# Application of series with recurrently calculated coefficients for solving initial-boundary value problems for nonlinear wave equations

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**M. Yu. Filimonov, and N. A. Vaganova**

# Application of Series with Recurrently Calculated Coefficients for Solving Initial-Boundary Value Problems for Nonlinear Wave Equations

M.Yu. Filimonov[1,2,a)] and N.A. Vaganova[1,2,b)]

[1]*Ural Federal University, Yekaterinburg, Russia*
[2]*Krasovskii Institute of Mathematics and Mechanics, Yekaterinburg, Russia*

[a)]Corresponding author: fmy@imm.uran.ru
[b)]vna@imm.uran.ru

**Abstract.** For one class of nonlinear wave equations with a small parameter, an initial-boundary value problem with zero boundary conditions is considered. The solution of such a problem is constructed with using series with recurrently calculated coefficients in two ways. In the first case, the method of special series is considered, which is based on the choice of some functions (basic functions), by the powers of these functions the solution of the original problem is presented into a series with recurrently calculated coefficients. In the other case to represent solutions of the problem a combination of Fourier and small parameter methods is used. It is shown that both proposed constructions of series with recurrently calculated coefficients converge to the solution of the initial-boundary value problem on a finite time interval.

## INTRODUCTION

There are various analytical methods for representing solutions of partial differential equations (PDE) by R. Courant [1], G.F. Duff [2], D. Ludwig [3], V.M. Babich [4]. In this paper an approach to construct solutions of PDE is closer to the approach of solving ordinary differential equations (ODE). The ideas to represent solutions of nonlinear ODE as a power series with respect to functions defined sequentially from another equations was considered by, for example, A.M. Lyapunov [5] and N.P. Yerugin [6]. In this case series with recurrently computed coefficients are also obtained.

Method of special series [7, 8, 9] is an analytical method of representation of solutions of nonlinear PDE in the form of series by the powers of one [8, 10, 11] or several functions [12, 13, 14, 15] chosen in a special way, which allow the series coefficients to be calculated recurrently without applying any truncation procedures. In contrast to the power Taylor series, which converge only for Cauchy–Kovalevskaya equations under the conditions of analyticity of the problem initial conditions, the constructed series can converge for wider classes of differential equations and systems.

This choice of functions allows one to find the coefficients of the series recurrently. These functions we will call *basic functions*. In some cases it is possible to prove global convergence of the constructed series in unlimited domains [16, 17, 18], where the numerical methods are hard to be used.

A theorem on the local convergence on time variable of the series to the solution of the initial-boundary value problem is proved for a wide class of nonlinear wave equations [8, 12, 19]. Note that in some cases the combination of Galerkin method and the expansions of the solution by the powers of a small parameter also leads to recurrent construction of the coefficients of the expansions of the solution into a series [20, 21], that allows to prove convergence of the series. The method of special series has been successfully applied to representation of solutions for Lin-Reissner-Tsien equation [22, 23, 24, 25], describing transonic gas flows. In some cases, special series are terminated and exact solutions are obtained [18, 26, 27, 28]. Sometimes, on the basis of an exact solution, it is possible to construct a new class of solution for a nonlinear filtration equation [29], for which solutions also exist in the form of series with recurrently calculated coefficients [30, 31, 32, 33].

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The basic functions can also contain an arbitrary function [34], which can be used in proving the existence of a solution of the initial-boundary value problem for nonlinear PDE [35, 36].

In this paper, we consider the representation of solutions of an initial-boundary value problem with zero boundary conditions for a certain class of nonlinear wave equations in the form of a convergent double series and we compare this solution with the solution obtained by Fourier method.

## METHOD OF SPECIAL SERIES FOR REPRESENTATION OF SOLUTIONS OF NONLINEAR WAVE EQUATION WITH ZERO BOUNDARY CONDITIONS

Following [17, 19, 37], let consider an approach to construction of solutions of nonlinear PDE in the form of special series.

Consider the method of special series for constructing solutions for following nonlinear wave equation of two independent variables $t$ and $x$:

$$u_{tt} = F(t, u, u_x, u_{xx}), \tag{1}$$

where $F$ is a polynomial of the unknown function $u(x, t)$ and the derivatives $u_x$, $u_{xx}$. The coefficients of this polynomial are continuous and bounded functions for $t \geq 0$.

Consider the following double series:

$$u(x, t) = \sum_{i,j=0}^{\infty} \alpha_{ij}(t) P^i(x) Q^j(x). \tag{2}$$

Let the functions $P(x)$ and $Q(x)$ satisfy the system of ODE

$$\begin{aligned} P' &= a_{10}P + a_{01}Q + a_{20}P^2 + a_{11}PQ + a_{02}Q^2 + \ldots = W_1(P, Q), \\ Q' &= b_{01}Q + b_{20}P^2 + b_{11}PQ + b_{20}P^2 + \ldots = W_2(P, Q). \end{aligned} \tag{3}$$

Here $a_{ij}$ = const, $b_{ij}$ = const, $i, j \geq 0$ and the functions $W_1(P, Q)$, $W_2(P, Q)$ are analytical with the conditions $W_1(0, 0) = W_2(0, 0) = \partial W_2(0, 0)/\partial P = 0$.

The following assertion is valid [37].

**Assertion 1.** *There exist the coefficients $\alpha_{ij}(t)$, such that the duble series* (2)*,* (3) *is a formal solution of equation* (1).

This assertion 1 is proved due to constructing the coefficients $\alpha_{ij}(t)$ by substituting series (2) into equation (1), differentiation, and multiplication of the series with taking into account relations (3). Then, equating expressions with the same powers of $P^i(x)$, $Q^j(x)$ the series coefficients $\alpha_{ij}(t)$ are found from a sequence of ODE. The equation for the free term $\alpha_{00}(t)$ of the series may be nonlinear. Note that the recurrence of obtaining the coefficient of the series is achieved by a special form of the linear part of system (3).

The following numbering function for the sequence of calculation of coefficients $\alpha_{ij}(t)$ is used:

$$c(m, n) = m + \frac{1}{2}(m + n)(m + n + 1).$$

According to this function, $\alpha_{pr}(t)$ will have been calculated before $\alpha_{mn}(t)$, if $c(p, r) < c(m, n)$.

Thus $c(0, 0) = 0$, $c(1, 0) = 0$, $c(0, 1) = 0, \ldots$ Therefore, the coefficients of the series will be calculated as follows: $\alpha_{00}(t)$, $\alpha_{10}(t)$, $\alpha_{01}(t), \ldots$

We consider method of special series [8, 37] for representation of solutions of initial-boundary value problem for following class of nonlinear wave equation:

$$u_{tt} = u_{xx} + u_{xx}\left(\sum_{m=1}^{K_1} \gamma_{2m} u^{2m} + \sum_{m=1}^{K_2} \beta_m u_x^{2m}\right) + \sum_{m+n=1}^{K_3} \omega_{2m+1,n} u^{2m+1} u_x^m, \quad \gamma_{2m}, \beta_m, \omega_{2m+1,n} = \text{const} \tag{4}$$

with initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad 0 \leq x \leq 1 \tag{5}$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \geq 0. \tag{6}$$

Consider a special case of system (3)

$$\begin{aligned} P' &= \sum_{m+2n\leq N-1} a_{m,2n+1} P^m Q^{2n+1}, \\ Q' &= \sum_{m+2n\leq N} b_{m,2n} P^m Q^{2n} \end{aligned} \tag{7}$$

and a special case of series (2)

$$u(x,t) = \sum_{i=0}^{\infty} \sum_{2j+1=1}^{\infty} \alpha_{ij}(t) P^i(x) Q^{2j+1}(x). \tag{8}$$

The following assertion is valid.

**Assertion 2.** *The double series* (8), (7) *is a formal solution of Cauchy problem* (4)–(5)*, if initial data* (5) *have the following form:*

$$u_\nu(x) = \sum_{i,j=0}^{\infty} \alpha^\nu_{i,2j+1} P^i(x) Q^{2j+1}(x), \quad \alpha^\nu_{i,2j+1} = \text{const}, \quad \nu = 0,1. \tag{9}$$

**Proof.** This assertion is verified by substituting series (2) into equation (4), differentiation, and multiplication of the series with taking into account relations (7). The coefficients $\alpha_{ij}(t)$ of series (8) will be calculated recurently from linear ODE

$$\alpha''_{ij} = R_{ij}(t, \alpha_{pr}), \tag{10}$$

in which the right sides $R_{ij}(t, \alpha_{pr})$ of equations (10) are determined by the form of the right side of equation (4) and the right side of system (7). The coefficients $\alpha_{pr}$ are calculated earlier, than the coefficients $\alpha_{ij}$, since for the corresponding numbering functions relation $c(p,r) < c(i,j)$ is true.

The identities are valid

$$\alpha_{i,2m}(t) \equiv 0, \quad t \geq 0, \quad i,m = 0,1,\ldots \tag{11}$$

The proof of these identities (11) is based on an analysis of the parity of the second indices of the coefficients $\alpha_{pr}$ included in the right-hand side of the equation (10). In this case, the index $r$ is always either even, or the coefficients with even second indices are factors in members containing odds with odd second indices. This fact is easy to check using the form of system (4). Therefore, if at the initial moment $\alpha_{i,2m}(0) = 0$, $\alpha'_{i,2m}(0) = 0$, then for $t > 0$ $\alpha_{i,2m}(t) \equiv 0$.

Thus, series (2), which is used to solve equation (4) does not include terms with coefficients for which the second indices are even. Therefore, series (2) coincides with series (8).

Consequently double series (8), (7) is a formal solution of Cauchy problem (4)–(5).

The following theorem is valid.

**Theorem 1.** *Let the following conditions are satisfied:*

1) *basic functions $P(x)$, $Q(x)$ satisfy system of differential equations* (7)*;*
2) $|P(x)| \leq 1$, $|Q(x)| \leq 1$ *for all $x$;*
3) $P(0) \neq 0$, $Q(0) = Q(1) = 0$*;*
4) *initial data* (5) *are presented in the form of series* (9) *and the following inequalities are valid:*

$$|\alpha^{(\nu)}_{ij}| \leq \frac{M \exp[-(i+j)]}{3(i+1)^4(j+1)^4}, \quad \nu = 0,1, \quad i,j \geq 0, \quad M = \text{const}.$$

*Then a solution of initial-boundary value problem* (4)–(6) *is represented in the form of the series* (8), (7) *in domain $G = \{(x,t) : 0 \leq x \leq 1, \quad 0 \leq t \leq T_1\}$, $T_1 > 0$.*

**Proof.** Series (8), (7) is a special case of series (2). But series (8), (7) can also be used to solve initial-boundary value problem (4)–(6), because it contains multiplier $Q(x)$, with condition 3 of Theorem 1 $Q(0) = Q(1) = 0$.

Taking into account Assertion 2 we can calculate the coefficients of series (8), (7) from second order linear ODE (10).

By the method of mathematical induction, taking into account that the solutions of equations (10) have the form

$$\alpha_{ij}(t) = \int_0^t \int_0^\tau R_{ij}(\sigma, u_{pr}) d\sigma d\tau + \alpha_{ij}^{(1)} t + \alpha_{ij}^{(0)},$$

we prove the following inequalities:

$$|\alpha_{pr}(t)| \leq \frac{M \exp[-q(p+r)t]}{(p+1)^4 (r+1)^4}, \quad 0 \leq t \leq (qN)^{-1}, \quad q = \text{const}, \quad p, r \geq 0. \tag{12}$$

Inequalities (12) are proved similarly [16].

Using inequality (12), we can estimate series (8), (7) and the series corresponding to partial derivatives $u_x$, $u_{xx}$, $u_t$, $u_{tt}$.

**Example of basic functions (7).** Consider functions

$$P(x) = \cos \pi x, \quad Q(x) = \sin \pi x. \tag{13}$$

These functions satisfy all conditions of Theorem 1. System (7) has the form

$$\begin{aligned} P' &= -\pi Q(P^2 + Q^2), \\ Q' &= \pi P(P^2 + Q^2). \end{aligned} \tag{14}$$

In system (14) we used the equality $P^2 + Q^2 \equiv 1$, which allows us to find the coefficients of series (8) from a sequence of second order ODE.

## JUSTIFICATION OF FOURIER METHOD FOR NONLINEAR WAVE EQUATIONS

Consider a special case of a nonlinear wave equation with small parameter $\varepsilon$

$$u_{tt} = u_{xx} + \varepsilon \sum_{k=1}^{K} \left( b_k u^{2k+1} + a_k u_{xx} u_x^k \right), \quad a_k, b_k = \text{const}. \tag{15}$$

Let initial data (5) for equation (15) has the form

$$u_\nu(x) = \sum_{j=1}^{J} \alpha_j^\nu \sin(\pi j x), \quad \alpha_j^\nu = \text{const}, \quad \nu = 0, 1 \tag{16}$$

and boundary conditions are (6).

We find the solution of problem (15), (5), (6) in the form

$$u(x,t) = \sum_{s=1}^{N} z_s(\varepsilon, t) X_s(x) + v(\varepsilon, t, x, N), \tag{17}$$

where $X_s(x) = \sin \pi s x$, $N \geq J$. If we substitute (17) into (15) and equate the expressions in front of identical $X_s(x)$, $s = \overline{1, N}$, we obtain the system of nonlinear ODE for the coefficients $z_s(\varepsilon, t)$

$$\begin{aligned} z_i'' &= -\omega_i^{2n} z_i + \varepsilon P_i(z_1, \ldots, z_N), \quad \omega_i = \pi i \\ z_i'(0) &= \alpha_i^1, z_i(0) = \alpha_i^0, \quad i = \overline{1, N}. \end{aligned} \tag{18}$$

In the following this system is called the *leading system*. Function $v(\varepsilon, t, x, N)$ satisfies the following equation:

$$v_{tt}=v_{xx}+\varepsilon\left(f(x,v,v_x,v_{xx})+\sum_{s=N+1}^{m_1}P_s(z_1,\ldots,z_N)X_s(x)\right), \tag{19}$$

but with the zero initial and boundary conditions

$$v(x,0)=v_t(x,0)=0, \tag{20}$$

$$v(0,t)=v(1,t)=0, \quad t\geq 0. \tag{21}$$

Here number $m_1$, polynomials $f(x,v,v_x,v_{xx})$, $P_s(z_1,\ldots,z_N)$, $s=\overline{N+1,m_1}$ are determined by equation (15) and the number $N$ in (17).

We find a solution of initial-boundary value problem (19)–(21) in the form of a power series of $\varepsilon$

$$v(x,t)=\sum_{i=1}^{\infty}\varepsilon^i v_i(x,t,\varepsilon). \tag{22}$$

Here the functions $v_i$ depend on $\varepsilon$, since the functions $z_j$, $j=\overline{1,N}$, wich are a solution of the leading system (17) also depend on $\varepsilon$.

If we substitute (22) into (19), we obtain the linear non-uniform equations for the functions $v_i(x,t,\varepsilon)$

$$\frac{\partial^2 v_i}{\partial t^2}-\frac{\partial^2 v_i}{\partial x^2}=F_i(t,x,v_1,\ldots,v_{i-1}), \tag{23}$$

$$v_i(x,0)=\frac{\partial v_i(x,0)}{\partial t}=0, \tag{24}$$

$$v_i(0,t)=v_i(1,t)=0, \quad t\geq 0. \tag{25}$$

For $i=1$ we have

$$F_1=\sum_{j=N+1}^{m_1}P_j(z_1,\ldots,z_N)X_j.$$

We may find the solution of problem (23)–(25) in the form of the sums

$$v_i=\sum_{j=1}^{m_i}q_{ij}(t,\varepsilon)X_j, \quad i\geq 1,$$

where $m_i=[(r_f-1)i+1]N+N_0$, $r_f$ is the degree of the polynomials $P_s(z_1,\ldots,z_N)$, the number $N_0\geq 0$ is determined by function $f$, and the coefficients $q_{ij}$ are determined successively as solutions of linear ODE.

Thus, we can rewrite the solution of problem (15)–(17) in the form of the series

$$u(x,t)=\sum_{i=1}^{N}z_i(\varepsilon,t)X_i+\sum_{i=1}^{\infty}\varepsilon^i\sum_{j=1}^{m_i}q_{ij}(t,\varepsilon)X_j. \tag{26}$$

We investigate convergence of series (26) to the solution of problem (15)–(17). The following theorems are valid.

**Theorem 2.** *Let the initial data* $\alpha_i^\nu$ *satisfy the conditions*

$$|\alpha_i^0|\leq\frac{M}{\omega_i^4},\ |\alpha_i^1|\leq\frac{M}{i^3},\ i=\overline{1,N},\ M\geq 0,$$

$$|a_k|+|b_k|\leq M_1, \quad M_1\geq 0$$

*and* $|\varepsilon|\leq\varepsilon_0(f,M,M_1)$. *Then the solutions of the corresponding leading systems* (18) *are bounded for all* $t\geq 0$.

To prove the boundedness of the solutions $z_i$ of the corresponding leading systems (18), Lyapunov functions are constructed for any $N$. We can estimate the functions $z_i$, if $|\varepsilon|\leq\varepsilon_0(f,M,M_1)$

$$|z_i| \le \frac{M}{\omega_i^4}, \quad |\dot{z}_i| \le \frac{M}{i^3}, \quad i = \overline{1, N}, \quad t \ge 0. \tag{27}$$

**Theorem 3.** *When the conditions of Theorem 2 are satisfied series* (26) *uniformly converges to the solution of initial-boundary value problem* (15), (5), (6), (16) *for all* $0 \le x \le 1$ *and* $0 \le t \le T$ $(T \sim \varepsilon^{-1})$.

If we use the method of mathematical induction and estimates (27), we can evaluate the functions $q_{ij}$ as follows

$$|q_{ij}| \le \frac{M^i t^i}{N^2 i^2 \omega_j^4}, \quad i \ge 1, \quad j = \overline{1, m_i}. \tag{28}$$

The estimates (27), (28) allow us to prove that series (26) converges to the solution of the initial-boundary value problem (15), (5), (6), (16) for all $0 \le x \le 1$ and $0 \le t \le T$ $(T = (M\varepsilon)^{-1})$.

**Remark.** *The additional function* $v$ *are estimated as follows*

$$|v(\varepsilon, t, x, N)| \le \frac{1}{N^2}\varepsilon t\Big(1 + \varepsilon t\Big)C, \quad C = \text{const.}$$

## NUMERICAL RESULTS

We consider an equation which describes nonlinear vibrations of a string

$$u_{tt} = u_{xx}(1 + \varepsilon u_x^2) \tag{29}$$

with fixed endpoints

$$u(0, t) = u(\pi, t) = 0, \quad t \ge 0 \tag{30}$$

and initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = 0. \tag{31}$$

We find the solution of problem (29)–(31) in the form of the finite Fourier series

$$u(t, x) = \sum_{i=1}^{N} z_i(t) \sin(ix) \tag{32}$$

and in the form of double series (8) with the functions $P(x)= \cos x$, $Q(x)= \sin x$

$$u(t, x) = \sum_{i=0}^{\infty} \sum_{2j+1=1}^{\infty} \alpha_{i,2j+1}(t) \cos^i x \sin^{2j+1} x. \tag{33}$$

We obtain equations for $z_i(t)$ $(\alpha_{ij}(t))$ by substituting finite sums (32) and series (33) into equation (29) and equate the multipliers of the same functions.

For the coefficients $z_i(t)$ we get a nonlinear system of equation of the form (18). For $N = 4$ this system is as follows:

$$z_1''(t)=-\Big[z_1(t)\pi^2+\varepsilon\pi^4\Big(\tfrac{1}{4}z_1^3+\tfrac{3}{4}z_1^2z_3+2z_1z_2^2+4z_1z_2z_4+\tfrac{9}{2}z_1z_3^2+8z_1z_4^2+3z_2^2z_3+12z_2z_3z_4\Big)\Big],$$
$$z_2''(t)=-\Big[4z_2(t)\pi^2+\varepsilon\pi^4\Big(2z_1^2z_2+2z_1^2z_4+6z_1z_2z_3+12z_1z_3z_4+4z_2^3+18z_2z_3^2+32z_2z_4^2+18z_3^2z_4\Big)\Big],$$
$$z_3''(t)=-\Big[9z_3(t)\pi^2+\varepsilon\pi^4\Big(\tfrac{1}{4}z_1^3+\tfrac{9}{2}z_1^2z_3+3z_1z_2^2+12z_1z_2z_4+18z_2^2z_3+36z_2z_3z_4+\tfrac{81}{4}z_3^3+72z_3z_4^2\Big)\Big],$$
$$z_4''(t)=-\Big[16z_4(t)\pi^2+\varepsilon\pi^4\Big(2z_1^2z_2+8z_1^2z_4+12z_1z_2z_3+32z_2^2z_4+18z_2z_3^2+72z_3^2z_4+64z_4^3\Big)\Big].$$

Here $z_i = z_i(t)$, $i = \overline{1, 4}$.

We obtain equations for the coefficients $\alpha_{ij}(t)$ of the series (33) by substituting the series into (29) and applying the equality $\sin^2 x + \cos^2 x \equiv 1$. As a result we get the sequence of linear ODE from which we recurrently compute the coefficients $\alpha_{ij}(t)$

$$
\begin{aligned}
&\alpha''_{00}(t) = -\alpha_{00}(t),\\
&\alpha''_{10}(t) = -4\alpha_{10}(t),\\
&\alpha''_{01}(t) = -3\alpha_{01}(t),\\
&\alpha''_{20}(t) = -7\alpha_{20}(t) - \varepsilon\alpha^3_{00}(t),\\
&\alpha''_{11}(t) = -10\alpha_{11}(t) + 2\varepsilon\alpha^2_{00}(t)\alpha_{10}(t),\\
&\alpha''_{02}(t) = -5\alpha_{02}(t) - \varepsilon\alpha^2_{10}(t)\alpha_{00}(t),\\
&\alpha''_{30}(t) = -10\alpha_{30}(t) - 6\varepsilon\alpha^2_{00}(t)\alpha_{10}(t),\\
&\alpha''_{21}(t) = -17\alpha_{21}(t)+2\alpha_{20}(t)+6\alpha_{01}(t)-\varepsilon[\alpha^2_{00}(t)\big(9\alpha_{10}(t)+4\alpha_{20}(t)\big)+10\alpha_{00}(t)\alpha^2_{10}(t)],\\
&\ldots.
\end{aligned}
$$

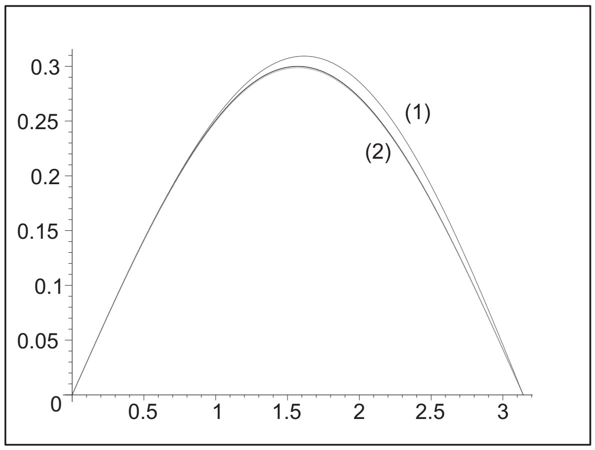


**FIGURE 1.** $u(0, x) = \sin x$, $t = \frac{2\pi}{5}$.

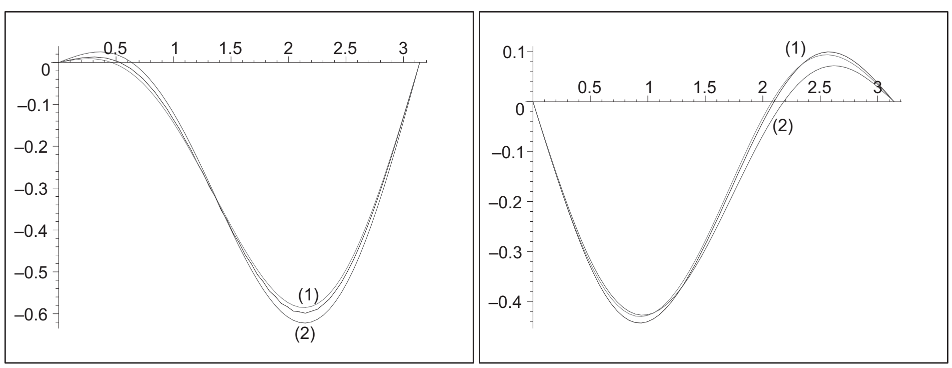


**FIGURE 2.** $u(0, x) = \frac{1}{2}\sin x + \frac{1}{2}\sin 2x$. The solutions at the time moments: $t_1 = \frac{5\pi}{6}$ (left), $t_2 = \frac{2\pi}{3}$ (right).

In Figures 1–2 the results of numerical solutions of initial-boundary value problem (29)–(31) with $\varepsilon = 0.1$ are presented. The solutions are computed in the form of a finite Fourier series (32) with $N = 6$ and in the form of the special series (33).

Figure 1 shows the solutions at $t = \frac{2\pi}{5}$. In the figure (1) and (2) denote the finite special series (33) and the finite Fourier series (32), respectively. The solutions computed by implicit finite difference method and solution computed by Fourier method differ imperceptibly.

Figure 2 shows the solutions at $t = \frac{5\pi}{6}$ and $t = \frac{2\pi}{3}$. In the figure (1) and (2) denote the finite sum of series (33) and the finite Fourier series (32), respectively. The third line between (1) and (2) is the solution computed by an implicit finite difference method. In this case the special series is more close to the finite difference solution than the Fourier series.

## CONCLUSION

Thus, it is shown that with the help of special double series we can construct some classes of solutions for initial-boundary value problem for nonlinear wave equations. Classes of nonlinear wave equations with a small parameter were described, for which it is possible to justify the applicability of Fourier method. The results of numerical calculations showed that the method of special series and Fourier method can be used to represent the solution of initial-boundary value problem for nonlinear wave equations.

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