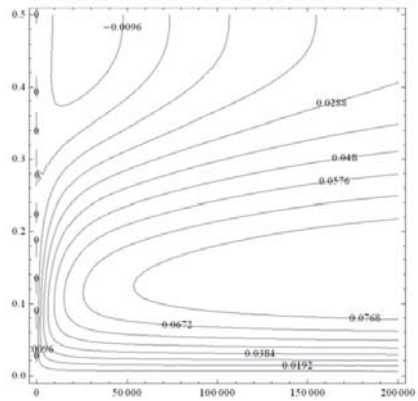

FIGURE 2. The level lines of the function $U(t,z)$ with $A=-0.08,\ B=0.048$

In Figs. 1 and 2, the stagnation point disappears with time, and the flow becomes homogeneous.

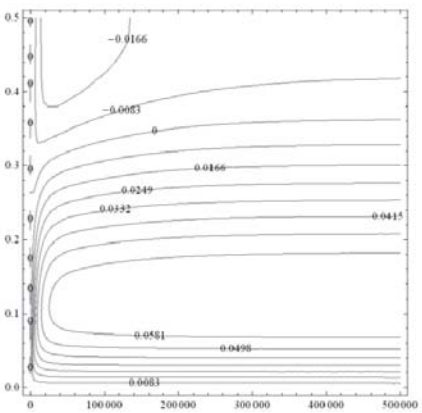

FIGURE 3. The level lines of the function $U(t,z)$ with $A=-0.08, B=0.05$

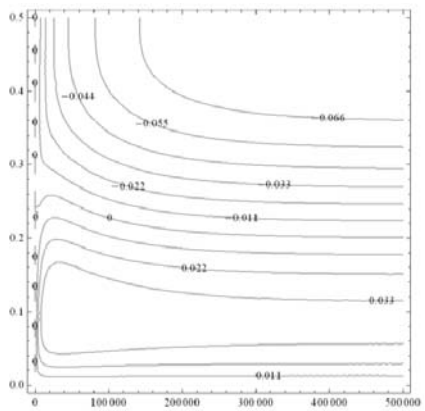

FIGURE 4. The level lines of the function $U(t,z)$ with $A=-0.08, B=0.053$

The stagnation point is preserved, and so does the counterflow.

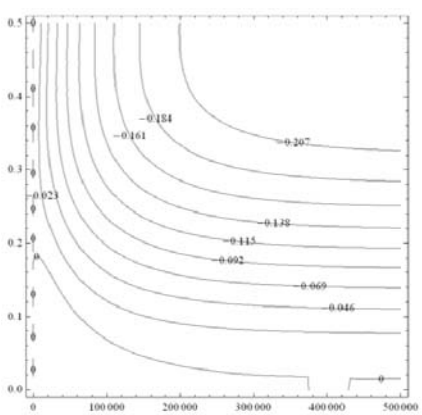

FIGURE 5. The level lines of the function $U(t,z)$ with $A=-0.08, B=0.06$

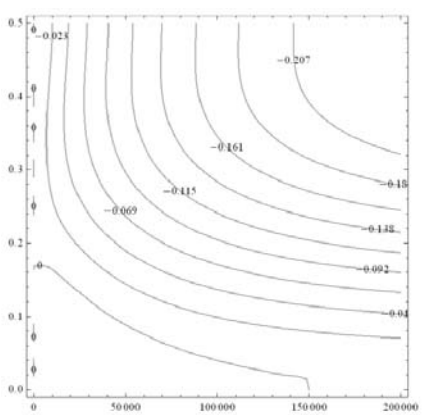

FIGURE 6. The level lines of the function $U(t,z)$ with $A=-0.08, B=0.0615$

CONCLUSION

The paper has discussed the temperature-gradient-induced nonstationary laminar flows of the Bénard-Marangoni Convection of a viscose incompressible fluid. Solutions have been obtained for the third-kind boundary conditions on the free surface (Newtonian heat exchange). The solutions enable the transition process duration to be evaluated. The example has shown the evolution of the stagnation point and the counterflow process.

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