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Semiring identities of the Brandt monoid*

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Abstract

The 6-element Brandt monoid B_2^1 admits a unique addition under which it becomes an additively idempotent semiring. We show that this addition is a term operation of B_2^1 as an inverse semigroup. As a consequence, we exhibit an easy proof that the semiring identities of B_2^1 are not finitely based.

We assume the reader's acquaintance with basic concepts of universal algebra such as an identity and a variety; see, e.g., [1, Chapter II].

The 6-element Brandt monoid B_2^1 can be represented as a semigroup of the following zero-one 2×2-matrices

$$\underset{0}{\begin{pmatrix}0&0\\0&0\end{pmatrix}}\quad \underset{E}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}\quad \underset{E_{12}}{\begin{pmatrix}0&1\\0&0\end{pmatrix}}\quad \underset{E_{21}}{\begin{pmatrix}0&0\\1&0\end{pmatrix}}\quad \underset{E_{11}}{\begin{pmatrix}1&0\\0&0\end{pmatrix}}\quad \underset{E_{22}}{\begin{pmatrix}0&0\\0&1\end{pmatrix}} \tag{1}$$

under the usual matrix multiplication $\cdot$ or as a monoid with presentation

$$\langle E_{12}, E_{21} \mid E_{12}E_{21}E_{12} = E_{12},\ E_{21}E_{12}E_{21} = E_{21},\ E_{12}^2 = E_{21}^2 = 0\rangle.$$

Quoting from a recent paper [3], 'This Brandt monoid is perhaps the most ubiquitous harbinger of complex behaviour in all finite semigroups'. In particular, $(B_2^1,\cdot)$ has no finite basis for its identities (Perkins [13, 14]) and is one of the four smallest semigroups with this property (Lee and Zhang [10]).

The monoid $(B_2^1,\cdot)$ has a natural involution that swaps E_{12} and E_{21} and fixes all other elements. In terms of the matrix representation (1) this involution is nothing but the usual matrix transposition; we will, however, use the notation $x \mapsto x^{-1}$ for

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the involution, emphasizing that x^{-1} is the unique inverse of x. Recall that elements x,y of a semigroup $(S,\cdot)$ are said to be *inverses* of each other if $xyx=x$ and $yxy=y$. A semigroup is called *inverse* if every its element has a unique inverse; inverse semigroups can therefore be thought of as algebras of type (2,1). Being considered as an inverse semigroup, the monoid $(B_2^1,\cdot,{}^{-1})$ retains its complex equational behaviour: B_2^1 has no finite basis for its inverse semigroup identities (Kleiman [6]) and is the smallest inverse semigroup with this property (Kleiman [5, 6]).

In the present note we consider equational properties of yet another enhancement of the monoid $(B_2^1,\cdot)$ with an additional operation, this time binary. Recall that an *additively idempotent semiring* an algebra $(S,+,\cdot)$ of type $(2,2)$ such that the additive reduct $(S,+)$ is a *semilattice* (that is, a commutative idempotent semigroup), the multiplicative reduct $(S,\cdot)$ is a semigroup, and multiplication distributes over addition on the left and on the right, that is, $(S,+,\cdot)$ satisfies the identities $x(y+z)\approx xy+xz$ and $(y+z)x\approx yx+zx$. In papers which motivation comes from semigroup theory, objects of this sort sometimes appear under the name *semilattice-ordered semigroups*, see, e.g., [8] or [12]. We will stay with the term 'additively idempotent semiring', abbreviated to 'ai-semiring' in the sequel.

Our key observation is the following:

Lemma 1. *Let* $(S,\cdot,{}^{-1})$ *be an inverse semigroup satisfying the identity*

$$x^n \approx x^{n+1} \tag{2}$$

for some n. Define

$$x\oplus y := (xy^{-1})^n x.$$

Then $(S,\cdot,\oplus)$ *is an ai-semiring.*

Proof. Let $E(S)$ stand for the set of all idempotents of S. The relation

$$\leq := \{(a,b)\in S\times S \mid a=eb \text{ for some } e\in E(S)\}$$

is a partial order on S referred to as the *natural partial order*; see [15, Section II.1] or [9, pp. 21–23]. We need two basic properties of the natural partial order:

1) $\leq$ is compatible with both multiplication and inversion;

2) $a\leq b$ if and only if $a=bf$ for some $f\in E(S)$.

Take any $a,b\in S$ and suppose that $c\leq a$ and $c\leq b$. Then $c^{-1}\leq b^{-1}$ whence by the compatibility with multiplication

$$c=(cc^{-1})^n c\leq (ab^{-1})^n a = a\oplus b.$$

In presence of the identity (2), $(ab^{-1})^n=(ab^{-1})^{n+1}=\cdots=(ab^{-1})^{2n}$. Hence

$$a \oplus b = (ab^{-1})^n \cdot a \leq a.$$

Further,

$$\begin{aligned} a \oplus b = (ab^{-1})^n a = (ab^{-1})^{n+1} a = \cdots = (ab^{-1})^{2n-1} a = \\ (ab^{-1})^n \cdot (ab^{-1})^{n-1} a = (ab^{-1})^n \cdot a(b^{-1}a)^{n-1} = \ \text{(using } b^{-1} = b^{-1}bb^{-1}\text{)} \\ (ab^{-1})^n \cdot b \cdot (b^{-1}a)^n \leq b \cdot (b^{-1}a)^n \leq b \end{aligned}$$

since $(b^{-1}a)^n \in E(S)$. We see that $a \oplus b$ is nothing but the infimum of $\{a, b\}$ with respect to the natural partial order. Thus, $(S, \oplus)$ is a semilattice. It is known [16, Proposition 1.22], see also [9, Proposition 19] that if a subset $H \subseteq S$ possesses an infimum under the natural partial order, then so do the subsets sH and Hs for any $s \in S$, and $\inf(sH) = s(\inf H)$, $\inf(Hs) = (\inf H)s$. This implies that multiplication distributes over $\oplus$ on the left and on the right. □

Remark 1. The essence of Lemma 1 is known. Leech, in the course of his comprehensive study of inverse monoids $(S, \cdot, ^{-1}, 1)$ that are inf-semilattices under the natural partial order, has verified that $(S, \leq)$ is a inf-semilattice whenever S is a periodic combinatorial[1] inverse monoid; see [11, Example 1.21(d), item (iv)]. Of course, the requirement of S being a monoid is not essential: if a semigroup S periodic and combinatorial then so is the monoid S^1 obtained by adjoining a formal identity to S. Clearly, if a semigroup satisfies (2), then it is both periodic and combinatorial whence Leech's observation applies. We have preferred the above direct proof of Lemma 1 because we need a $(\cdot, ^{-1})$-term for the semilattice operation, and such a term is not explicitly present in [11].

Obviously, the 6-element Brandt monoid satisfies the identity $x^2 \approx x^3$. Thus, Lemma 1 applies, and $(B_2^1, \oplus, \cdot)$ is an ai-semiring. It is known (and easy to verify) that $\oplus$ is the only addition on B_2^1 under which B_2^1 becomes an ai-semiring.

Our main result states that, similarly to the plain semigroup $(B_2^1, \cdot)$ and the inverse semigroup $(B_2^1, \cdot, ^{-1})$, the ai-semiring $(B_2^1, \oplus, \cdot)$ admits no finite identity basis. Its proof employs a series of inverse semigroups C_n, $n = 2, 3, \ldots$, constructed in [6] as semigroups of partial one-to-one transformations. Here, to align with the matrix representation chosen for the B_2^1, we describe them as semigroups of zero-one matrices.

The set R_m of all zero-one $m \times m$-matrices which have at most one entry equal to 1 in each row and column forms an inverse monoid under usual matrix multipli-

[1] A semigroup S is *periodic* if all monogenic subsemigroups of S are finite and *combinatorial* if all subgroups of S are trivial.

cation $\cdot$ and transposition. The inverse monoid R_m is called the *rook monoid*[2] as its matrices encode placements of nonattacking rooks on an $m \times m$ chessboard.

Let $m = 2n+1$ and define $m \times m$-matrices $c_1, \ldots, c_n$ by

$$c_k := E_{k+1\,k} + E_{n+k\;n+k+1}, \;\; k = 1, \ldots, n,$$

where, as usual, E_{ij} denotes the $m \times m$-matrix unit with an entry 1 in the (i,j) position and 0's elsewhere. For instance, if $n = 2$, then c_1 and c_2 are the following 5×5-matrices:

$$c_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let C_n be the inverse subsemigroup of the rook monoid R_m generated by the matrices $c_1, \ldots, c_n$. As a plain subsemigroup, C_n is generated by $c_1, \ldots, c_n$ and their inverses (i.e., transposes) $c_1^{-1}, \ldots, c_n^{-1}$.

The next lemma collects properties of the semigroups C_n that we need.

Lemma 2. (i) *The semigroup $(C_n, \cdot)$ does not belong to the semigroup variety generated by the monoid $(B_2^1, \cdot)$.*

(ii) *The semigroup $(C_n, \cdot)$ satisfies the identity $x^2 \approx x^3$.*

(iii) *For each $k = 1, \ldots n$, $M_k(n) := C_n \setminus \{c_k, c_k^{-1}\}$ forms an inverse subsemigroup of the inverse semigroup $(C_n, \cdot, ^{-1})$.*

(iv) *For each $k = 1, \ldots n$, the inverse semigroup $(M_k(n), \cdot, ^{-1})$ belongs to the inverse semigroup variety generated by the inverse monoid $(B_2^1, \cdot, ^{-1})$.*

Proof. (i) This property was established in [7, Lemma 3] by exhibiting, for each $n \geq 2$, a semigroup identity that holds in $(B_2^1, \cdot)$ and fails in $(C_n, \cdot)$.

(ii) This is easy to verify (and also follows from the proof of Lemma 1 in [6]).

(iii) This is clear (and is a part of Lemma 1 in [6]).

(iv) This is Property (C) in [6]. □

Remark 2. Items (i)–(iii) of Lemma 2 are easy. In contrast, the proof of (iv) in [6] is long and complicated. We mention in passing that now the proof can be radically simplified by using a deep result by Kaďourek [4] who provided an effective membership test for the inverse semigroup variety generated by $(B_2^1, \cdot, ^{-1})$.

[2]The rook monoid is nothing but the matrix representation of the *symmetric inverse monoid*; see [15, Section IV.1] or [9, p. 6]. The name 'rook monoid' was suggested by Solomon [17].

Theorem 3. *The semiring identities of the additively idempotent semiring $(B_2^1, \oplus, \cdot)$ admit no basis involving only finitely many variables, and hence, no finite basis.*

Proof. Arguing by contradiction, assume that $(B_2^1, \oplus, \cdot)$ has an identity basis Σ such that each identity $u \approx v$ in Σ involves less than n variables. Consider the inverse semigroup $(C_n, \cdot, ^{-1})$. By Lemmas 1 and 2(ii), the addition defined by $x \oplus y := (xy^{-1})^2 x$ makes $(C_n, \oplus, \cdot)$ an ai-semiring. Consider an arbitrary evaluation ε of variables $x_1, \ldots, x_\ell$ involved in the identity $u \approx v$ in this ai-semiring. By the pigeonhole principle, there exists an index $k \in \{1, \ldots n\}$ such that neither c_k nor c_k^{-1} belongs to the set $\{\varepsilon(x_1), \ldots, \varepsilon(x_\ell)\}$ as this set contains at most $\ell < n$ elements. Thus, $\{\varepsilon(x_1), \ldots, \varepsilon(x_\ell)\} \subset M_k(n)$.

Since $x \oplus y$ expresses as $(\cdot, ^{-1})$-term, one can rewrite the identity $u \approx v$ into an identity $u' \approx v'$ in which u' and v' are $(\cdot, ^{-1})$-terms. Since $u \approx v$ holds in $(B_2^1, \oplus, \cdot)$, the rewritten identity $u' \approx v'$ holds in the inverse semigroup $(B_2^1, \cdot, ^{-1})$. By Lemma 2(iv) the latter identity holds also in the inverse semigroup $(M_k(n), \cdot, ^{-1})$, and so u' and v' take the same value under every evaluation of the variables $x_1, \ldots, x_\ell$ in $M_k(n)$. Hence $\varepsilon(u) = \varepsilon(u') = \varepsilon(v') = \varepsilon(v)$. We conclude that the identity $u \approx v$ holds in the ai-semiring $(C_n, \oplus, \cdot)$. Since an arbitrary identity from Σ holds in $(C_n, \oplus, \cdot)$, this ai-semiring belongs to the ai-semiring variety generated by $(B_2^1, \oplus, \cdot)$. This, however, contradicts Lemma 2(i), according to which even the semigroup reduct $(C_n, \cdot)$, does not belongs to semigroup variety generated by $(B_2^1, \cdot)$. □

Remark 3. To the best of my knowledge, the result of Theorem 3 has not been published up to now. However, after preparing the present article I have learnt that the result has also been obtained by colleagues in Xi'an and Melbourne but with an entirely unrelated proof.

I mention also a related paper by Dolinka [2] where he introduces a 7-element ai-semiring denoted Σ_7 and proves that its identities are not finitely based. The semigroup reduct of Σ_7 is just the monoid B_2^1 with an extra zero adjoined so that $(\Sigma_7, \cdot, ^{-1})$ and $(B_2^1, \cdot, ^{-1})$ satisfy the same inverse semigroup identities. However, the addition in Σ_7 is not derived from its inverse semigroup structure, and one can easily see that the semiring identities of $(\Sigma_7, +, \cdot)$ and $(B_2^1, \oplus, \cdot)$ are essentially different. It should be also mentioned that in [2] Dolinka actually considers ai-semirings with 0 as algebras of type (2,2,0).

Remark 4. Leech [11] defined an *inverse algebra* as an algebra $(A, \cdot, \wedge, ^{-1}, 1)$ of type $(2,2,1,0)$ such that the reduct $(A, \cdot, ^{-1}, 1)$ is an inverse monoid, the reduct $(A, \wedge)$ is a meet semilattice, and the natural partial order of the inverse monoid coincides with that of the semilattice. Clearly, $(B_2^1, \cdot, \oplus, ^{-1}, E)$ constitutes an inverse algebra in Leech's sense, and the above proof of Theorem 3 can be easily adapted to show that $(B_2^1, \cdot, \oplus, ^{-1}, E)$ has no finite identity basis also as such algebra.

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