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A. N. Sesekin, and A. A. Frolova

Singular Linear-Quadratic Problem with a Terminal Condition

A. N. Sesekin[1,2,a)] and A. A. Frolova[1]

[1]*Ural Federal University, 19 Mira st., Ekaterinburg, 620002, Russia.*
[2]*N. N. Krasovskii Institute of Mathematics and Mechanics, Ural Division of Russian Academy of Sciences, 16 S. Kovalevskaya, Ekaterinburg, 620108, Russia.*

[a)]Corresponding author: sesekin@list.ru

Abstract. A singular linear-quadratic optimization problem with a terminal condition is considered. This problem in the space of absolutely continuous functions has no solutions. To ensure the existence of a solution, it is necessary to extend the set of admissible controls. In the case under consideration, we consider the generalized derivatives of functions of bounded variation as admissible controls. As a result, valid controls may contain impulse components. Under some assumptions, an optimal control is constructed containing impulse components at the initial and final instants of time. An illustrative example is provided.

INTRODUCTION

The linear-quadratic problem in optimal control is one of the most common and well-studied. Starting with the works of R. Kallman and A.M. Letov. It is well known that linear-quadratic optimization problems are of great practical importance. The problem is solved analytically almost to the end and only at the last stage it is necessary to solve the Riccati equation, which is analytically solved only in rare cases. The advantage of solving the linear-quadratic problem is that the control found is constructed on the basis of the feedback principle (it is able to work out perturbations that act on the object in the control process). The functional in this task has the following form:

$$J[u(\cdot)] = \frac{1}{2}x^T(t_1)S_f x^T(t_1) + \frac{1}{2}\int_{t_0}^{t_1} [x^T(t)Q(t)x(t) + u^T R(t)u(t)]\, dt. \tag{1}$$

The integrand contains two non-negative definite quadratic forms - one for x and the other for u. Minimization of such a functional minimizes the integral deviation of the trajectory from the zero solution and minimizes the cost of control, which is given by the quadratic form of the control u. Often in applied problems it happens that minimization of the quadratic component is not required (see, for example, [1]). But if we remove the quadratic form of the control in the functional (1), then the resulting problem in terms of control theory becomes degenerate (it is not possible to determine the optimal control from the maximum principle). A consequence of this is also the fact that the new problem, generally speaking, has no solution in the space of trajectories - absolutely continuous functions, and there is a need to expand the control space to generalized derivatives of functions of bounded variation.

In this paper, we consider the linear-quadratic optimization problem with the terminal condition. It inherited the main problems from the classical version of the linear-quadratic problem, which was considered in [2], but has some specifics. In this paper, control is constructed in the problem of minimizing the integral quadratic functional, which depends only on the phase coordinates. Note that there are many problems when it is important to ensure a small deviation of the phase vector from the zero solution, and there are no restrictions on the control. In solving the auxiliary problem, the Pontryagin maximum principle was used. On a specific example, the type of optimal control action was considered.

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FORMULATION OF THE PROBLEM

We consider the problem of minimizing the functional

$$J[u(\cdot)] = \frac{1}{2}\int_{t_0}^{t_1}[x^T(t)Q(t)x(t)]dt, \tag{2}$$

where $Q(t)$ is the non-negative definite for each $t \in [t_0, t_1]$ matrix function of dimension $n\times n$ with continuous elements, along the trajectories of the system of differential equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3}$$

with boundary conditions

$$x(t_0) = x_0, \;\; Hx(t_1) = g. \tag{4}$$

Here $x(t)$ is a n-dimensional function, $u(t)$ — m–dimensional vector-function, ($t_0 < t_1$), $A(t)$ is a continuous matrix function of dimension $n\times n$, $B(t)$ is a continuously differentiable matrix function of dimension $n\times m$, with continuously differentiable elements, H is a dimension matrix $k \times n$, $g \in R^k$, $k \leq n$.

A feature of the problem (2) — (4) is that, in the class of measurable controls, the problem under consideration, generally speaking, has no solution. In this regard, it is necessary to expand the set of feasible controls and build an extended task.

Let $u(t) = \dot{v}(t)$, where $v(t)$ is a function of bounded variation, and the derivative $\dot{v}(t)$ is understood in a generalized sense. For the first time such an extension in optimal control problems was considered in the monograph [3]. Note that the solution $x(t)$ in this case will be a function of bounded variation that satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} A(s)x(s)\,ds + \int_{t_0}^{t} B(s)\,dv(s),$$

where the last integral is understood in the Stieltjes sense. A non-degenerate version of this problem was considered in [2].

AUXILIARY PROBLEM

In the problem (2) — (4) we make a change of variable

$$y(t) = x(t) - B(t)v(t). \tag{5}$$

This change of variable was used, in particular, in constructing various extensions of optimal control problems, in particular, in [4, 5, 6].

As a result of the replacement (5) in the problem (2)—(4) $u(t) = \dot{v}(t)$, we obtain the following auxiliary problem. Required to minimize functional

$$J[u(\cdot)] = \frac{1}{2}\int_{t_0}^{t_1}[(B(t)v(t) + y(t)]^T Q(t)[B(t)v(t) + y(t)]dt \tag{6}$$

along the trajectories of the system of differential equations

$$\dot{y}(t) = A(t)y(t) + B_1(t)v(t), \tag{7}$$

where $B_1(t) = A(t)B(t) - \dot{B}(t)$.

The boundary condition for $t = t_0$ will have the form $y(t_0) = x_0$, $v(t_0) = 0$ and for $t = t_1$

$$H(y(t_1) + B(t_1)v(t_1)) = g. \tag{8}$$

Further, we assume that the matrices $B(t)$ and H have the structure

$$B(t) = \begin{pmatrix} \hat{0} \\ \bar{B} \end{pmatrix}, H = \begin{pmatrix} \hat{H}_1 \\ \bar{H}_2 \end{pmatrix}. \tag{9}$$

Here $\bar{B}(t)$ — $m \times m$ is a continuous non-degenerate matrix, $\hat{0}$ is the zero $(n-m) \times m$ matrix. The matrix H_1 consists of the first $k-m$ matrix H, and the matrix H_2 consists of the remaining m rows of the matrix H. We represent the vectors y and g in the form $y = (\tilde{y}^T, \bar{y}^T)^T$, where $\tilde{y}$ is the first $n-m$ coordinates of the vector y, and the subsequent m coordinates will be denoted as $\bar{y}$, and $g = (\breve{g}^T, \bar{g}^T)^T$, where $\breve{g}$ is the first $k-m$ of the coordinates of the vector g and $\bar{g}$ is the last m of the coordinates of the vector g. Then from (8), taking into account that g is a nonzero vector, it follows that the first $n-m$ coordinates of the vector $y(t_1)$ will be zero. With the notation introduced, the equation (8) can be written as

$$H_1 y(t_1) = \breve{g} \tag{10}$$

$$H_2 y(t_1) + \bar{B}(t_1)v(t_1) = \bar{g}. \tag{11}$$

Therefore, without imposing any restrictions on $\hat{y}(t_1)$, the condition (11) can always be ensured by setting

$$v(t_1) = -\bar{B}^{-1}(H_2 y(t_1) - \bar{g}). \tag{12}$$

Thus, the boundary condition $Hx(t_1) = g$ of the original problem in the auxiliary problem turns into the condition (10), and the remaining coordinates of the vector $y(t_1)$ can be arbitrary. For the original problem, the condition (12) means that the last m coordinates of the vector $x(t_1)$ will receive the required values due to the momentum at a finite moment in time.

SOLUTION OF AN AUXILIARY PROBLEM

Auxiliary problem is solved using the L.S. Pontryagins maximum principle [7]. Pontryagin function will be next

$$\mathcal{H}(t, y(t), v(t), \psi(t)) = \psi^T[A(t)y(t) + B_1(t)v(t)] - \frac{1}{2}[y^T(t)Q(t)y(t) + 2v^T B^T(t)Q(t)y(t) + v^T B^T(t)Q(t)B(t)v(t)].$$

The conjugate system will be as follows

$$\dot{\psi} = Q(t)y - A^T\psi + Q(t)B(t)v(t). \tag{13}$$

Due to the fact that the optimal control delivers the maximum of the function $H(t, y(t), v, \psi(t))$, the following necessary condition must be satisfied

$$\frac{\partial \mathcal{H}}{\partial v} = B_1(t)^T\psi - B^T Q(t)y(t) - B(t)^T B(t)v(t) = 0.$$

We will assume that $\det(B^T(t)Q(t)B(t)) \neq 0$. Then from the previous equation it follows that

$$v(t) = (B^T(t)Q(t)B(t))^{-1}[(B_1(t)^T\psi - B^T(t)Q(t)y(t)]. \tag{14}$$

Substitute control (14) into (7) and (13). As a result, we get

$$\dot{y}(t) = (A(t) - M(t)B^T(t)Q(t))y(t) + M(t)B_1^T(t)\psi(t), \tag{15}$$

$$\dot{\psi}(t) = [Q(t) - N(t)B^T Q(t)]y(t) - [A(t)^T - N(t)B_1^T(t)]\psi, \tag{16}$$

where

$$M(t) = B_1(t)^T(B(t)^T Q(t)B(t))^{-1}, \quad N(t) = Q(t)B(t)(B^T(t)Q(t)B(t))^{-1}. \tag{17}$$

The optimal process should satisfy the system (15), (16) with boundary conditions $y(t_0) = x_0$, (10) and $\psi(t_1) = H_1^T\lambda^0$. The last condition is obtained from the transversality condition.

Similarly [2], it is not difficult to show that the unique vector of Lagrange multipliers, and, accordingly, the unique solution to the conjugate system.

The system of differential equations (15), (16) is homogeneous with respect to the vector $(y^T, \psi^T)^T$. The matrix of the system has the form

$$\overline{A}(t) = \begin{pmatrix} A(t) - M(t)B^T(t)Q(t) & M(t)B_1^T(t) \\ Q(t) - N(t)B^T(t)Q(t) & -A^T(t) + N(t)B_1^T(t). \end{pmatrix}.$$

We introduce the fundamental matrix $\Phi(t, t_0)$ as a solution to the initial problem

$$\dot{\Phi}(t) = \overline{A}(t)\Phi(t),\ \Phi(t_0) = E_{2n}. \tag{18}$$

The matrix Φ can be divided into blocks of equal dimension:

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}.$$

As a result, we get

$$\begin{cases} H_1\Phi_{11}(t_1, t_0)x_0 + H_1\Phi_{12}(t^1, t_0)\vartheta = H_1 y(t_1) = \breve{g} \\ \Phi_{21}(t^1, t_0)x_0 + \Phi_{22}(t^1, t_0)\vartheta = \psi(t_1) = H_1^T\lambda^0, \end{cases} \tag{19}$$

where $\vartheta = \psi(t_0)$. For the fundamental matrix $\Phi(t, \tau)$ the equality $\Phi(t_1, t) = F(t)$ holds for any $t \in [t_o, t_1]$, where the matrix $F(t)$ there is a solution to the conjugate system

$$\dot{F} = -F\overline{A}(t) \tag{20}$$

with the boundary condition $F(t_1) = E_{2n}$. We represent this matrix as a matrix consisting of four blocks of equal dimension:

$$F(t) = \begin{pmatrix} F_{11}(t) & F_{12}(t) \\ F_{21}(t) & F_{22}(t). \end{pmatrix}.$$

Then the system of algebraic equations (19) can be represented as

$$\begin{cases} H_1F_{12}(t_0)\vartheta = g - H_1F_{11}(t_0)x_0 \\ F_{22}(t_0)\vartheta - H_1^T\lambda^0 = -F_{21}(t_0)x_0. \end{cases} \tag{21}$$

From the second equation of the system (21) we obtain

$$\vartheta = F_{22}^{-1}(t_0)[H_1^T\lambda^0 - F_{21}(t_0)x_0]. \tag{22}$$

Substitute ϑ from (22) into the first equation of systems (21) and solving the obtained equation with respect to to λ^0 as a result, we get

$$\lambda^0 = R^{-1}(t_0)\left((H_1F_{12}(t_0)F_{22}^{-1}(t_0)F_{21}(t_0) - H_1F_{11}(t_0))x_0 + \breve{g}\right), \tag{23}$$

where

$$R(t_0) = H_1F_{12}(t_0)F_{22}^{-1}(t_0)H_1^T.$$

The non-degeneracy of the matrix $R(t_0)$ can be shown similarly to [2]. Substituting λ^0 from (23) into (22) we have

$$\vartheta = F_{22}^{-1}(t_0)\left(H_1^T\left(R^{-1}(t_0)(H_1^TF_{12}(t_0)F_{22}(t_0))^{-1}F_{21}(t_0) - H_1F_{11}(t_0)\right)x_0 - F_{21}(t_0)x_0 + H_1^T\breve{g}\right). \tag{24}$$

Replacing t_0 in (24) by $t \in [t_0, t_1]$, x_0 by $y(t)$ and substituting the resulting expression in (14) instead of ψ, get optimal control

$$v(t) = S(t)y(t) + G(t)\breve{g}, \tag{25}$$

where

$$S(t) = (B^T(t)Q(t)B(t))^{-1}\left(B_1(t)^TF_{22}^{-1}(t)[H_1^TR^{-1}(t)H_1F_{12}(t)F_{22}^{-1}(t)F_{21}(t) - H_1F_{11}(t) - F_{21}(t)] - B^T(t)Q(t)\right),$$

$$G(t) = (B^T(t)Q(t)B(t))^{-1}B_1^T(t)F_{22}^{-1}(t)H_1^TR^{-1}(t),$$

solving the auxiliary problem.

Theorem 1. The optimal control in the auxiliary problem has the form (25).

SOLUTION OF THE INITIAL PROBLEM

The optimal control in the original problem is obtained by differentiating the function $v(t)$ in the sense of the theory of generalized functions. It should be borne in mind that the function $v(t)$ is equal to zero for $t < t_0$ and equal to $v(t_1)$ for $t \geq t_1$. As a result, we have

Theorem 2. The optimal control structure will be as follows:

$$\dot{v}(t) = \Delta v(t_0)\delta(t - t_0) + \dot{v}_r(t) + \Delta v(t_1)\delta(t - t_1), \tag{26}$$

where $\delta(t)$ is Dirac Delta Function, $\dot{v}_r(t)$ is a regular component of a generalized derivative of a function $\dot{v}(t)$,

$$\Delta v(t_0) = S(t_0)x_0 + G(t_0)\breve{g}; \tag{27}$$

$$\Delta v(t_1) = -S(t_1)y(t_1) - G(t_1)\breve{g} - \bar{B}^{-1}(H_2 y(t_1) - \breve{g}).$$

The regular component of the optimal control $\dot{v}_r(t)$ can be obtained by integrating the system of differential equations

$$\begin{cases} \dot{y}(t) &= A(t)y(t) + B_1(t)v(t) \\ \dot{v}(t) &= \dot{S}(t)y(t) + S(t)(A(t)y(t) + B_1 v(t)) + \dot{G}(t)F_{22}(t_0)\breve{g} \end{cases} \tag{28}$$

with initial conditions $y(t_0 + 0) = x_o,\ v(t_0 + 0) = \Delta v(t_0)$.

You can get the regular control component in another way. The structure of sorting control is such that at the initial moment a pulse action acts on the phase point, and then the movement to the final moment occurs without impulses. Therefore, by virtue of the optimality principle, expression (27), if we replaced t_0 ony an arbitrary moment $t \in (t_0, t_1)$, must be zero. We differentiate this expression by virtue of system (1). As a result, we obtain the expression

$$\dot{S}(t)x(t) + S(t)(A(t)x(t) + B(t)u) + \dot{G}(t)\breve{g} = 0.$$

From here it follows that

$$S(t)B(t)u = -(\dot{S}(t) + S(t)A(t))x(t) - \dot{G}(t)\breve{g}.$$

From this expression, you can try to get the regular component of control. Note that in the example below, the regular control component was found in this way.

Note that the optimal control (26) is a programmed control. In order to build positional control, it is necessary, similarly to [8], to switch to impulse-sliding regimes.

EXAMPLE

We consider the problem of minimizing a functional

$$J[u(\cdot)] = \frac{1}{2}\int_0^1 (x^2(t)_1 + x^2(t)_2)dt$$

on the trajectories of the system

$$\begin{cases} \dot{x}_1(t) = x_2 \\ \dot{x}_2(t) = u. \end{cases}$$

Boundary conditions are set at the start and end points

$$x_1(0) = x_{10}, x_2(0) = x_{20},$$

$$x_1(1) = 0, x_2(1) = 0.$$

For this problem, the optimal control will be

$$u(t) = \Delta v(0)\delta(t) + \dot{v}_r(t) + \Delta v(1)\delta(t - 1)$$

where

$$\Delta v(0) = \frac{x_{10}}{1 - e^2}(1 + e^2) - x_{20},\ \Delta v(1) = 2\frac{x_{10}}{e^2 - 1}e,\ \dot{v}_r(t) = \frac{x_{10}}{e^2 - 1}(e^{2-t} - e^t).$$

CONCLUSIONS

The article considers a degenerate linear-quadratic optimal control problem with a terminal condition. Under certain assumptions, an optimal control is obtained. Impulse components are contained in the initial and final moments. The regular control component in $(t_0, t)1)$ is an absolutely continuous function. An illustrative example is considered.

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