Dynamic Programming in Routing Problems (Nuclear power, Engineering)

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Abstract. In this investigation, extremal routing problems oriented to engineering applications are considered. These applications can be associated with dismantling of radioactive sources under accidents at nuclear power plants and with control of sheet cutting on CNC machines. Issues related to construction of optimal processes are investigated, including the choice of starting point, regular succession of tasks, and determination of concrete trajectories. To solve this problem, a nonstandard variant of dynamic programming is used. Precedence conditions play an important role in issues of reducing computational complexity. A theoretical procedure is implemented in the form of an algorithm and a standard program for a PC and a supercomputer. Model problems oriented to the above-mentioned engineering applications are resolved.

INTRODUCTION

Routing problems arise in various fields of human activity. We are limited to some engineering applications for which sequential traversal problems arise. We keep in mind the dismantling problem in nuclear power and control problem under sheet cutting on CNC machines. Certainly, the natural prototype of our problem is the known traveling salesman problem (TSP); see [1–3]. However, essential quality features arise; namely, the natural setting of our problem contains constraints and complicated cost functions. Now, we note [4–6]. Serious formalization is required. Moreover, it is required to use advanced mathematical methods. We focus on widely understood dynamic programming (DP). The very general procedure of DP for routing problems was considered in [7, 8]. In this connection, we note the studies [9, 10] devoted to TSP. Our scheme is a wide development of [9]. A more specific variant of DP application is implemented in [11] devoted to studying issues related to dismantling of radioactive sources. Certainly, many other methods are used in TSP and in problems of the TSP type; now we note only a widely known branch and a bound method (see [12]).

THE MEGACITIES MODEL IN ROUTING PROBLEMS

Now, we consider a general model associated with visiting for nonempty finite sets called megacities. In this model, we implement the system of multivariate movements with choice of the regular succession. Fix two nonempty sets X and X^0; suppose that X^0 is a finite set and $X^0 \subset X$. Let $N \in \mathbb{N} \triangleq \{1;2;\ldots\}$ (hereinafter $\triangleq$ is an equality by definition). Let $N \geq 2$. We fix pairwise disjoint nonempty finite sets $M_1,\ldots,M_N$ termed megacities; let $M_j \subset X$ and $X^0 \cap M_j = \varnothing$ under $j \in \overline{1,N} \triangleq \{k \in \mathbb{N} | k \leq N\}$. Elements of X_0 are used as possible starting points and megacities $M_1,\ldots,M_N$ are visiting objects. The next setting corresponds to [13]. We use designations of [13] recalling only a few. Thus, we consider the process [13](1), where $\mathbb{M}_1,\ldots,\mathbb{M}_N$ are nonempty relations: $\mathbb{M}_j \subset M_j \times M_j$ under $j \in \overline{1,N}$. We denote by $\mathbb{P}$ the (nonempty finite) set of all permutations of the set $\overline{1,N}$. Then, process [13](1) is reduced to the scheme

$$(z_0 = (x,x)) \longrightarrow z_1 \in \mathbb{M}_{\alpha(1)} \longrightarrow \ldots \longrightarrow z_N \in \mathbb{M}_{\alpha(N)}, \tag{1}$$

where $x \in X^0$ and $\alpha \in \mathbb{P}$. Moreover, the choice of $\alpha \in \mathbb{P}$ must satisfy to precedence conditions. Namely, a set $\mathbf{K}$, $\mathbf{K} \subset \overline{1,N} \times \overline{1,N}$, is fixed; we suppose that Condition 2.2.1 in [7] is fulfilled. Then [7] (Part 2)

$$\mathbf{A} \triangleq \{\ \alpha \in \mathbb{P} |\ \forall t_1 \in \overline{1,N}\ \forall t_2 \in \overline{1,N} \quad ((\alpha(t_1),\alpha(t_2)) \in \mathbf{K}) \Longrightarrow (t_1 < t_2)\} \neq \varnothing. \tag{2}$$

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The elements of $\mathbf{A}$ are called routes admissible by precedence. We must use only routes of (2) as variants of regular succession choice. However, we must introduce trajectories coordinated with these routes.

For every ordered pair (OP) h, we denote the first and the second elements of h by $\mathrm{pr}_1(h)$ and $\mathrm{pr}_2(h)$ respectively. We introduce (see [13]) the sets $\mathfrak{M}_j \triangleq \{\ \mathrm{pr}_1(z) :\ z \in \mathbb{M}_j\}$ and $\mathbf{M}_j \triangleq \{\ \mathrm{pr}_2(z) :\ z \in \mathbb{M}_j\}$, where $j \in \overline{1,N}$. Then (as in [13]), we denote by $\mathfrak{M}$ the union of all sets $\mathfrak{M}_j,\ j \in \overline{1,N}$. Analogously, by $\mathbf{M}$ we denote the union of all sets $\mathbf{M}_j,\ j \in \overline{1,N}$. Naturally $\mathfrak{M}$ and $\mathbf{M}$ are nonempty finite sets. Let $\mathbb{X} \triangleq \mathfrak{M} \cup X^0$ and $\mathbf{X} \triangleq X^0 \cup \mathbf{M}$. Then $\mathbb{M}_j \subset \mathfrak{M}_j \times \mathbf{M}_j$ under $j \in \overline{1,N}$. Therefore, by (1) we can consider $\mathbb{X} \times \mathbf{X}$ as phase space of process (1). In this connection, by $\mathbb{Z}$ we denote the set of all mappings from $\overline{0,N} \triangleq \{0\} \cup \overline{1,N}$ into $\mathbb{X} \times \mathbf{X}$. As in [13], we suppose that $\mathbb{Z}_\alpha[x]$ is the set of all trajectories coordinated with $\alpha \in \mathbb{P}$ and starting from point $x \in X^0$: $\mathbb{Z}_\alpha[x]$ is the set of all processions $(z_t)_{t\in\overline{0,N}} \in \mathbb{Z}$ for which $z_0 = (x,x)$ and (see (1)) $z_t \in \mathbb{M}_{\alpha(t)}\ \ \forall t \in \overline{1,N}$. Certainly, under $\alpha \in \mathbb{P}$ and $x \in X^0$, in the form of $\mathbb{Z}_\alpha[x]$, we obtain a nonempty finite set. As in [13], we suppose that $\tilde{\mathbf{D}}[x] \triangleq \{(\alpha,(z_t)_{t\in\overline{0,N}}) \in \mathbf{A} \times \mathbb{Z} | (z_t)_{t\in\overline{0,N}}) \in \mathbb{Z}_\alpha[x]\}$ is the set of all admissible solutions $(\alpha,(z_t)_{t\in\overline{0,N}})$ starting from x. Moreover, we use $\mathbf{D}$ from [13] as the set of all admissible solutions of our complete problem (see [13]). Now, we introduce $\mathbb{R}_+ \triangleq \{t \in \mathbb{R} | 0 \le t\}$, where $\mathbb{R}$ is real line, and by $\mathscr{R}_+[T]$ we denote the set of all functions from a nonempty set T into $\mathbb{R}_+$ (therefore, $\mathscr{R}_+[T]$ is the set of all nonnegative real-valued functions on T). Let $\mathfrak{N}$ be the family of all nonempty subsets of $\overline{1,N}$ and

$$\mathbf{c} \in \mathscr{R}_+[\mathbf{X} \times \mathbb{X} \times \mathfrak{N}],\ c_1 \in \mathscr{R}_+[\mathbb{M}_1 \times \mathfrak{N}], \ldots, c_N \in \mathscr{R}_+[\mathbb{M}_N \times \mathfrak{N}],\ f \in \mathscr{R}_+[\mathbf{M}]. \tag{3}$$

In connection with $c_1,\ldots,c_N$, we recall that, under $x_1 \in X$, $x_2 \in X$, and $\tilde{K} \in \mathfrak{N}$, by [14] (Ch. 1) $(x_1,x_2,\tilde{K}) \triangleq ((x_1,x_2),\tilde{K})$; therefore for $j \in \overline{1,N}$, $z \in \mathbb{M}_j$, and $K \in \mathfrak{N}$, we have the representation $c_j(z,K) = c_j(\mathrm{pr}_1(z),\mathrm{pr}_2(z),K)$. We use $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$ and $\overline{p,q} \triangleq \{k \in \mathbb{N}_0 | (p \le k)\&(k \le q)\}\ \ \forall p \in \mathbb{N}_0\ \forall q \in \mathbb{N}_0$. Using (1) and (3), we suppose that

$$\mathfrak{C}_\alpha[(z_t)_{t\in\overline{0,N}}] \triangleq \sum_{t=1}^{N} [\mathbf{c}(\mathrm{pr}_2(z_{t-1}),\mathrm{pr}_1(z_t),\{\alpha(k) :\ k \in \overline{t,N}\}) + c_{\alpha(t)}(z_t,\{\alpha(k) :\ k \in \overline{t,N}\})] + f(\mathrm{pr}_2(z_N)),$$

where $\alpha \in \mathbb{P}$ and $(z_t)_{t\in\overline{0,N}} \in \mathbb{Z}_\alpha[x]$ under $x \in X^0$. Then, under $x \in X^0$, in the form

$$\mathfrak{C}_\alpha[(z_t)_{t\in\overline{0,N}}] \longrightarrow \min, \qquad (\alpha,(z_t)_{t\in\overline{0,N}}) \in \tilde{\mathbf{D}}[x], \tag{4}$$

we obtain the natural x-problem of routing with extremum $V[x]$ (see [13]) and the nonempty set

$$(\mathrm{sol})[x] \triangleq \{(\alpha^0,(z^0_t)_{t\in\overline{0,N}}) \in \tilde{\mathbf{D}}[x] | \mathfrak{C}_{\alpha^0}[(z^0_t)_{t\in\overline{0,N}}] = V[x]\} \tag{5}$$

of optimal solutions. Moreover, we consider the complete routing problem

$$\mathfrak{C}_\alpha[(z_t)_{t\in\overline{0,N}}] \longrightarrow \min, \qquad (\alpha,(z_t)_{t\in\overline{0,N}},x) \in \mathbf{D}, \tag{6}$$

with extremum $\mathbb{V}$ and the nonempty set $\mathbf{SOL} \triangleq \{(\alpha^0,(z^0_t)_{t\in\overline{0,N}},x^0) \in \mathbf{D} | \mathfrak{C}_{\alpha^0}[(z^0_t)_{t\in\overline{0,N}}] = \mathbb{V}\}$. Moreover, we have the natural problem of the starting point optimization

$$V[x] \longrightarrow \min, \qquad x \in X^0 \tag{7}$$

with extremum $\mathbb{V}$ and the set $X^0_{\mathrm{opt}} \triangleq \{x^0 \in X^0 | V[x^0] = \mathbb{V}\} \neq \varnothing$ of optimal starting points (we recall the definition of $\mathbf{D}$ in [13]). The following property is obvious. Namely, let, under $x \in X^0_{\mathrm{opt}}$,

$$(\overline{\mathrm{sol}})[x] \triangleq \{(\alpha,(z_t)_{t\in\overline{0,N}},x) :\ ((\alpha,(z_t)_{t\in\overline{0,N}}) \in (\mathrm{sol})[x]\}. \tag{8}$$

Proposition 1. The set $\mathbf{SOL}$ of all optimal solutions of the complete routing problem coincides with the union of all sets $(\overline{\mathrm{sol}})[x]$, $x \in X^0_{\mathrm{opt}}$.

DYNAMIC PROGRAMMING

In the following, we use the DP procedure of [13] (moreover, see [5, 6]). In this connection, we note that some variations of (3) with respect to [13] are unessential. By [13], we introduce the families $\mathscr{C}$, $\mathscr{C}_1, \ldots, \mathscr{C}_N$ elements of which are essential lists. Naturally, $\mathscr{C}_N$ is a singleton $\{\overline{1,N}\}$ and $(\mathscr{C}_i)_{i\in\overline{1,N}}$ is a partition of the family $\mathscr{C}$. In terms of $(\mathscr{C}_i)_{i\in\overline{1,N}}$, we define (see [5, 6, 13]) the sets $D_0, \ldots, D_N$ the elements of which are OP (x,K), where $x \in \mathbf{X}$ and K is a subset of $\overline{1,N}$. Thus, $D_0, \ldots, D_N$ are sets in the position space (we recall the rule [6] (p.1963)). Next, we construct the functions $v_0 \in \mathscr{R}_+[D_0]$, $v_1 \in \mathscr{R}_+[D_1], \ldots, v_N \in \mathscr{R}_+[D_N]$, for which the recurrence procedure [13] ((5)) is used. In addition, v_0 is defined by f; $v_N(x,\overline{1,N}) = V[x]\ \forall x \in X^0$. We recall that, for $s \in \overline{1,N}$, the transformation v_{s-1} to v_s is defined by [6] (Proposition 4.1). We determine $\mathbb{V}$ as the least of all numbers $v_N(x,\overline{1,N})$, $x \in X^0$. This calculation completes construction of the Bellman function layers (in this connection, we note that $v_0, v_1, \ldots, v_N$ are restrictions of the uniform Bellman function to the sets $D_0, D_1, \ldots, D_N$ respectively).

CONSTRUCTION OF OPTIMAL SOLUTION

Thus, we have already built the functions $v_0, v_1, \ldots., v_N$ and found the value $\mathbb{V}$. Moreover, we choose $x^0 \in X^0$ such that $v_N(x^0,\overline{1,N}) = \mathbb{V}$. Now, we consider the optimal solution construction. Here, we note that x^0 is a solution of problem (7): $x^0 \in X^0_{\text{opt}}$. Now, we suppose that $\mathbf{z}^{(0)} \overset{\triangle}{=} (x^0, x^0)$. Then, $V[x^0] = \mathbb{V}$. Therefore, by [6] ((4.12))

$$\mathbb{V} = \min_{j\in\mathbf{I}(\overline{1,N})} \min_{z\in\mathbb{M}_j} [\mathbf{c}(x^0, \text{pr}_1(z), \overline{1,N}) + c_j(z,\overline{1,N}) + v_{N-1}(\text{pr}_2(z), \overline{1,N}\setminus\{j\})], \tag{9}$$

where $\mathbf{I} : \mathfrak{N} \longrightarrow \mathfrak{N}$ is defined by [6] ((3.9)), moreover, see [7] (Part 2). Using (9), we find $\eta_1 \in \mathbf{I}(\overline{1,N})$ and $\mathbf{z}^{(1)} \in \mathbb{M}_{\eta_1}$ such that

$$\mathbb{V} = \mathbf{c}(x^0, \text{pr}_1(\mathbf{z}^{(1)}), \overline{1,N}) + c_{\eta_1}(\mathbf{z}^{(1)}, \overline{1,N}) + v_{N-1}(\text{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\eta_1\}); \tag{10}$$

see [6] ((4.13)). We use the property $(x^0,\overline{1,N}) \in D_N$. Then (see [6] (Section 4)) $(\text{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\eta_1\}) \in D_{N-1}$. Therefore [6] (Section4)

$$\begin{aligned} v_{N-1}(\text{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\eta_1\}) = \min_{j\in\mathbf{I}(\overline{1,N}\setminus\{\eta_1\})} \min_{z\in\mathbb{M}_j} [&\mathbf{c}(\text{pr}_2(\mathbf{z}^{(1)}), \text{pr}_1(z), \overline{1,N}\setminus\{\eta_1\}) \\ &+ c_j(z, \overline{1,N}\setminus\{\eta_1\}) + v_{N-2}(\text{pr}_2(z), \overline{1,N}\setminus\{\eta_1; j\})]. \end{aligned} \tag{11}$$

Using (11), we choose $\eta_2 \in \mathbf{I}(\overline{1,N}\setminus\{\eta_1\})$ and $\mathbf{z}^{(2)} \in \mathbb{M}_{\eta_2}$ for which

$$\begin{aligned} &v_{N-1}(\text{pr}_2(\mathbf{z}^{(1)}), \overline{1,N}\setminus\{\eta_1\}) \\ &= \mathbf{c}(\text{pr}_2(\mathbf{z}^{(1)}), \text{pr}_1(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\eta_1\}) + c_{\eta_2}(\mathbf{z}^{(2)}, \overline{1,N}\setminus\{\eta_1\}) + v_{N-2}(\text{pr}_2(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\eta_1;\eta_2\}). \end{aligned} \tag{12}$$

In addition, $(\text{pr}_2(\mathbf{z}^{(2)}), \overline{1,N}\setminus\{\eta_1;\eta_2\}) \in D_{N-2}$. Later, operations similar to (10) and (12) should continue up to the exhaustion of the set $\overline{1,N}$. Then, for $\eta \overset{\triangle}{=} (\eta_i)_{i\in\overline{1,N}}$ we have the property: $(\eta, (\mathbf{z}^{(i)})_{i\in\overline{0,N}}) \in (\text{sol})[x^0]$ is obtained. Obviously, $(\eta, (\mathbf{z}^{(i)})_{i\in\overline{0,N}}, x^0) \in (\widetilde{\text{sol}})[x^0]$. Then, by **Proposition 1**

$$(\eta, (\mathbf{z}^{(i)})_{i\in\overline{0,N}}, x^0) \in \mathbf{SOL}. \tag{13}$$

Now, we recall basic steps of procedure implementation (13).

1) Construction the Bellman function layers $v_0, v_1, \ldots, v_N$.

2) Finding global extremum $\mathbb{V}$ and the optimal starting point $x^0 \in X^0_{\text{opt}}$.

3) Constructing (13) by a step-by-step procedure (see (10),(12)).

We note that for building only $\mathbb{V}$ and $x^0 \in X^0_{\text{opt}}$, the next scheme modification can be implemented.

1') Construction v_N with overwriting functions-layers $v_1, \ldots, v_N$. In this case, we use [6] (Proposition4.1) for determination of the transformation $v_{s-1} \longrightarrow v_s$ under $s \in \overline{1,N}$. If $s < N$, then, after this transformation, the value array for v_{s-1} is destroyed; it is replaced by the value array for the constructed function v_s. So, in computer memory, only one of the Bellman function layers is situated.

2') Minimization of values of the function v_N for determination $\mathbb{V}$ and $x^0 \in X^0_{\text{opt}}$.

The employment of $\mathbb{V}$ and x^0 can be associated with heuristic testing in the case of the large dimension of our problem.

OPTIMIZING MULTI-INSERTIONS IN PROBLEMS OF LARGE DIMENSION

For routing problems with large dimension (first of all, under high value N), serious difficulties associated with computing implementation of DP arise, although the structure of the optimal solution is clear (see previous section). A critical difficulty is connected with memory resource (of course, some decrease of computing complexity is attained by applying the procedure with construction only Bellman function layers $v_0,\ v_1,...,v_N$ but not the entire array of Bellman function; these decrease is realized in the problem with precedence conditions). Nevertheless, in routing problem of large dimension, the use of heuristics inevitably. However, in this case, we can use DP for optimization “in window". Now, we note only one variant of such a procedure using a substantial way of reasoning (more detailed construstions are reduced in [15]). We consider optimizing multi-insertions for given “window" system. It is supposed that we have some admissible heuristic solution (it is supposed here that the starting point is fixed). Such a solution can be found by greedy algorithm. It is supposed that the number of megacities is sufficiently large. In the index set, we form some disjoint “window" system obtaining a procession of fragments. In these fragments, we realize local DP procedures. In addition, global precedence conditions generate a system of local precedence conditions. Moreover, the “global" cost functions (see (3)) with dependence on the task lists generate a system of local cost functions with similar dependence. Optimization procedures for our multi-insertion are implemented by a parallel algorithm for a supercomputer (see [15]; program of A.M. Grigoryev).

CONCLUSION

A model problem oriented to dismantling of radioactive sources under accidents at nuclear power plants has been considered. The used cost functions are defined by integration of instantaneous radiation actions (nonlinear functions). A variant with $N = 255$ has been considered. It is supposed that megacities are 30-element sets. The case when the size of every window is 20 was used. The number of “windows" is 13. The original result has been improved by 15.8%. The computation time was 25 min 55 sec.

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