TWO WEAKER VARIANTS OF CONGRUENCE PERMUTABILITY FOR MONOID VARIETIES

SERGEY V. GUSEV AND BORIS M. VERNIKOV

Abstract. We completely determine all varieties of monoids on whose free objects all fully invariant congruences or all fully invariant congruences contained in the least semilattice congruence permute. Along the way, we find several new monoid varieties with the distributive subvariety lattice (only a few examples of varieties with such a property are known so far).

1. Introduction and summary

In this article, we study varieties of monoids as semigroups equipped with an additional 0-ary operation that fixes the identity element. But we start with the universal-algebraic notions that closely connected with our research.

For congruences α and β on an algebra A, we denote by $\alpha\beta$ their *relational product*, that is, the relation

$$\{(a,b) \in A \times A \mid a\,\alpha\,c\,\beta\,b \text{ for some } c \in A\}.$$

The congruences α and β on an algebra A are said to *permute* if $\alpha\beta = \beta\alpha$. It is well-known that, exactly in this case, the relation $\alpha\beta$ is again a congruence on A that coincides with the *lattice join* $\alpha \vee \beta$ of α and β, that is, the least congruence containing both α and β.

The family of *congruence permutable* varieties (that is, varieties on whose algebras all congruences permute) is very rich and important. In particular, it includes all varieties of groups and (not necessarily associative) rings. Unfortunately, proper semigroup or monoid varieties fail to belong to this family. Saying so, we refer to the following fact: a congruence permutable semigroup or monoid variety must consist entirely of groups. For semigroup varieties this claim first verified by Tully [22] and was then rediscovered and strengthened several times (see Evans [2, p. 35], Freese and Nation [3, Corollary on pp. 57–58], Jones [13, Theorem 1.2(iii)] or Lipparini [19, Corollary 0]). Analog of this fact is true for monoid varieties as well (see Section 7).

However, if we restrict to fully invariant congruences, the situation considerably improves since there already exist interesting varieties on whose semigroups all fully invariant congruences permute. For example, Pastijn [20] and Petrich and

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Reilly [21] have observed that fully invariant congruences on completely simple semigroups permute.

Considering fully invariant congruences is most natural for free objects of varieties. Indeed, if $\mathbf{V}$ is a variety of algebras, then the lattice of fully invariant congruences on $\mathbf{V}$-free object over a countably infinite alphabet is known to be anti-isomorphic to the subvariety lattice of $\mathbf{V}$. Thus, any "positive" information about the fully invariant congruences on $\mathbf{V}$-free objects contributes to clarifying the structure of the subvariety lattice of $\mathbf{V}$. In particular, the permutability of fully invariant congruences on $\mathbf{V}$-free objects reflects in the very important Arguesian property of the corresponding subvariety lattice. (Recall that, by results of Jónsson, any lattice of permuting equivalences is Arguesian [14] and the class of Arguesian lattices is self-dual [15].) For brevity, we call a variety of algebras $\mathbf{V}$ *fi-permutable* if every two fully invariant congruences on any $\mathbf{V}$-free object permute.

In order to introduce one more interesting class of varieties closely related with fi-permutable ones, we need some definitions and notation. Commutative idempotent semigroups usually are called *semilattices*. We call commutative idempotent monoids *semilattice monoids*. Recall that an element a of a lattice L is called *neutral* if, for any $x, y \in L$, the sublattice of L generated by a, x and y is distributive. Let $\mathbf{SL}_{\mathsf{sem}}$ [respectively, $\mathbf{SL}$] be the variety of all semilattices [semilattice monoids] and $\mathbb{SEM}$ [respectively, $\mathbb{MON}$] be the lattice of all semigroup [monoid] varieties. It is well known that $\mathbf{SL}_{\mathsf{sem}}$ [respectively, $\mathbf{SL}$] is an atom of the lattice $\mathbb{SEM}$ [respectively, $\mathbb{MON}$] and a neutral element of this lattice (see Volkov [29, Proposition 4.1] for the semigroup case and Gusev [5, Theorem 1.1] for the monoid one). These claims together with well-known properties of neutral elements in lattices (see [4, Theorem 254], for instance) imply that the lattice $\mathbb{SEM}$ [respectively, $\mathbb{MON}$] is decomposed into a subdirect product of the 2-element chain and the interval $[\mathbf{SL}_{\mathsf{sem}}, \mathbf{SEM}]$ [the interval $[\mathbf{SL}, \mathbf{MON}]$, respectively], where $\mathbf{SEM}$ [respectively, $\mathbf{MON}$] is the variety of all semigroups [monoids]. On every semigroup [monoid] S, there exists the least congruence σ such that the quotient S/σ is a semilattice [semilattice monoid]. The congruence σ is called the *least semilattice congruence* on S. The aforementioned observations show that it is natural to consider semigroup and monoid varieties $\mathbf{V}$ such that, on any $\mathbf{V}$-free object F, not all fully invariant congruences but only those of them that are contained in the least semilattice congruence on F permute. We call a semigroup or monoid variety with such a property *almost fi-permutable*. In view of the aforementioned properties of the varieties $\mathbf{SL}_{\mathsf{sem}}$ and $\mathbf{SL}$, an almost fi-permutable variety of semigroups or monoids has the Arguesian subvariety lattice. The class of almost fi-permutable semigroup varieties is quite wide; in particular, it contains all completely regular varieties [20, 21]. As we will prove below, the same is true for almost fi-permutable monoid varieties (see Theorem 1.2).

A complete classification of fi-permutable and almost fi-permutable semigroup varieties is given by Vernikov and Volkov in [27] and [28], respectively. The second result contains a minor inaccuracy that is fixed by Vernikov and Shaprynskiĭ [26]. Semigroup varieties with other multiplicative restrictions to fully invariant congruences on their free objects were examined in Vernikov [23–25], Vernikov and Shaprynskiĭ [26] and some other articles; further details see in Section 7. The present article is devoted to a complete determination of fi-permutable and almost fi-permutable monoid varieties.

To formulate the main results of the article, we need some definitions and notation. Let $\mathfrak{X}$ be a countably infinite set called an *alphabet*. As usual, we denote by $\mathfrak{X}^+$ [respectively, by $\mathfrak{X}^*$] the free semigroup [monoid] over the alphabet $\mathfrak{X}$; elements of $\mathfrak{X}^+$ and $\mathfrak{X}^*$ are called *words*, while elements of $\mathfrak{X}$ are said to be *letters*. Words unlike letters are written in bold. The two words forming an identity are connected by the symbol $\approx$, while the symbol $=$ denotes, among other things, the equality relation on $\mathfrak{X}^+$ or $\mathfrak{X}^*$.

For a possibly empty set W of words, we denote by $I(W)$ the set of all words that are not subwords of words from W. It is clear that $I(W)$ is an ideal of $\mathfrak{X}^*$. Let $S(W)$ denote the Rees quotient monoid $\mathfrak{X}^*/I(W)$. If $W = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$, then we write $S(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k)$ rather than $S(\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\})$. The monoids of the kind $S(W)$ often appeared in the literature (see [7, 8, 11, 12, 18], for instance).

As usual, $\mathbb{N}$ denotes the set of all natural numbers. We denote by $\operatorname{var} M$ the monoid variety generated by a monoid M. Put

$$\mathbf{D}_k = \begin{cases} \operatorname{var} S(xy) & \text{if } k = 1, \\ \operatorname{var} S(xt_1xt_2 \cdots xt_{k-1}x) & \text{if } k > 1 \end{cases} \quad \text{and} \quad \mathbf{D}_\infty = \bigvee_{k\in\mathbb{N}} \mathbf{D}_k.$$

For any $n \in \mathbb{N}$, we denote by S_n the full symmetric group on the set $\{1, 2, \dots, n\}$. For convenience, we put $S_0 = S_1$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $n, m \in \mathbb{N}_0$, $\pi, \tau \in S_n$ and $\rho \in S_{n+m}$, we define the words

$$\begin{aligned}
\mathbf{c}_{n,m}[\rho] &= \biggl(\prod_{i=1}^{n} z_i t_i\biggr) xyt \biggl(\prod_{i=n+1}^{n+m} z_i t_i\biggr) x \biggl(\prod_{i=1}^{n+m} z_{i\rho}\biggr) y, \\
\mathbf{c}'_{n,m}[\rho] &= \biggl(\prod_{i=1}^{n} z_i t_i\biggr) yxt \biggl(\prod_{i=n+1}^{n+m} z_i t_i\biggr) x \biggl(\prod_{i=1}^{n+m} z_{i\rho}\biggr) y, \\
\mathbf{w}_n[\pi, \tau] &= \biggl(\prod_{i=1}^{n} z_i t_i\biggr) x \biggl(\prod_{i=1}^{n} z_{i\pi} z_{n+i\tau}\biggr) x \biggl(\prod_{i=n+1}^{2n} t_i z_i\biggr), \\
\mathbf{w}'_n[\pi, \tau] &= \biggl(\prod_{i=1}^{n} z_i t_i\biggr) x^2 \biggl(\prod_{i=1}^{n} z_{i\pi} z_{n+i\tau}\biggr) \biggl(\prod_{i=n+1}^{2n} t_i z_i\biggr).
\end{aligned}$$

We denote by $\mathbf{d}_{n,m}[\rho]$ and $\mathbf{d}'_{n,m}[\rho]$ the words that are obtained from the words $\mathbf{c}_{n,m}[\rho]$ and $\mathbf{c}'_{n,m}[\rho]$, respectively, when reading the last words from right to left.

We fix notation for the following six identities:

$$\begin{aligned}
\sigma_1 &: \ xysxty \approx yxsxty, \\
\sigma_2 &: \ xsytxy \approx xsytyx, \\
\sigma_3 &: \ xsxyty \approx xsyxty, \\
\alpha_1 &: \ xysxtxhy \approx yxsxtxhy, \\
\alpha_2 &: \ xysxtyhx \approx yxsxtyhx, \\
\alpha_3 &: \ xysytxhx \approx yxsytxhx.
\end{aligned}$$

For any $i = 1, 2, 3$, we denote by β_i the identity dual to α_i. The identity σ_1 [respectively, σ_2, σ_3] allows us to swap adjacent occurrences of two letters in a word whenever both the occurrences are non-last [both the occurrences are non-first, one occurrence is non-first and another one is non-last]. We will use these facts throughout the article many times without explicitly specifying this.

Let $\operatorname{var}\Sigma$ denote the monoid variety given by an identity system Σ. We fix notation for the following monoid varieties:

$$\mathbf{K} = \operatorname{var}\{x^2y \approx x^2yx,\, xyx \approx xyx^2,\, x^2y^2 \approx y^2x^2\},$$

$$\mathbf{N} = \operatorname{var}\{x^2 \approx x^3,\, x^2y \approx yx^2,\, xyxzx \approx x^2yz,\, \sigma_2,\, \sigma_3\},$$

$$\mathbf{P}_n = \operatorname{var}\{x^n \approx x^{n+1},\, x^ny \approx yx^n,\, x^2y \approx xyx\}, \text{ where } n \in \mathbb{N},$$

$$\mathbf{Q}_{r,s} = \operatorname{var}\left\{\begin{array}{l|l} x^2 \approx x^3,\, x^2y \approx yx^2, & 1 \le i, j \le 3, \\ \sigma_3,\, \alpha_i,\, \beta_j, & i \ne r,\, j \ne s, \\ \mathbf{c}_{n,m}[\rho] \approx \mathbf{c}'_{n,m}[\rho], & n, m \in \mathbb{N}_0, \\ \mathbf{d}_{n,m}[\rho] \approx \mathbf{d}'_{n,m}[\rho] & \rho \in S_{n+m} \end{array}\right\}, \text{ where } 1 \le r, s \le 3,$$

$$\mathbf{R} = \operatorname{var}\{x^2 \approx x^3,\, x^2y \approx yx^2,\, \sigma_1,\, \sigma_2,\, \mathbf{w}_n[\pi,\tau] \approx \mathbf{w}'_n[\pi,\tau] \mid n \in \mathbb{N},\, \pi, \tau \in S_n\}.$$

By $\mathbf{V}^\delta$ we denote the monoid variety *dual* to the variety $\mathbf{V}$ (in other words, $\mathbf{V}^\delta$ consists of monoids dual to members of $\mathbf{V}$).

Our first main result is the following

Theorem 1.1. *A variety of monoids $\mathbf{V}$ is fi-permutable if and only if one of the following holds:*

(i) $\mathbf{V}$ *is a group variety;*
(ii) $\mathbf{V}$ *is a variety of idempotent monoids;*
(iii) $\mathbf{V}$ *is contained in one of the following varieties:* $\mathbf{D}_\infty \vee \mathbf{N}$, $\mathbf{D}_\infty \vee \mathbf{N}^\delta$, $\mathbf{K}$, $\mathbf{K}^\delta$, $\mathbf{P}_n$, $\mathbf{P}_n^\delta$, $\mathbf{Q}_{r,s}$ *or* $\mathbf{R}$, *where* $n \in \mathbb{N}$ *and* $1 \le r, s \le 3$.

A variety of semigroups [monoids] is called *completely regular* if it consists of *completely regular* semigroups [monoids], that is, unions of groups. The second main result of the article is the following

Theorem 1.2. *A variety of monoids $\mathbf{V}$ is almost fi-permutable if and only if $\mathbf{V}$ either is a completely regular variety or is contained in one of the varieties listed in the item (iii) of Theorem 1.1.*

As we have mentioned above, any [almost] fi-permutable semigroup variety has the Arguesian subvariety lattice. It follows from results of [27, 28] that, beyond the completely simple [completely regular] case, subvariety lattices of such varieties possesses the much stronger distributive law. Results of the present article imply the following analogs of these claims.

Corollary 1.3. *The subvariety lattice of any non-group fi-permutable variety of monoids is distributive.*

Corollary 1.4. *The subvariety lattice of any non-completely regular almost fi-permutable variety of monoids is distributive.*

The claims that the varieties $\mathbf{D}_\infty \vee \mathbf{N}$, $\mathbf{D}_\infty \vee \mathbf{N}^\delta$, $\mathbf{P}_n$, $\mathbf{P}_n^\delta$, $\mathbf{Q}_{r,s}$ and $\mathbf{R}$ with $n \in \mathbb{N}$ and $1 \le r, s \le 3$ have distributive subvariety lattices are new. Moreover, we find below some monoid variety that properly contains $\mathbf{Q}_{r,s}$ and $\mathbf{R}$ and has the distributive subvariety lattice (see Proposition 3.15 and proof of Corollaries 1.3 and 1.4 given in Section 6). All these facts are of some independent interest because only a few examples of monoid varieties with the distributive subvariety lattice are known so far.

The main result of the article [28] shows that the class of almost fi-permutable semigroup varieties is not closed under taking of subvarieties. This fact contrasts with the following assertion which immediately follows from Theorem 1.2.

Corollary 1.5. *Every subvariety of any almost fi-permutable monoid variety is almost fi-permutable variety too.* $\square$

One more immediate corollary of Theorems 1.1 and 1.2 is the following claim.

Corollary 1.6. *If a non-completely regular monoid variety is almost fi-permutable, then it is fi-permutable.* $\square$

Results of the articles [27] and [28] show that the analog of the last claim for semigroup varieties is not the case.

The article consists of seven sections. Section 2 contains definitions, notation, certain known results and its simple corollaries. In Section 3 we prove a number of auxiliary assertions. Section 4 contains several examples of non-permutative fully invariant congruences on the free monoid $\mathfrak{X}^*$, while in Section 5 we prove the fi-permutability of several concrete monoid varieties. Section 6 is devoted to verification of Theorems 1.1 and 1.2 and Corollaries 1.3 and 1.4. Finally, in Section 7 we discuss some generalizations of the notion of fi-permutability.

2. Preliminaries

We start with a general remark which can be straightforwardly checked.

Lemma 2.1. *Let α, β and ν be equivalences on a set S such that $\alpha, \beta \supseteq \nu$. Then α and β permute if and only if the equivalences α/ν and β/ν on the quotient set S/ν permute.* $\square$

Lemma 2.1 shows that, when studying permuting fully invariant congruences, we may consider congruences on the free monoid $\mathfrak{X}^*$ that contain the fully invariant congruence ν on $\mathfrak{X}^*$ corresponding to a variety $\mathbf{V}$ instead of congruences on the $\mathbf{V}$-free object $\mathfrak{X}^*/\nu$. This is convenient for it is easier to deal with elements of $\mathfrak{X}^*$ (that is, words) than with elements of an arbitrary free objects of monoid varieties.

The well-known result by Jónsson [14] (see also [4, Theorem 410], for instance) immediately implies the following

Lemma 2.2. *Every fi-permutable monoid variety has a modular and moreover, Arguesian subvariety lattice.* $\square$

We denote the empty word by λ. The *content* of a word $\mathbf{w}$, that is, the set of all letters occurring in $\mathbf{w}$ is denoted by $\mathrm{con}(\mathbf{w})$. The following assertion is well known (see [8, Lemma 2.1], for instance).

Lemma 2.3. *Let $\mathbf{V}$ be a monoid variety. The following are equivalent:*

a) $\mathbf{V}$ *is a group variety;*
b) $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$ *with* $\mathrm{con}(\mathbf{u}) \neq \mathrm{con}(\mathbf{v})$*;*
c) $\mathbf{SL} \nsubseteq \mathbf{V}$. $\square$

The following fact is well-known. It was explicitly noted, for example, in [5, Proposition 2.1] or [12, Subsection 1.1].

Lemma 2.4. *The map from the lattice $\mathbb{MON}$ of all monoid varieties to the lattice $\mathbb{SEM}$ of all semigroup varieties that maps a monoid variety generated by a monoid M to the semigroup variety generated by the semigroup reduct of M is an embedding from $\mathbb{MON}$ to $\mathbb{SEM}$.* $\square$

The fully invariant congruence on the free monoid $\mathfrak{X}^*$ corresponding to a variety $\mathbf{V}$ will be denoted by $\theta_{\mathbf{V}}$. It is evident that congruences α and β on an algebra A permute whenever they are comparable in the congruence lattice of A. Therefore, the following claim is true.

Lemma 2.5. *If the varieties of monoids $\mathbf{X}$ and $\mathbf{Y}$ are comparable in the lattice $\mathbb{MON}$, then the congruences $\theta_{\mathbf{X}}$ and $\theta_{\mathbf{Y}}$ permute.* □

Lemma 2.6. *If $\mathbf{X}$ and $\mathbf{Y}$ are completely regular monoid varieties and $\mathbf{X}, \mathbf{Y} \supseteq \mathbf{SL}$, then the congruences $\theta_{\mathbf{X}}$ and $\theta_{\mathbf{Y}}$ permute.*

Proof. Let $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. Then $\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{w}\,\theta_{\mathbf{Y}}\,\mathbf{v}$ for some word $\mathbf{w}$. Now Lemma 2.3 applies with the conclusion that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{w}) = \mathrm{con}(\mathbf{v})$. Therefore, either $\mathbf{u} = \mathbf{v} = \mathbf{w} = \lambda$ or all the words $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are non-empty. Clearly, $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{Y}}\theta_{\mathbf{X}}$ in the first case. Let us consider the second one.

Let φ be the embedding from $\mathbb{MON}$ to $\mathbb{SEM}$ mentioned in Lemma 2.4. Put $\mathbf{X}' = \varphi(\mathbf{X})$ and $\mathbf{Y}' = \varphi(\mathbf{Y})$. Let α and β denote the fully invariant congruences on the free semigroup $\mathfrak{X}^+$ corresponding to the varieties $\mathbf{X}'$ and $\mathbf{Y}'$, respectively. Since every monoid satisfies the identities $x \approx 1 \cdot x \approx x \cdot 1$ and the words $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are non-empty, we may assume that these words do not contain the symbol of 0-ary operation. Hence the identities $\mathbf{u} \approx \mathbf{w}$ and $\mathbf{w} \approx \mathbf{v}$ hold in the varieties $\mathbf{X}'$ and $\mathbf{Y}'$, respectively. Thus $\mathbf{u}\,\alpha\,\mathbf{w}\,\beta\,\mathbf{v}$. It is verified independently by Pastijn [20] and Petrich and Reilly [21] that each completely regular semigroup variety is almost fi-permutable. Then the congruences α and β permute, whence there is a word $\mathbf{w}' \in \mathfrak{X}^+$ with $\mathbf{u}\,\beta\,\mathbf{w}'\,\alpha\,\mathbf{v}$. Therefore, the identities $\mathbf{u} \approx \mathbf{w}'$ and $\mathbf{w}' \approx \mathbf{v}$ hold in the varieties $\mathbf{Y}$ and $\mathbf{X}$, respectively. Thus, $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{Y}}\theta_{\mathbf{X}}$, and we are done. □

A word $\mathbf{w}$ is called an *isoterm* for a class of monoids if no monoid in the class satisfies any non-trivial identity of the form $\mathbf{w} \approx \mathbf{w}'$. It is evident that if a word $\mathbf{u}$ is an isoterm for a monoid variety $\mathbf{V}$, then every subword of $\mathbf{u}$ is an isoterm for $\mathbf{V}$ too. Below we will use this fact many times, as a rule, without mentioning it explicitly. The following statement explains an important role that monoids of the form $S(W)$ play.

Lemma 2.7 (Jackson [11, Lemma 3.3])**.** *Let $\mathbf{V}$ be a monoid variety and W be a set of words. Then $S(W)$ lies in $\mathbf{V}$ if and only if each word in W is an isoterm for $\mathbf{V}$.* □

For any $n \geq 2$, we put $\mathbf{C}_n = \mathrm{var}\{x^n \approx x^{n+1},\, xy \approx yx\}$. The following statement is well-known. It readily follows from [1, Corollary 6.1.5], for instance.

Lemma 2.8. *Let $n \in \mathbb{N}$. For a monoid variety $\mathbf{V}$, the following are equivalent:*

a) *x^n is not an isoterm for $\mathbf{V}$;*

b) *$\mathbf{V}$ satisfies the identity*

$$x^n \approx x^m \tag{2.1}$$

for some $m > n$;

c) *$\mathbf{C}_{n+1} \nsubseteq \mathbf{V}$.* □

It is well known that a monoid variety is completely regular if and only if it satisfies an identity of the form

$$x \approx x^{n+1} \tag{2.2}$$

for some $n \in \mathbb{N}$. Then Lemma 2.8 implies

Corollary 2.9 (Gusev and Vernikov [8, Corollary 2.6])**.** *A variety of monoids* $\mathbf{V}$ *is completely regular if and only if* $\mathbf{C}_2 \nsubseteq \mathbf{V}$*.* □

A variety of monoids is called *aperiodic* if all its groups are singletons. It is well known that a variety is aperiodic if and only if it satisfies an identity of the form

$$x^n \approx x^{n+1} \tag{2.3}$$

for some $n \in \mathbb{N}$.

Lemma 2.10. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be aperiodic monoid varieties. If* $\mathbf{X} \wedge \mathbf{Y}$ *satisfies a non-trivial identity of the form* (2.1)*, then this identity holds in either* $\mathbf{X}$ *or* $\mathbf{Y}$*.*

Proof. We may assume without loss of generality that $n < m$. Lemma 2.8 implies that one of the varieties $\mathbf{X}$ and $\mathbf{Y}$, say, $\mathbf{X}$ satisfies an identity of the form $x^n \approx x^r$ for some $r > n$. But the variety $\mathbf{X}$ is aperiodic, whence it satisfies the identity (2.3) and therefore, the identity (2.1). □

For any $k \in \mathbb{N}$, we fix notation for the following identity:

$$\delta_k: \ \ xt_1xt_2x \cdots t_kx \approx x^2t_1t_2 \cdots t_k.$$

For convenience, we denote by δ_∞ the trivial identity.

Lemma 2.11 (Lee [18, proof of Proposition 4.1])**.** *Let* $k \in \mathbb{N} \cup \{\infty\}$*. Then*

$$\mathbf{D}_k = \mathrm{var}\{x^2 \approx x^3,\, x^2y \approx yx^2,\, \sigma_1,\, \sigma_2,\, \sigma_3,\, \delta_k\}.$$ □

Lemma 2.12. *Let* $\mathbf{V}$ *be a monoid variety. If* $\mathbf{D}_{k+1} \nsubseteq \mathbf{V}$ *for some* $k \in \mathbb{N}$*, then* $\mathbf{V}$ *satisfies an identity of the form*

$$xt_1xt_2x \cdots t_kx \approx x^{e_0}t_1x^{e_1}t_2x^{e_2} \cdots t_kx^{e_k}, \tag{2.4}$$

where $e_0, e_1, \ldots, e_k \in \mathbb{N}_0$ *and* $e_i > 1$ *for some* $i \in \{0, 1, \ldots, k\}$*.*

Proof. If $\mathbf{V}$ is non-completely regular, then this claim is proved in Gusev and Vernikov [8, Lemma 2.15]. Suppose now that $\mathbf{V}$ is completely regular. Then $\mathbf{V}$ satisfies the identity (2.2) for some $n \in \mathbb{N}$. Hence the identity $xt_1xt_2x \cdots t_kx \approx x^{n+1}t_1xt_2x \cdots t_kx$ holds in $\mathbf{V}$. □

Put $\mathbf{E} = \mathrm{var}\{x^2 \approx x^3,\, x^2y^2 \approx y^2x^2,\, \delta_1\}$.

Lemma 2.13. *Let* $\mathbf{V}$ *be a monoid variety satisfying the identity* (2.3) *with* $n \geq 2$*. If* $\mathbf{C}_2 \subseteq \mathbf{V}$ *and* $\mathbf{E} \nsubseteq \mathbf{V}$*, then* $\mathbf{V}$ *satisfies the identity*

$$x^nyx^n \approx yx^n. \tag{2.5}$$

Proof. If $\mathbf{D}_1 \nsubseteq \mathbf{V}$, then $\mathbf{V}$ is commutative by Gusev and Vernikov [8, Lemma 2.14]. Then $\mathbf{V}$ satisfies the identities $x^nyx^n \approx yx^{2n} \approx yx^n$ and therefore, the identity (2.5). Suppose now that $\mathbf{D}_1 \subseteq \mathbf{V}$. Then $\mathbf{V}$ satisfies an identity of the form $x^pyx^q \approx yx^r$ for some $p, q \in \mathbb{N}$ and $r \geq 2$ by [8, Lemma 4.1 and Proposition 4.2]. One can substitute x^n for x in this identity and apply the identity (2.3). As a result, we obtain the identity (2.5). □

The following statement immediately follows from Gusev and Vernikov [8, Proposition 4.2].

Lemma 2.14. *If the variety* $\mathbf{E}$ *satisfies an identity of the form* $yx^k \approx \mathbf{w}$ *with* $k \geq 2$*, then* $\mathbf{w} = yx^\ell$ *for some* $\ell \geq 2$*.* □

Put $\mathbf{L} = \operatorname{var} S(xtxysy)$ and $\mathbf{M} = \operatorname{var} S(xytxsy)$.

Lemma 2.15 (Gusev and Vernikov [8, Lemma 4.9])**.** *Let* $\mathbf{V}$ *be a monoid variety with* $\mathbf{D}_2 \subseteq \mathbf{V}$.

(i) *If* $\mathbf{M} \nsubseteq \mathbf{V}$, *then* $\mathbf{V}$ *satisfies the identity* σ_1.
(ii) *If* $\mathbf{M}^\delta \nsubseteq \mathbf{V}$, *then* $\mathbf{V}$ *satisfies the identity* σ_2.
(iii) *If* $\mathbf{L} \nsubseteq \mathbf{V}$, *then* $\mathbf{V}$ *satisfies the identity* σ_3. □

Put $\mathbf{O} = \operatorname{var}\{\sigma_2,\, \sigma_3\}$. Following Lee [17], we call an identity of the form

$$\mathbf{u}_0\biggl(\prod_{i=1}^r t_i\mathbf{u}_i\biggr) \approx \mathbf{v}_0\biggl(\prod_{i=1}^r t_i\mathbf{v}_i\biggr),$$

where $\{t_0, t_1, \ldots, t_r\} = \operatorname{sim}\bigl(\prod_{i=0}^r t_i\mathbf{u}_i\bigr) = \operatorname{sim}\bigl(\prod_{i=0}^r t_i\mathbf{v}_i\bigr)$ *efficient* if $\mathbf{u}_i\mathbf{v}_i \ne \lambda$ for any $i = 0, 1, \ldots, r$.

Lemma 2.16 (Lee [17, Lemma 8 and Remark 11])**.** *Every non-commutative subvariety of the variety* $\mathbf{O}$ *can be given within* $\mathbf{O}$ *by a finite number of efficient identities of the form either*

$$x^{e_0}\biggl(\prod_{i=1}^r t_i x^{e_i}\biggr) \approx x^{f_0}\biggl(\prod_{i=1}^r t_i x^{f_i}\biggr), \tag{2.6}$$

where $r, e_0, f_0, e_1, f_1, \ldots, e_r, f_r \in \mathbb{N}_0$ *or*

$$x^{e_0}y^{f_0}\biggl(\prod_{i=1}^r t_i x^{e_i} y^{f_i}\biggr) \approx y^{f_0}x^{e_0}\biggl(\prod_{i=1}^r t_i x^{e_i} y^{f_i}\biggr), \tag{2.7}$$

where $r \in \mathbb{N}_0$, $e_0, f_0 \in \mathbb{N}$, $e_1, f_1, \ldots, e_r, f_r \in \mathbb{N}_0$, $\sum_{i=0}^r e_i \ge 2$ *and* $\sum_{i=0}^r f_i \ge 2$. □

For a word $\mathbf{w}$ and a letter x, let $\operatorname{occ}_x(\mathbf{w})$ denote the number of occurrences of x in $\mathbf{w}$. A letter x is called *simple* [*multiple*] *in a word* $\mathbf{w}$ if $\operatorname{occ}_x(\mathbf{w}) = 1$ [respectively, $\operatorname{occ}_x(\mathbf{w}) > 1$]. The set of all simple [multiple] letters of a word $\mathbf{w}$ is denoted by $\operatorname{sim}(\mathbf{w})$ [respectively, $\operatorname{mul}(\mathbf{w})$]. Let $\mathbf{w}$ be a word and $\operatorname{sim}(\mathbf{w}) = \{t_1, t_2, \ldots, t_m\}$. We will assume without loss of generality that $\mathbf{w}(t_1, t_2, \ldots, t_m) = t_1t_2\cdots t_m$. Then $\mathbf{w} = \mathbf{w}_0t_1\mathbf{w}_1\cdots t_m\mathbf{w}_m$ for some words $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m$. The words $\mathbf{w}_0$, $\mathbf{w}_1$, …, $\mathbf{w}_m$ are called *blocks* of the word $\mathbf{w}$. The representation of the word $\mathbf{w}$ as a product of alternating simple in $\mathbf{w}$ letters and blocks is called a *decomposition* of the word $\mathbf{w}$.

The following statement follows from Lemma 2.7 and the definition of the variety $\mathbf{D}_1$.

Lemma 2.17. *Let* $\mathbf{u} \approx \mathbf{v}$ *be an identity that holds in the variety* $\mathbf{D}_1$. *Suppose that*

$$\mathbf{u}_0t_1\mathbf{u}_1\cdots t_m\mathbf{u}_m \tag{2.8}$$

is the decomposition of the word $\mathbf{u}$. *Then* $\operatorname{con}(\mathbf{u}) = \operatorname{con}(\mathbf{v})$ *and the decomposition of the word* $\mathbf{v}$ *has the form*

$$\mathbf{v}_0t_1\mathbf{v}_1\cdots t_m\mathbf{v}_m \tag{2.9}$$

for some words $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$. □

As usual, the subvariety lattice of a variety $\mathbf{V}$ is denoted by $L(\mathbf{V})$.

Lemma 2.18 (Gusev and Vernikov [8, Proposition 6.1])**.** *The lattice* $L(\mathbf{K})$ *is a chain.* □

Let $\mathbf{V}\Sigma$ denote the variety given by the identity system Σ within the variety $\mathbf{V}$.

Lemma 2.19. *Let* $\mathbf{V}$ *be a variety of algebras and* $\mathbf{W} \subseteq \mathbf{V}$. *Suppose that there is an identity system* Σ *such that:*

(i) *if* $\mathbf{W} \subseteq \mathbf{U} \subseteq \mathbf{V}$, *then* $\mathbf{U} = \mathbf{V}\Phi$ *for some identity system* $\Phi \subseteq \Sigma$;
(ii) *if* $\mathbf{U}, \mathbf{U}' \in [\mathbf{W}, \mathbf{V}]$ *and* $\mathbf{U} \wedge \mathbf{U}'$ *satisfies an identity* $\sigma \in \Sigma$, *then* σ *holds in either* $\mathbf{U}$ *or* $\mathbf{U}'$.

Then the interval $[\mathbf{W}, \mathbf{V}]$ *of the lattice* $L(\mathbf{V})$ *is distributive.*

Proof. Arguing by contradiction, we suppose that there are varieties $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in [\mathbf{W}, \mathbf{V}]$ such that the sublattice L of the interval $[\mathbf{W}, \mathbf{V}]$ generated by $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ is one of the two 5-element non-distributive lattices. We may assume without loss of generality that the varieties $\mathbf{X}$ and $\mathbf{Y}$ are atoms in the lattice L and either $\mathbf{Z}$ also is an atom in L or $\mathbf{Y} \subset \mathbf{Z}$. In either case there is an identity σ that holds in $\mathbf{Y}$ and fails in $\mathbf{Z}$. By the claim (i), we may assume that $\sigma \in \Sigma$. The identity σ holds in the variety $\mathbf{X} \wedge \mathbf{Y} = \mathbf{X} \wedge \mathbf{Z}$. Since this identity fails in $\mathbf{Z}$, the claim (ii) implies that it holds in $\mathbf{X}$. Therefore, σ holds in $\mathbf{X} \vee \mathbf{Y} = \mathbf{X} \vee \mathbf{Z}$. But this contradicts the fact that σ fails in $\mathbf{Z}$. $\square$

3. Auxiliary results

3.1. **Linear-balanced identities.** If $\mathbf{w}$ is a word and $X \subseteq \mathrm{con}(\mathbf{w})$, then we denote by $\mathbf{w}(X)$ the word obtained from $\mathbf{w}$ by deleting all letters except letters from X. If $X = \{x_1, x_2, \dots, x_k\}$, then we write $\mathbf{w}(x_1, x_2, \dots, x_k)$ rather than $\mathbf{w}(\{x_1, x_2, \dots, x_k\})$. If $X \subseteq \mathrm{con}(\mathbf{w})$ and $x_1, x_2, \dots, x_k \in \mathrm{con}(\mathbf{w}) \setminus X$, then we write $\mathbf{w}(x_1, x_2, \dots, x_k, X)$ instead of $\mathbf{w}(X \cup \{x_1, x_2, \dots, x_k\})$. Clearly, if $\mathbf{u}$ and $\mathbf{v}$ are words with $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ and $X \subseteq \mathrm{con}(\mathbf{u})$, then the identity $\mathbf{u} \approx \mathbf{v}$ implies the identity $\mathbf{u}(X) \approx \mathbf{v}(X)$. We will use this fact throughout the rest of the article many times without explicitly specifying this.

A non-empty word $\mathbf{w}$ is called *linear* if $\mathrm{occ}_x(\mathbf{w}) \le 1$ for each letter x. Let $\mathbf{u}$ and $\mathbf{v}$ be words and (2.8) and (2.9) be decompositions of $\mathbf{u}$ and $\mathbf{v}$, respectively. A letter x is called *linear-balanced in the identity* $\mathbf{u} \approx \mathbf{v}$ if x is multiple in $\mathbf{u}$ and $\mathrm{occ}_x(\mathbf{u}_i) = \mathrm{occ}_x(\mathbf{v}_i) \le 1$ for all $i = 0, 1, \dots, m$; the identity $\mathbf{u} \approx \mathbf{v}$ is called *linear-balanced* if any letter $x \in \mathrm{mul}(\mathbf{u}) \cup \mathrm{mul}(\mathbf{v})$ is linear-balanced in this identity.

Lemma 3.1. *Let* $\mathbf{V}$ *be a monoid variety such that the word* $\bigl(\prod_{i=1}^{k} xt_i\bigr)x$ *is an isoterm for* $\mathbf{V}$, $\mathbf{u}$ *be a word such that all its blocks are linear words and* $\mathrm{occ}_x(\mathbf{u}) \le k+1$ *for every letter* x. *Then every identity of the form* $\mathbf{u} \approx \mathbf{v}$ *that holds in the variety* $\mathbf{V}$ *is linear-balanced.*

Proof. Suppose that an identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{V}$. Let (2.8) be the decomposition of $\mathbf{u}$. Lemma 2.7 implies that $\mathbf{D}_1 \subseteq \mathbf{V}$. Then the decomposition of $\mathbf{v}$ has the form (2.9) and $\mathrm{mul}(\mathbf{u}) = \mathrm{mul}(\mathbf{v})$ by Lemma 2.17. Let $x \in \mathrm{mul}(\mathbf{u})$ and $r = \mathrm{occ}_x(\mathbf{u})$. Then x occurs in exactly r blocks of $\mathbf{u}$, say, in blocks $\mathbf{u}_{i_1}, \mathbf{u}_{i_2}, \dots, \mathbf{u}_{i_r}$ with $i_1 < i_2 < \cdots < i_r$. Let $T = \{t_{i_1+1}, t_{i_2+1}, \dots, t_{i_{r-1}+1}\}$. Clearly, $\mathbf{u}(x, T) = \bigl(\prod_{s=1}^{r-1} xt_{i_s+1}\bigr)x$. Since $r \le k+1$ and the word $\bigl(\prod_{i=1}^{k} xt_i\bigr)x$ is an isoterm for $\mathbf{V}$, the word $\mathbf{u}(x, T)$ is an isoterm for $\mathbf{V}$ too. Therefore, $\mathbf{u}(x, T) = \mathbf{v}(x, T)$. It follows that $\mathrm{occ}_x(\mathbf{v}) = r$ and x occurs at most once in each block of $\mathbf{v}$. Suppose that x occurs in blocks $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \dots, \mathbf{v}_{j_r}$ of $\mathbf{v}$ and $j_1 < j_2 < \cdots < j_r$. If $i_1 < j_1$, then $\mathbf{u}(x, T) \ne \mathbf{v}(x, T)$ because the first occurrence of x is preceded by t_{i_1+1} in $\mathbf{v}$. We have a contradiction again. Therefore, $j_1 \le i_1$. By symmetry, $i_1 \le j_1$, whence $i_1 = j_1$. Analogous considerations show that

$i_s = j_s$ for all $s = 2, 3, \ldots, r$. This means that the letter x is linear-balanced in the identity $\mathbf{u} \approx \mathbf{v}$. $\square$

3.2. **The variety $\mathbf{A}$ and its subvarieties.** Put

$$\mathbf{A} = \mathrm{var}\{x^2 \approx x^3,\ x^2y \approx yx^2\}.$$

The following assertion follows from Lemma 3.3 of the work by Lee [18], its proof and Lemma 4.2 of the same work.

Lemma 3.2. *Let $\mathbf{u} \approx \mathbf{v}$ be a non-trivial identity of the form (2.6) with $r, e_0, f_0, e_1, f_1, \ldots, e_r, f_r \in \mathbb{N}_0$, $\sum_{i=0}^r e_i \geq 2$ and $\sum_{i=0}^r f_i \geq 2$. Then*

(i) *if $e_i, f_j > 1$ for some $0 \leq i, j \leq r$, then the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{A}$;*
(ii) *if $e_0, e_1, \ldots, e_r \leq 1$ and $e = \sum_{i=0}^r e_i$, then $\mathbf{A}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{A}\{\delta_{e-1}\}$.* $\square$

We denote the trivial variety of monoids by $\mathbf{T}$.

Lemma 3.3.

(i) *The lattice $L(\mathbf{A})$ is a set-theoretical union of the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{D}_2$ and the interval $[\mathbf{D}_2, \mathbf{A}]$.*
(ii) *The interval $[\mathbf{D}_2, \mathbf{A}]$ is a disjoint union of intervals of the form $[\mathbf{D}_k, \mathbf{A}\{\delta_k\}]$, where $2 \leq k \leq \infty$.*

Proof. (i) The lattice $L(\mathbf{D}_2)$ is the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{D}_2$ by Jackson [11, Fig. 1]. Let $\mathbf{V}$ be a subvariety of $\mathbf{A}$ with $\mathbf{D}_2 \nsubseteq \mathbf{V}$. We have to check that $\mathbf{V} \subseteq \mathbf{D}_2$. In view of Lemma 2.12, the variety $\mathbf{V}$ satisfies an identity of the form $xyx \approx x^k yx^\ell$, where either $k \geq 2$ or $\ell \geq 2$. Since the identities

$$x^2 \approx x^3, \tag{3.1}$$

$$x^2y \approx yx^2 \tag{3.2}$$

hold in the variety $\mathbf{V}$, this variety satisfies the identities $x^2y \approx xyx \approx yx^2$. Clearly, these identities imply the identities σ_1, σ_2, σ_3 and δ_1. Now we can apply Lemma 2.11 and conclude that $\mathbf{X} \subseteq \mathbf{D}_1 \subset \mathbf{D}_2$.

(ii) Let $\mathbf{V} \in [\mathbf{D}_2, \mathbf{A}]$. If $\mathbf{D}_\infty \subseteq \mathbf{V}$, then $\mathbf{V} \in [\mathbf{D}_\infty, \mathbf{A}\{\delta_\infty\}]$. Otherwise, there is a natural number $k \geq 2$ such that $\mathbf{D}_k \subseteq \mathbf{V}$ but $\mathbf{D}_{k+1} \nsubseteq \mathbf{V}$. Now Lemma 2.12 applies with the conclusion that $\mathbf{V}$ satisfies an identity of the form (2.4), where $e_i > 1$ for some i. Then $\mathbf{A}\{(2.4)\} = \mathbf{A}\{\delta_k\}$ by Lemma 3.2(ii). Hence $\mathbf{V} \in [\mathbf{D}_k, \mathbf{A}\{\delta_k\}]$. $\square$

Corollary 3.4. *Let $\mathbf{X}$ and $\mathbf{Y}$ be subvarieties of the variety $\mathbf{A}$. If $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identity δ_k for some $2 \leq k \leq \infty$, then this identity is true in either $\mathbf{X}$ or $\mathbf{Y}$.*

Proof. Lemma 3.3(i) allows us to assume that $\mathbf{X}, \mathbf{Y} \in [\mathbf{D}_2, \mathbf{A}]$. Then Lemma 3.3(ii) implies that there are r, s such that $2 \leq r, s \leq \infty$, $\mathbf{X} \in [\mathbf{D}_r, \mathbf{A}\{\delta_r\}]$ and $\mathbf{Y} \in [\mathbf{D}_s, \mathbf{A}\{\delta_s\}]$. We may assume without loss of generality that $r \leq s$. Then $\mathbf{X} \wedge \mathbf{Y} \in [\mathbf{D}_r, \mathbf{A}\{\delta_r\}]$. If $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identity δ_k, then $k \geq r$ and therefore, δ_k holds in $\mathbf{X}$. $\square$

If $\mathbf{w}$ is a word and $X \subseteq \mathrm{con}(\mathbf{w})$, then we denote by $\mathbf{w}_X$ the word obtained from $\mathbf{w}$ by deleting all letters from X. If $X = \{x\}$, then we write $\mathbf{w}_x$ rather than $\mathbf{w}_{\{x\}}$. Clearly, if $\mathbf{u}$ and $\mathbf{v}$ are words with $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ and $X \subseteq \mathrm{con}(\mathbf{u})$, then the identity $\mathbf{u} \approx \mathbf{v}$ implies the identity $\mathbf{u}_X \approx \mathbf{v}_X$. We will use this fact throughout the rest of the article many times without explicitly specifying this.

Lemma 3.5. *Let $\mathbf{V} \subseteq \mathbf{A}$. If $\mathbf{V}$ does not contain the monoid $S(\mathbf{p}xy\mathbf{q})$, where*

$$\begin{aligned}&\mathbf{p} = a_1t_1 \cdots a_kt_k \text{ and } \mathbf{q} = t_{k+1}a_{k+1} \cdots t_{k+\ell}a_{k+\ell} \text{ for some } k, \ell \in \mathbb{N}_0\\ &\text{and } a_1, a_2, \dots, a_{k+\ell} \text{ are letters such that } \{a_1, a_2, \dots, a_{k+\ell}\} = \{x, y\},\end{aligned} \tag{3.3}$$

then $\mathbf{V}$ satisfies the identity

$$\mathbf{p}xy\mathbf{q} \approx \mathbf{p}yx\mathbf{q}. \tag{3.4}$$

Proof. It is clear that the identity (3.4) holds in the monoid $S(xyx)$ and therefore, in the variety $\mathbf{D}_2$. In view of Lemma 3.3(i), we may assume that $\mathbf{D}_2 \subseteq \mathbf{V}$. Put $\mathbf{u} = \mathbf{p}xy\mathbf{q}$. We note that $\mathrm{sim}(\mathbf{u}) = \{t_1, t_2, \dots, t_{k+\ell}\}$. If the word $\mathbf{u}_y$ is not an isoterm for $\mathbf{V}$, then $\mathbf{V}$ satisfies a non-trivial identity of the form $\mathbf{u}_y \approx \mathbf{u}'$. Lemma 2.17 implies that $\mathbf{u}' = x^{f_0}\bigl(\prod_{i=1}^{k+\ell} t_ix^{f_i}\bigr)$ for some $f_0, f_1, \dots, f_{k+\ell} \in \mathbb{N}_0$. Then we can apply Lemma 3.2(ii) with the conclusion that $\mathbf{V}$ satisfies the identity $\delta_{\mathrm{occ}_x(\mathbf{u})-1}$. This identity implies the identities $\mathbf{u} \approx x^2\mathbf{u}_x \approx \mathbf{p}yx\mathbf{q}$, and we are done. Analogous considerations show that if the word $\mathbf{u}_x$ is not an isoterm for $\mathbf{V}$, then this variety satisfies the identity (3.4).

Finally, suppose that both the words $\mathbf{u}_x$ and $\mathbf{u}_y$ are isoterms for $\mathbf{V}$. Lemma 2.7 implies that $\mathbf{V}$ satisfies a non-trivial identity of the form $\mathbf{u} \approx \mathbf{v}$. In view of Lemma 3.1, the identity $\mathbf{u} \approx \mathbf{v}$ is linear-balanced. This means that $\mathrm{sim}(\mathbf{v}) = \{t_1, t_2, \dots, t_{k+\ell}\}$ and blocks of the word $\mathbf{v}$ (in order of their appearance from left to right) are a_1, a_2, …, a_k, $\mathbf{w}$, a_{k+1}, …, $a_{k+\ell}$, where $\mathbf{w} \in \{xy, yx\}$. Since the identity $\mathbf{u} \approx \mathbf{v}$ is non-trivial, $\mathbf{w} = yx$, whence $\mathbf{v} = \mathbf{p}yx\mathbf{q}$. □

Corollary 3.6. *Let $\mathbf{X}$ and $\mathbf{Y}$ be subvarieties of the variety $\mathbf{A}$. If $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identity (3.4), where the equalities (3.3) hold, then this identity is true in either $\mathbf{X}$ or $\mathbf{Y}$.*

Proof. Lemma 2.7 implies that one of the varieties $\mathbf{X}$ or $\mathbf{Y}$, say $\mathbf{X}$, does not contain the monoid $S(\mathbf{p}xy\mathbf{q})$. Then Lemma 3.5 applies with the conclusion that $\mathbf{X}$ satisfies (3.4). □

3.3. Identities of the form $\mathbf{w}_n[\pi, \tau] \approx \mathbf{w}'_n[\pi, \tau]$.

Lemma 3.7. *Let $n \in \mathbb{N}$, $\pi, \tau, \xi, \eta \in S_n$ with $\mathbf{w}_n[\pi, \tau] \neq \mathbf{w}_n[\xi, \eta]$. Then the monoid $S(\mathbf{w}_n[\xi, \eta])$ satisfies the identity*

$$\mathbf{w}_n[\pi, \tau] \approx \mathbf{w}'_n[\pi, \tau]. \tag{3.5}$$

Proof. We are going to show that if we substitute elements of the monoid $S(\mathbf{w}_n[\xi, \eta])$ for letters in the identity (3.5), then we always obtain a right equality.

If we substitute 1 for x in the identity (3.5), then no matter what is substituted for the other letters, the resulting identity will be trivial, whence it will be true in $S(\mathbf{w}_n[\xi, \eta])$. Suppose now that we substitute some other element of the monoid $S(\mathbf{w}_n[\xi, \eta])$ for x. Then value of the word $\mathbf{w}'_n[\pi, \tau]$ equals 0 because $\mathbf{w}_n[\xi, \eta]$ is square free. One can denote value of the word $\mathbf{w}_n[\pi, \tau]$ under the substitutions by $\overline{\mathbf{w}_n[\pi, \tau]}$ and verify that $\overline{\mathbf{w}_n[\pi, \tau]}$ equals 0 as well. Non-zero elements of the monoid $S(\mathbf{w}_n[\xi, \eta])$ are subwords of the word $\mathbf{w}_n[\xi, \eta]$. Suppose that we substitute some word $\mathbf{a} \in S(\mathbf{w}_n[\xi, \eta]) \setminus \{0\}$ for x in $\mathbf{w}_n[\pi, \tau]$. If $z_i \in \mathrm{con}(\mathbf{a})$ for some $1 \le i \le 2n$, then $\overline{\mathbf{w}_n[\pi, \tau]}$ equals 0 because occurrences of the letter z_i in the word $\mathbf{w}_n[\xi, \eta]$ lie in different blocks of this word, while the both occurrences of the letter x in the word $\mathbf{w}_n[\pi, \tau]$ lie in the same block. Further, if $t_i \in \mathrm{con}(\mathbf{a})$ for some $1 \le i \le 2n$, then $\overline{\mathbf{w}_n[\pi, \tau]}$ equals 0 again because $t_i \in \mathrm{sim}(\mathbf{w}_n[\xi, \eta])$, while $x \in \mathrm{mul}(\mathbf{w}_n[\pi, \tau])$.

It remains to consider the case when $\mathbf{a} = x^k$ for some $k \in \mathbb{N}$. We may assume that $k = 1$ because $\mathbf{a}$ is not a subword of $\mathbf{w}_n[\xi, \eta]$ otherwise. Then $\mathbf{w}_n[\pi, \tau]$ is a subword of the word $\mathbf{w}_n[\xi, \eta]$ only whenever $\mathbf{w}_n[\xi, \eta] = \mathbf{w}_n[\pi, \tau]$. But this is not the case. $\square$

The arguments arising in the proof of Lemma 3.7 are very typical. There are many places below, where it will be necessary to establish that a particular identity is true in a monoid of the form $S(W)$. We will omit the corresponding calculations, since they are very similar in concept to the proof of Lemma 3.7 and simpler than this proof.

If $\mathbf{u}$ and $\mathbf{v}$ are words and Σ is an identity or system of identities, then we will write $\mathbf{u} \stackrel{\Sigma}{\approx} \mathbf{v}$ in the case when the identity $\mathbf{u} \approx \mathbf{v}$ follows from Σ.

Lemma 3.8. *Let* $\mathbf{V}$ *be a monoid variety satisfying the identities* (3.2) *and* (3.5) *for any* $n \in \mathbb{N}$ *and* $\pi, \tau \in S_n$. *If* $\mathbf{w} = \mathbf{p}x\mathbf{q}x\mathbf{r}$ *and* $\mathrm{con}(\mathbf{q}) \subseteq \mathrm{mul}(\mathbf{w})$, *then* $\mathbf{V}$ *satisfies the identity* $\mathbf{w} \approx \mathbf{p}x^2\mathbf{q}\mathbf{r}$.

Proof. For any k and m with $k + m > 0$ and any $\rho \in S_{k+m}$, we put

$$\mathbf{w}_{k,m}[\rho] = \biggl(\prod_{i=1}^{k} z_i t_i\biggr) x \biggl(\prod_{i=1}^{k+m} z_{i\rho}\biggr) x \biggl(\prod_{i=k+1}^{k+m} t_i z_i\biggr),$$
$$\mathbf{w}'_{k,m}[\rho] = \biggl(\prod_{i=1}^{k} z_i t_i\biggr) x^2 \biggl(\prod_{i=1}^{k+m} z_{i\rho}\biggr) \biggl(\prod_{i=k+1}^{k+m} t_i z_i\biggr).$$

The proof of Lemma 4.4 in Gusev and Vernikov [8] implies that any identity of the form

$$\mathbf{w}_{k,m}[\rho] \approx \mathbf{w}'_{k,m}[\rho] \tag{3.6}$$

follows from an identity of the form (3.5) for some $n \in \mathbb{N}$ and $\pi, \tau \in S_n$. Therefore, $\mathbf{V}$ satisfies the identity (3.6) for any $k, m \in \mathbb{N}_0$ and $\rho \in S_{k+m}$.

We may assume that $\mathbf{q} \neq \lambda$ because the required conclusion is evident otherwise. Further considerations are divided into two cases.

Case 1: $x \notin \mathrm{con}(\mathbf{q})$. Suppose that the word $\mathbf{q}$ is linear and depends on pairwise different letters $z_1, z_2, \dots, z_n$. Then $z_i \in \mathrm{con}(\mathbf{pr})$ for each $i = 1, 2, \dots, n$. For any $i = 1, 2, \dots, n$, we fix one occurrence of the letter z_i in the word $\mathbf{pr}$. We may assume without loss of generality that $z_1, z_2, \dots, z_k \in \mathrm{con}(\mathbf{p})$ and $z_{k+1}, \dots, z_{k+m} \in \mathrm{con}(\mathbf{r})$ for some k and m with $k + m = n$ (if it is not the case, we can rename letters). Then $\mathbf{w} = \mathbf{p}x z_{1\rho} z_{2\rho} \cdots z_{(k+m)\rho} x\mathbf{r}$ for an appropriate permutation $\rho \in S_{k+m}$, where

$$\mathbf{p} = \mathbf{w}_0 \biggl(\prod_{i=1}^{k} z_i \mathbf{w}_i\biggr) \quad \text{and} \quad \mathbf{r} = \biggl(\prod_{i=k+1}^{k+m} \mathbf{w}_i z_i\biggr) \mathbf{w}_{k+m+1}$$

for some words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_{k+m+1}$. Therefore, $\mathbf{V}$ satisfies the identity

$$\mathbf{w} \stackrel{(3.6)}{\approx} \mathbf{p}x^2 z_{1\rho} z_{2\rho} \cdots z_{(k+m)\rho} \mathbf{r},$$

and we are done.

It remains to consider the case when the word $\mathbf{q}$ is non-linear. Then there is a letter $y_1 \in \mathrm{con}(\mathbf{q})$ such that $\mathbf{q} = \mathbf{v}_1 y_1 \mathbf{v}_2 y_1 \mathbf{v}_3$, where $y_1 \notin \mathrm{con}(\mathbf{v}_2)$ and the word $\mathbf{v}_2$ is either empty or linear. Then the same arguments as in the previous paragraph show that $\mathbf{V}$ satisfies the identities

$$\mathbf{w} = \mathbf{p}x\mathbf{v}_1 y_1 \mathbf{v}_2 y_1 \mathbf{v}_3 x\mathbf{r} \stackrel{(3.6)}{\approx} \mathbf{p}x\mathbf{v}_1 y_1^2 \mathbf{v}_2 \mathbf{v}_3 x\mathbf{r} \stackrel{(3.2)}{\approx} \mathbf{p}y_1^2 x\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 x\mathbf{r}.$$

In other words, we may delete the letter y_1 from the word $\mathbf{q}$. Repeating these considerations, we may delete from $\mathbf{q}$ all multiple letters. As a result, we obtain that $\mathbf{V}$ satisfies the identity $\mathbf{w} \approx \mathbf{p}y_1^2y_2^2\cdots y_\ell^2 x\mathbf{q}'x\mathbf{r}$ for some letters $y_1, y_2, \dots, y_\ell$ and some word $\mathbf{q}'$ that is either empty or linear. Then we may repeat considerations from the previous paragraph and conclude that $\mathbf{V}$ satisfies the identities

$$\mathbf{w} \approx \mathbf{p}y_1^2y_2^2\cdots y_\ell^2 x\mathbf{q}'x\mathbf{r} \approx \mathbf{p}y_1^2y_2^2\cdots y_\ell^2 x^2\mathbf{q}'\mathbf{r} \stackrel{(3.2)}{\approx} \mathbf{p}x^2y_1^2y_2^2\cdots y_\ell^2\mathbf{q}'\mathbf{r}.$$

It remains to return the letters $y_1, y_2, \dots, y_\ell$ to their original places using the identities (3.2) and (3.6). As a result, we obtain that $\mathbf{V}$ satisfies the identity $\mathbf{w} \approx \mathbf{p}x^2\mathbf{q}\mathbf{r}$.

Case 2: $x \in \operatorname{con}(\mathbf{q})$. Then $\mathbf{q} = \mathbf{q}_0 \prod_{i=1}^r (x\mathbf{q}_i)$, where $x \notin \operatorname{con}(\mathbf{q}_0\mathbf{q}_1\cdots\mathbf{q}_r)$. The same arguments as in Case 1 implies that we can step by step swap x and $\mathbf{q}_r$, $\mathbf{q}_{r-1}$, $\dots$, $\mathbf{q}_0$. As a result, we get that $\mathbf{V}$ satisfies the identities

$$\begin{aligned}\mathbf{w} = \mathbf{p}x\mathbf{q}_0\left(\prod_{i=1}^{r-1} x\mathbf{q}_i\right)x\mathbf{q}_r x\mathbf{r} &\approx \mathbf{p}x\mathbf{q}_0\left(\prod_{i=1}^{r-1} x\mathbf{q}_i\right)x^2\mathbf{q}_r\mathbf{r} \approx \\ \approx \mathbf{p}x\mathbf{q}_0\left(\prod_{i=1}^{r-2} x\mathbf{q}_i\right)x^2\mathbf{q}_{r-1}x\mathbf{q}_r\mathbf{r} &\approx \cdots \approx \mathbf{p}x\mathbf{q}_0x^2\mathbf{q}_1\left(\prod_{i=2}^{r} x\mathbf{q}_i\right)\mathbf{r} \approx \mathbf{p}x^2\mathbf{q}\mathbf{r},\end{aligned}$$

and we are done. □

Put $\mathbf{A}^* = \mathbf{A}\{\mathbf{w}_n[\pi,\tau] \approx \mathbf{w}_n'[\pi,\tau] \mid n \in \mathbb{N},\ \pi,\tau \in S_n\}$.

Lemma 3.9. *Let $\mathbf{X} \in [\mathbf{D}_p, \mathbf{A}^*\{\delta_p\}]$ and $\mathbf{Y} \in [\mathbf{D}_q, \mathbf{A}^*\{\delta_q\}]$, where $2 \le p, q \le \infty$. Suppose that $\mathbf{X} \wedge \mathbf{Y}$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$. Then there are a linear-balanced identity $\mathbf{u}' \approx \mathbf{v}'$ that holds in $\mathbf{X} \wedge \mathbf{Y}$ and a word $\mathbf{p}$ such that $\mathbf{A}^*\{\delta_p\}$ and $\mathbf{A}^*\{\delta_q\}$ satisfy the identities $\mathbf{u} \approx \mathbf{p}\mathbf{u}'$ and $\mathbf{v} \approx \mathbf{p}\mathbf{v}'$, respectively, and* $\operatorname{con}(\mathbf{p}) \cap \operatorname{con}(\mathbf{u}') = \varnothing$.

Proof. We will assume without loss of generality that $p \le q$. Let (2.8) is the decomposition of $\mathbf{u}$. Then the decomposition of $\mathbf{v}$ has the form (2.9) by Lemma 2.17. Suppose that exactly k letters are not linear-balanced in the identity $\mathbf{u} \approx \mathbf{v}$. We use induction on k.

Induction base. If $k = 0$, then the required statement is evident.

Induction step. Suppose that $k > 0$. Let x be a letter that is not linear-balanced in the identity $\mathbf{u} \approx \mathbf{v}$. Put $e_i = \operatorname{occ}_x(\mathbf{u}_i)$ and $f_i = \operatorname{occ}_x(\mathbf{v}_i)$ for $i = 0, 1, \dots, m$. Let $e = \sum_{i=0}^m e_i$ and $f = \sum_{i=0}^m f_i$. Further considerations are naturally divided into two cases.

Case 1: $e_i, f_j > 1$ for some i and j. Being a subvariety of $\mathbf{A}$, the variety $\mathbf{A}^*$ satisfies the identity (3.2). Therefore, Lemma 3.8 implies that $\mathbf{A}^*\{\delta_p\}$ and $\mathbf{A}^*\{\delta_q\}$ satisfy the identities $\mathbf{u} \approx x^2\mathbf{u}_x$ and $\mathbf{v} \approx x^2\mathbf{v}_x$, respectively.

The identity $\mathbf{u}_x \approx \mathbf{v}_x$ holds in $\mathbf{X} \wedge \mathbf{Y}$ and only $k-1$ letters are not linear-balanced in this identity. By the induction assumption, there are a linear-balanced identity $\mathbf{u}' \approx \mathbf{v}'$ that holds in $\mathbf{X} \wedge \mathbf{Y}$ and a word $\mathbf{p}$ such that $\mathbf{A}^*\{\delta_p\}$ and $\mathbf{A}^*\{\delta_q\}$ satisfy the identities $\mathbf{u}_x \approx \mathbf{p}\mathbf{u}'$ and $\mathbf{v}_x \approx \mathbf{p}\mathbf{v}'$, respectively, and $\operatorname{con}(\mathbf{p}) \cap \operatorname{con}(\mathbf{u}') = \varnothing$. Then $\mathbf{A}^*\{\delta_p\}$ and $\mathbf{A}^*\{\delta_q\}$ satisfy the identities $\mathbf{u} \approx x^2\mathbf{u}_x \approx x^2\mathbf{p}\mathbf{u}'$ and $\mathbf{v} \approx x^2\mathbf{v}_x \approx x^2\mathbf{p}\mathbf{v}'$, respectively, and we are done.

Case 2: either $e_i \le 1$ for all $i = 0, 1, \dots, m$ or $f_i \le 1$ for all $i = 0, 1, \dots, m$. We may assume without loss of generality that the first claim is true. The identity

$$\mathbf{u}(x, t_1, t_2, \dots, t_m) \approx \mathbf{v}(x, t_1, t_2, \dots, t_m)$$

is non-trivial because the letter x is not linear-balanced in the identity $\mathbf{u}\approx\mathbf{v}$. Then Lemma 3.2(ii) implies that the variety $\mathbf{X}\wedge\mathbf{Y}$ satisfies the identity δ_{e-1}. It is clear that $p\le e-1$ because δ_{e-1} is false in $\mathbf{D}_p$ otherwise. Hence $\mathbf{A}^*\{\delta_p\}$ satisfies the identity δ_{e-1}.

Suppose that $f_i\le 1$ for all $i=0,1,\dots,m$. Then we may assume that $e\le f$ by symmetry. Put $\Psi=\{(3.1),\ \delta_{e-1}\}$. Then $\mathbf{A}^*\{\delta_p\}$ satisfies the identity $\mathbf{u}\stackrel{\Psi}{\approx}\mathbf{w}$, where

$$\mathbf{w}=x^{f_0}(\mathbf{u}_0)_x\biggl(\prod_{i=1}^{m}t_ix^{f_i}(\mathbf{u}_i)_x\biggr).$$

The identity $\mathbf{w}\approx\mathbf{v}$ holds in $\mathbf{X}\wedge\mathbf{Y}$ because $\mathbf{w}\approx\mathbf{u}$ holds in $\mathbf{A}^*\{\delta_p\}$ and $\mathbf{u}\approx\mathbf{v}$ holds in $\mathbf{X}\wedge\mathbf{Y}$. Besides that, the letter x is linear-balanced in the identity $\mathbf{w}\approx\mathbf{v}$, whence only $k-1$ letters are not linear-balanced in this identity. By the induction assumption, there are linear-balanced identity $\mathbf{u}'\approx\mathbf{v}'$ that holds in $\mathbf{X}\wedge\mathbf{Y}$ and a word $\mathbf{p}$ such that $\mathbf{A}^*\{\delta_p\}$ and $\mathbf{A}^*\{\delta_q\}$ satisfy the identities $\mathbf{w}\approx\mathbf{p}\mathbf{u}'$ and $\mathbf{v}\approx\mathbf{p}\mathbf{v}'$, respectively, and $\mathrm{con}(\mathbf{p})\cap\mathrm{con}(\mathbf{u}')=\varnothing$. Then $\mathbf{A}^*\{\delta_p\}$ and $\mathbf{A}^*\{\delta_q\}$ satisfy the identities $\mathbf{u}\approx\mathbf{w}\approx\mathbf{p}\mathbf{u}'$ and $\mathbf{v}\approx\mathbf{p}\mathbf{v}'$, respectively, and we are done.

Finally, suppose that $f_j>1$ for some $j\in\{0,1,\dots,m\}$. It is clear that $\mathbf{A}^*\{\delta_p\}$ satisfies the identity $\mathbf{u}\stackrel{\Psi}{\approx}x^2\mathbf{u}_x$. Further, Lemma 3.8 implies that $\mathbf{A}^*\{\delta_q\}$ satisfies the identity $\mathbf{v}\approx x^2\mathbf{v}_x$. The identity $\mathbf{u}_x\approx\mathbf{v}_x$ holds in $\mathbf{X}\wedge\mathbf{Y}$ and only $k-1$ letters are not linear-balanced in this identity. This allows us to complete the proof by repeating literally arguments from the second paragraph of Case 1. □

Let $n\in\mathbb{N}$, $0\le k\le\ell\le n$ and $\pi,\tau\in S_n$. Put

$$\begin{aligned}\mathbf{w}_n^{k,\ell}[\pi,\tau]=\ &\biggl(\prod_{i=1}^{n}z_it_i\biggr)\biggl(\prod_{i=1}^{k}z_{i\pi}z_{n+i\tau}\biggr)x\biggl(\prod_{i=k+1}^{\ell}z_{i\pi}z_{n+i\tau}\biggr)x\\ &\cdot\biggl(\prod_{i=\ell+1}^{n}z_{i\pi}z_{n+i\tau}\biggr)\biggl(\prod_{i=n+1}^{2n}t_iz_i\biggr).\end{aligned}$$

Note that $\mathbf{w}_n^{0,n}[\pi,\tau]=\mathbf{w}_n[\pi,\tau]$ and $\mathbf{w}_n^{0,0}[\pi,\tau]=\mathbf{w}_n'[\pi,\tau]$.

Lemma 3.10. *Let $n\in\mathbb{N}$, $\pi,\tau\in S_n$ and $\mathbf{X}$ be a monoid variety such that $\mathbf{L}\subseteq\mathbf{X}\subseteq\mathbf{A}\{\sigma_1,\sigma_2\}$. If $S(\mathbf{w}_n[\pi,\tau])\notin\mathbf{X}$, then $\mathbf{X}$ satisfies a non-trivial identity of the form $\mathbf{w}_n[\pi,\tau]\approx\mathbf{w}_n^{k,\ell}[\pi,\tau]$ for some $0\le k\le\ell\le n$.*

Proof. Suppose that $S(\mathbf{w}_n[\pi,\tau])\notin\mathbf{X}$. Then $\mathbf{X}$ satisfies a non-trivial identity of the form $\mathbf{w}_n[\pi,\tau]\approx\mathbf{w}$ by Lemma 2.7. Repeating literally arguments from the proof of Lemma 4.10 in Gusev and Vernikov [8], we can check that

$$\mathbf{w}_x=\biggl(\prod_{i=1}^{n}z_it_i\biggr)\biggl(\prod_{i=1}^{n}z_{i\pi}z_{n+i\tau}\biggr)\biggl(\prod_{i=n+1}^{2n}t_iz_i\biggr).$$

Lemma 2.8 and inclusions $\mathbf{C}_2\subset\mathbf{L}\subseteq\mathbf{X}$ imply that x is an isoterm for $\mathbf{V}$, whence $\mathrm{occ}_x(\mathbf{w})\ge 2$. Therefore,

$$\mathbf{w}=\biggl(\prod_{i=1}^{n}\mathbf{p}_{2i-1}z_i\mathbf{p}_{2i}t_i\biggr)\mathbf{q}_0\biggl(\prod_{i=1}^{n}z_{i\pi}\mathbf{q}_{2i-1}z_{n+i\tau}\mathbf{q}_{2i}\biggr)\biggl(\prod_{i=n+1}^{2n}t_i\mathbf{r}_{2i-2n-1}z_i\mathbf{r}_{2i-2n}\biggr),$$

where $\mathbf{p}_1\cdots\mathbf{p}_{2n}\mathbf{q}_0\mathbf{q}_1\cdots\mathbf{q}_{2n}\mathbf{r}_1\cdots\mathbf{r}_{2n}=x^m$ with $m\ge 2$. If $m=2$, then we can complete the proof by the same arguments as in the proof of Lemma 4.10 in [8]. Now we suppose that $m>2$.

Suppose that x appears at most once in all blocks of the word $\mathbf{w}$. Then $\mathbf{X}$ satisfies the identity δ_{m-1} by Lemma 3.2(ii). Therefore, $\mathbf{X}$ satisfies the identity

$$\mathbf{w}_n[\pi,\tau] \stackrel{\Psi}{\approx} \left(\prod_{i=1}^{n} z_i t_i\right) x^2 \left(\prod_{i=1}^{n} z_{i\pi} z_{n+i\tau}\right)\left(\prod_{i=n+1}^{2n} t_i z_i\right) = \mathbf{w}_n^{0,0}[\pi,\tau],$$

where $\Psi = \{(3.1), (3.2), \delta_{m-1}\}$. Thus, we may assume that x appears at least twice in some block of the word $\mathbf{w}$. Further considerations are divided into three cases.

Case 1: $\mathrm{occ}_x(\mathbf{p}_{2j-1}\mathbf{p}_{2j}) > 1$ for some $j \in \{1, 2, \ldots, n\}$. If either $\mathrm{occ}_x(\mathbf{p}_{2j-1}) > 1$ or $\mathrm{occ}(\mathbf{p}_{2j}) > 1$, then we can use the identities (3.1) and (3.2) and obtain the identity $\mathbf{w}_n[\pi,\tau] \approx \mathbf{w}_n^{0,0}[\pi,\tau]$.

It remains to consider the case when $\mathbf{p}_{2j-1} = \mathbf{p}_{2j} = x$. Here we use the identity σ_1 and swap the word $\mathbf{p}_{2j-1}$ and the letter z_j. Then we obtain the situation considered in the previous paragraph.

Case 2: $\mathrm{occ}_x(\mathbf{r}_{2j-1}\mathbf{r}_{2j}) > 1$ for some $j \in \{1, 2, \ldots, n\}$. This case is dual to the previous one.

Case 3: $\mathrm{occ}_x(\mathbf{q}_0\mathbf{q}_1 \cdots \mathbf{q}_{2n}) > 1$. If $\mathrm{occ}_x(\mathbf{q}_i) > 1$ for some $0 \le i \le 2n$, then, as well as in Case 1, we can use the identities (3.1) and (3.2) and obtain the identity $\mathbf{w}_n[\pi,\tau] \approx \mathbf{w}_n^{0,0}[\pi,\tau]$.

Thus, we may assume that $\mathrm{occ}_x(\mathbf{q}_i) \le 1$ for $i = 0, 1, \ldots, 2n$. Then there are s and t such that $s < t$, $\mathbf{q}_s = \mathbf{q}_t = x$ and $\mathbf{q}_{s+1} \cdots \mathbf{q}_{t-1} = \lambda$. Since the letter x appears at most twice in the word $\mathbf{w}$, we have that either $x \in \mathrm{con}\big(\prod_{i=1}^{2n} \mathbf{p}_i \cdot \prod_{i=0}^{s-1} \mathbf{q}_i\big)$ or $x \in \mathrm{con}\big(\prod_{i=t+1}^{2n} \mathbf{q}_i \cdot \prod_{i=1}^{2n} \mathbf{r}_i\big)$. By symmetry, it suffices to consider the former case. Then we apply the identities σ_1 and σ_2 and replace the word $\mathbf{q}_s$ so that it is next to the word $\mathbf{q}_t$. Then we obtain the situation considered in the previous paragraph. $\square$

Lemma 3.11. *The identities σ_3 and* (3.2) *imply the identity* (3.5) *for any $n \in \mathbb{N}$ and $\pi, \tau \in S_n$.*

Proof. Indeed, we have

$$\mathbf{w}_n[\pi,\tau] = \mathbf{p}x\mathbf{q}x\mathbf{r} \stackrel{\sigma_3}{\approx} \mathbf{p}x\mathbf{ab}x\mathbf{r} \stackrel{\sigma_3}{\approx} \mathbf{pa}x^2\mathbf{br} \stackrel{(3.2)}{\approx} \mathbf{p}x^2\mathbf{abr} \stackrel{\sigma_3}{\approx} \mathbf{p}x^2\mathbf{qr} = \mathbf{w}'_n[\pi,\tau],$$

where

$$\mathbf{p} = \prod_{i=1}^{n}(z_i t_i),\ \mathbf{q} = \prod_{i=1}^{n}(z_{i\pi} z_{n+i\tau}),\ \mathbf{r} = \prod_{i=n+1}^{2n}(t_i z_i),\ \mathbf{a} = \prod_{i=1}^{n} z_{i\pi} \quad \text{and} \quad \mathbf{b} = \prod_{i=1}^{n} z_{n+i\tau}.$$

Lemma is proved. $\square$

3.4. **Identities of the form $\mathbf{c}_{n,m,k}[\rho] \approx \mathbf{c}'_{n,m,k}[\rho]$.** For any $n, m, k \in \mathbb{N}_0$ and $\rho \in S_{n+m+k}$, we define the words

$$\mathbf{c}_{n,m,k}[\rho] = \left(\prod_{i=1}^{n} z_i t_i\right) xyt \left(\prod_{i=n+1}^{n+m} z_i t_i\right) x \left(\prod_{i=1}^{n+m+k} z_{i\rho}\right) y \left(\prod_{i=n+m+1}^{n+m+k} t_i z_i\right),$$

$$\mathbf{c}'_{n,m,k}[\rho] = \left(\prod_{i=1}^{n} z_i t_i\right) yxt \left(\prod_{i=n+1}^{n+m} z_i t_i\right) x \left(\prod_{i=1}^{n+m+k} z_{i\rho}\right) y \left(\prod_{i=n+m+1}^{n+m+k} t_i z_i\right).$$

Note that $\mathbf{c}_{n,m,0}[\rho] = \mathbf{c}_{n,m}[\rho]$ and $\mathbf{c}'_{n,m,0}[\rho] = \mathbf{c}'_{n,m}[\rho]$ for all $n, m \in \mathbb{N}_0$ and $\rho \in S_{n+m}$.

Lemma 3.12. *Let* $\mathbf{V}$ *be a monoid variety satisfying the identities* (3.2), (3.5) *and*

$$\mathbf{c}_{n,m,k}[\rho] \approx \mathbf{c}'_{n,m,k}[\rho] \tag{3.7}$$

for all $n, m, k \in \mathbb{N}_0$, $\pi, \tau \in S_n$ *and* $\rho \in S_{n+m+k}$. *If* $\mathbf{w} = \mathbf{p}xy\mathbf{q}x\mathbf{r}y\mathbf{s}$ *and* $\mathrm{con}(\mathbf{r}) \subseteq \mathrm{mul}(\mathbf{w})$, *then* $\mathbf{V}$ *satisfies the identity* $\mathbf{w} \approx \mathbf{p}yx\mathbf{q}x\mathbf{r}y\mathbf{s}$.

Proof. If $\mathbf{r} = \lambda$, then the identity $\mathbf{w} \approx \mathbf{p}yx\mathbf{q}xy\mathbf{s}$ follows from the identity $\mathbf{c}_{0,0,0}[\varepsilon] \approx \mathbf{c}'_{0,0,0}[\varepsilon]$, where ε is the trivial permutation, that is, from the identity

$$xytxy \approx yxtxy. \tag{3.8}$$

Thus, we may assume that $\mathbf{r} \neq \lambda$. Further considerations are divided into two cases.

Case 1: $x, y \notin \mathrm{con}(\mathbf{r})$. Suppose that the word $\mathbf{r}$ is linear and depends on pairwise different letters $z_1, z_2, \dots, z_s$. Then $z_i \in \mathrm{con}(\mathbf{pqs})$ for each $i = 1, 2, \dots, s$. Fix one occurrence of z_i in $\mathbf{pqs}$ for any $i = 1, 2, \dots, s$. Renaming the letters $z_1, z_2, \dots, z_s$ if necessary, we can achieve the inclusions $z_1, z_2, \dots, z_n \in \mathrm{con}(\mathbf{p})$, $z_{n+1}, \dots, z_{n+m} \in \mathrm{con}(\mathbf{q})$ and $z_{n+m+1}, \dots, z_{n+m+k} \in \mathrm{con}(\mathbf{s})$ for some n, m and k with $n + m + k = s$. Then $\mathbf{w} = \mathbf{p}xy\mathbf{q}xz_{1\rho}z_{2\rho}\cdots z_{s\rho}y\mathbf{s}$ for an appropriate permutation $\rho \in S_{n+m+k}$, where

$$\mathbf{p} = \mathbf{w}_0\biggl(\prod_{i=1}^{n} z_i\mathbf{w}_i\biggr),\ \mathbf{q} = \mathbf{w}_{n+1}\biggl(\prod_{i=n+1}^{n+m} z_i\mathbf{w}_{i+1}\biggr),\ \mathbf{s} = \biggl(\prod_{i=n+m+1}^{n+m+k} \mathbf{w}_{i+1}z_i\biggr)\mathbf{w}_{n+m+k+2}$$

for some words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_{n+m+k+2}$. Then the required identity $\mathbf{w} \approx \mathbf{p}yx\mathbf{q}x\mathbf{r}y\mathbf{s}$ follows from the identity (3.7).

It remains to consider the case when the word $\mathbf{r}$ is not linear. Then there is a letter $y_1 \in \mathrm{con}(\mathbf{r})$ such that $\mathbf{r} = \mathbf{v}_1y_1\mathbf{v}_2y_1\mathbf{v}_3$ for some words $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Then we can apply Lemma 3.8 and conclude that $\mathbf{V}$ satisfies the identities

$$\mathbf{w} = \mathbf{p}xy\mathbf{q}x\mathbf{v}_1y_1\mathbf{v}_2y_1\mathbf{v}_3y\mathbf{s} \approx \mathbf{p}xy\mathbf{q}x\mathbf{v}_1y_1^2\mathbf{v}_2\mathbf{v}_3y\mathbf{s} \stackrel{(3.2)}{\approx} \mathbf{p}xy\mathbf{q}y_1^2x\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3y\mathbf{s}.$$

In other words, we can eliminate two occurrences of the letter y_1 from the word $\mathbf{r}$. Repeating these arguments the necessary number of times, we obtain that $\mathbf{V}$ satisfies the identity $\mathbf{w} \approx \mathbf{p}xy\mathbf{q}y_1^2y_2^2\cdots y_\ell^2x\mathbf{r}'y\mathbf{s}$ for some letters $y_1, y_2, \dots, y_\ell$ and some empty or linear word $\mathbf{r}'$. Then we can repeat arguments from the previous paragraph and obtain that $\mathbf{V}$ satisfies the identity $\mathbf{w} \approx \mathbf{p}yx\mathbf{q}y_1^2y_2^2\cdots y_\ell^2x\mathbf{r}'y\mathbf{s}$. Finally, it remains to return the letters $y_1, y_2, \dots, y_\ell$ to their original places using Lemma 3.8 and the identity (3.2). As a result, we obtain that $\mathbf{V}$ satisfies the required identity $\mathbf{w} \approx \mathbf{p}yx\mathbf{q}x\mathbf{r}y\mathbf{s}$.

Case 2: either $x \in \mathrm{con}(\mathbf{r})$ or $y \in \mathrm{con}(\mathbf{r})$. Then there are the words $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ such that $x\mathbf{r}y = \mathbf{r}_1x\mathbf{r}_2y\mathbf{r}_3$ and $x, y \notin \mathrm{con}(\mathbf{r}_2)$. Then considerations from Case 1 imply that $\mathbf{V}$ satisfies the identities

$$\mathbf{w} = \mathbf{p}xy\mathbf{q}\mathbf{r}_1x\mathbf{r}_2y\mathbf{r}_3\mathbf{s} \approx \mathbf{p}yx\mathbf{q}\mathbf{r}_1x\mathbf{r}_2y\mathbf{r}_3\mathbf{s} = \mathbf{p}yx\mathbf{q}x\mathbf{r}y\mathbf{s},$$

and we are done. □

Lemma 3.13. *Let* $\mathbf{V}$ *be a monoid variety that contains the variety* $\mathbf{M}^\delta$ *and does not contain the monoid* $S(\mathbf{c}_{n,m}[\rho])$ *for some* $n, m \in \mathbb{N}_0$ *and* $\rho \in S_{n+m}$. *Then* $\mathbf{V}$ *satisfies the identity*

$$\mathbf{c}_{n,m}[\rho] \approx \mathbf{c}'_{n,m}[\rho]. \tag{3.9}$$

Proof. In view of Lemma 2.7, $\mathbf{V}$ satisfies a non-trivial identity of the form $\mathbf{c}_{n,m}[\rho] \approx \mathbf{c}$ and the word $xsytxy$ is an isoterm for $\mathbf{V}$. Then the word xyx also is an isoterm for $\mathbf{V}$. Now Lemma 3.1 applies and we conclude that

$$\mathbf{c} = \left(\prod_{i=1}^{n} z_i t_i\right)\mathbf{c}' t\left(\prod_{i=n+1}^{n+m} z_i t_i\right)\mathbf{c}'',$$

where $\mathbf{c}' \in \{xy, yx\}$ and $\mathbf{c}''$ is a linear word that depends on the letters z_1, z_2, ..., z_{n+m}, x and y. The assertion dual to Lemma 2.5 in Gusev [6] implies that

$$\mathbf{c}'' = x\left(\prod_{i=1}^{n+m} z_{i\rho}\right)y.$$

Then the claim that the identity $\mathbf{c}_{n,m}[\rho] \approx \mathbf{c}$ is non-trivial implies that $\mathbf{c}' = yx$, whence $\mathbf{c} = \mathbf{c}'_{n,m}[\rho]$. □

Lemma 3.14. *Let* $\mathbf{V}$ *be a monoid variety such that* $\mathbf{D}_2 \subseteq \mathbf{V}$*,* $\mathbf{N} \nsubseteq \mathbf{V}$ *and* $S(\mathbf{c}_{n,m}[\rho]) \notin \mathbf{V}$ *for some* $n, m \in \mathbb{N}_0$ *and* $\rho \in S_{n+m}$*. Then the identity* (3.9) *holds in* $\mathbf{V}$*.*

Proof. Let $\mathbf{u} \approx \mathbf{v}$ be an arbitrary identity that fails in $\mathbf{N}$. We denote by X the set of all letters x such that either $\operatorname{occ}_x(\mathbf{u}) > 2$ or x occurs more than one times in some block of the word $\mathbf{u}$. Lemma 2.7 implies that the word xyx is an isoterm for $\mathbf{V}$. Suppose that there is a letter $x \in X$ such that $\operatorname{occ}_x(\mathbf{v}) = 2$ and x occurs in two different blocks of $\mathbf{v}$. Let t be a simple in $\mathbf{v}$ letter that is located between the occurrences of x in $\mathbf{v}$. Then $\mathbf{V}$ satisfies the identity $\mathbf{v}(x,t) \approx \mathbf{u}(x,t)$ and $\mathbf{v}(x,t) = xtx$. Therefore, $\mathbf{u}(x,t) = xtx$, contradicting with the choice of the letter x. Therefore, the set X coincides with the set of all letters x such that either $\operatorname{occ}_x(\mathbf{v}) > 2$ or x occurs more than one times in some block of the word $\mathbf{v}$.

Let $X = \{x_1, x_2, \dots, x_s\}$. Clearly, $\mathbf{N}$ satisfies the identities $\mathbf{u} \approx x_1^2 x_2^2 \cdots x_s^2 \mathbf{u}_X$ and $\mathbf{v} \approx x_1^2 x_2^2 \cdots x_s^2 \mathbf{v}_X$. This means that the identity $\mathbf{u}_X \approx \mathbf{v}_X$ fails in the variety $\mathbf{N}$. Let (2.8) be the decomposition of the word $\mathbf{u}_X$. By Lemma 2.17, the decomposition of the word $\mathbf{v}_X$ has the form (2.9). Since $\mathbf{N}$ satisfies the identity σ_3, this variety satisfies also the identities

$$\mathbf{u}_X \approx \mathbf{p}_0\mathbf{q}_0\left(\prod_{i=1}^{m} t_i \mathbf{p}_i \mathbf{q}_i\right) \quad \text{and} \quad \mathbf{v}_X \approx \mathbf{p}'_0\mathbf{q}'_0\left(\prod_{i=1}^{m} t_i \mathbf{p}'_i \mathbf{q}'_i\right),$$

where $\mathbf{p}_i$ and $\mathbf{p}'_i$ consist of the first occurrences of letters in the words $\mathbf{u}_X$ and $\mathbf{v}_X$, respectively, while $\mathbf{q}_i$ and $\mathbf{q}'_i$ consist of the second occurrences of letters in the words $\mathbf{u}_X$ and $\mathbf{v}_X$, respectively. Suppose that $\operatorname{con}(\mathbf{p}_i) \neq \operatorname{con}(\mathbf{p}'_i)$ for some $i \in \{0, 1, \dots, m\}$. We may assume without loss of generality that there is a letter $x \in \operatorname{con}(\mathbf{p}_i) \setminus \operatorname{con}(\mathbf{p}'_i)$. Then the first occurrence of x in $\mathbf{v}_X$ lies in $\mathbf{p}'_j$ for some $j \neq i$. We may assume without loss of generality that $i < j$. Then $\mathbf{u}(x, t_{i+1}) = xt_{i+1}x$ but $\mathbf{v}(x, t_{i+1}) \neq xt_{i+1}x$. But this is impossible because $\mathbf{V}$ satisfies the identity $\mathbf{u}(x, t_{i+1}) \approx \mathbf{v}(x, t_{i+1})$ and xtx is an isoterm for $\mathbf{V}$. Thus, $\operatorname{con}(\mathbf{p}_i) = \operatorname{con}(\mathbf{p}'_i)$ for all $i = 0, 1, \dots, m$. Analogous arguments show that $\operatorname{con}(\mathbf{q}_i) = \operatorname{con}(\mathbf{q}'_i)$ for all $i = 0, 1, \dots, m$. Therefore, the identity $\mathbf{u}_X \approx \mathbf{v}_X$ is linear-balanced.

If $\mathbf{p}_i = \mathbf{p}'_i$ for all $i = 0, 1, \dots, m$, then

$$\mathbf{u}_X \stackrel{\sigma_3}{\approx} \mathbf{p}_0\mathbf{q}_0\left(\prod_{i=1}^{m} t_i \mathbf{p}_i \mathbf{q}_i\right) \stackrel{\sigma_2}{\approx} \mathbf{p}_0\mathbf{q}'_0\left(\prod_{i=1}^{m} t_i \mathbf{p}_i \mathbf{q}'_i\right) \stackrel{\sigma_3}{\approx} \mathbf{v}_X.$$

Since the identity $\mathbf{u}_X \approx \mathbf{v}_X$ fails in the variety $\mathbf{N}$ and the identities σ_2 and σ_3 hold in this variety, there is $i \in \{0, 1, \ldots, m\}$ with $\mathbf{p}_i \ne \mathbf{p}_i'$. All blocks in the words $\mathbf{u}_X$ and $\mathbf{v}_X$ are linear words. Hence there are $i \in \{0, 1, \ldots, m\}$ and $x, y \in \mathrm{con}(\mathbf{p}_i)$ such that x precedes y in $\mathbf{p}_i$ but y precedes x in $\mathbf{p}_i'$. Since the identity $\mathbf{u}_X \approx \mathbf{v}_X$ is linear-balanced and the letters x and y occur in the words $\mathbf{u}_X$ and $\mathbf{v}_X$ exactly two times, the identity $\mathbf{u}(x, y, t_{i+1}) \approx \mathbf{v}(x, y, t_{i+1})$ coincides (up to renaming of letters) with one of the identities

$$xytxy \approx yxtyx, \tag{3.10}$$

$$xytyx \approx yxtxy \tag{3.11}$$

or (3.8). Since $\mathbf{u} \approx \mathbf{v}$ is an arbitrary identity that fails in $\mathbf{N}$, this means that each variety that does not contain $\mathbf{N}$ satisfies one of the identities (3.8), (3.10) or (3.11). In particular, one of these identities holds in $\mathbf{V}$.

Lemma 3.13 allows us to suppose that $\mathbf{M}^\delta \nsubseteq \mathbf{V}$. Then Lemma 2.15(ii) applies with the conclusion that $\mathbf{V}$ satisfies the identity σ_2. Therefore, if $\mathbf{V}$ satisfies one of the identities (3.10) or (3.11), then $\mathbf{V}$ satisfies also the identity (3.8). Thus, $\mathbf{V}$ satisfies the identity (3.8) in any case. Therefore, $\mathbf{V}$ satisfies the identities

$$\begin{aligned}\mathbf{c}_{n,m}[\rho] &\stackrel{\sigma_2}{\approx} \biggl(\prod_{i=1}^{n} z_i t_i\biggr) xyt \biggl(\prod_{i=n+1}^{n+m} z_i t_i\biggr)\biggl(\prod_{i=1}^{n+m} z_{i\pi}\biggr) xy\\ &\stackrel{(3.8)}{\approx} \biggl(\prod_{i=1}^{n} z_i t_i\biggr) yxt \biggl(\prod_{i=n+1}^{n+m} z_i t_i\biggr)\biggl(\prod_{i=1}^{n+m} z_{i\pi}\biggr) xy \stackrel{\sigma_2}{\approx} \mathbf{c}_{n,m}'[\rho].\end{aligned}$$

Lemma is proved. □

3.5. **The variety $\mathbf{A}'$.** For any $n, m, k \in \mathbb{N}_0$ and $\rho \in S_{n+m+k}$, we denote by $\mathbf{d}_{n,m,k}[\rho]$ and $\mathbf{d}_{n,m,k}'[\rho]$ the words that are obtained from the words $\mathbf{c}_{n,m,k}[\rho]$ and $\mathbf{c}_{n,m,k}'[\rho]$, respectively, when reading the last words from right to left. Put

$$\mathbf{A}' = \mathbf{A}^*\{\mathbf{c}_{n,m,k}[\rho] \approx \mathbf{c}_{n,m,k}'[\rho],\ \mathbf{d}_{n,m,k}[\rho] \approx \mathbf{d}_{n,m,k}'[\rho] \mid n, m, k \in \mathbb{N}_0,\ \rho \in S_{n+m+k}\}.$$

The following statement indicates an important property of the lattice $L(\mathbf{A}')$, which will be useful for us in the proof of Corollaries 1.3 and 1.4.

Proposition 3.15. *The lattice $L(\mathbf{A}')$ is distributive.*

Proof. In view of Lemma 3.3(i), it suffices to verify that the interval $[\mathbf{D}_2, \mathbf{A}']$ is distributive. We are going to deduce this fact from Lemma 2.19 with $\mathbf{V} = \mathbf{A}'$, $\mathbf{W} = \mathbf{D}_2$ and the identity system Σ that consists of all identities of the form δ_n with $2 \le n \le \infty$ and all identities of the form (3.4) such that the equalities (3.3) hold. In view of Corollaries 3.4 and 3.6, to do this, it remains to prove only that each monoid variety from the interval $[\mathbf{D}_2, \mathbf{A}']$ may be given within the variety $\mathbf{A}'$ by the identity δ_n with some $2 \le n \le \infty$ and identities of the form (3.4) such that the equalities (3.3) hold.

So, let $\mathbf{V} \in [\mathbf{D}_2, \mathbf{A}']$ and $\mathbf{a} \approx \mathbf{b}$ be an identity that holds in $\mathbf{D}_2$. In view of Lemma 3.3(ii), $\mathbf{A}'\{\mathbf{a} \approx \mathbf{b}\} \in [\mathbf{D}_n, \mathbf{A}'\{\delta_n\}]$ for some $2 \le n \le \infty$. Lemma 3.9 with $\mathbf{X} = \mathbf{Y} = \mathbf{A}'\{\mathbf{a} \approx \mathbf{b}\}$ implies that $\mathbf{A}'\{\delta_n\}$ satisfies the identities $\mathbf{a} \approx \mathbf{pu}$ and $\mathbf{pv} \approx \mathbf{b}$ for some linear-balanced identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{A}'\{\mathbf{a} \approx \mathbf{b}\}$ and some word $\mathbf{p}$ with $\mathrm{con}(\mathbf{p}) \cap \mathrm{con}(\mathbf{u}) = \varnothing$. Clearly, $\mathbf{A}'\{\mathbf{a} \approx \mathbf{b}\} = \mathbf{A}'\{\delta_n,\ \mathbf{u} \approx \mathbf{v}\}$. To complete the proof, it suffices to verify that each linear-balanced identity $\mathbf{u} \approx \mathbf{v}$ is equivalent within $\mathbf{A}'$ to some system of identities of the form (3.4) such that the equalities (3.3) hold. We denote the set of all identities of such a kind by Γ.

We call an identity $\mathbf{c} \approx \mathbf{d}$ 1-*invertible* if $\mathbf{c} = \mathbf{w}'xy\mathbf{w}''$ and $\mathbf{d} = \mathbf{w}'yx\mathbf{w}''$ for some words $\mathbf{w}', \mathbf{w}''$ and letters $x, y \in \operatorname{con}(\mathbf{w}'\mathbf{w}'')$. Let $n > 1$. An identity $\mathbf{c} \approx \mathbf{d}$ is called n-*invertible* if there is a sequence of words $\mathbf{c} = \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n = \mathbf{d}$ such that the identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ is 1-invertible for each $i = 0, 1, \ldots, n-1$ and n is the least number with such a property. For convenience, we will call the trivial identity 0-*invertible*.

Since the identity $\mathbf{u} \approx \mathbf{v}$ is linear-balanced, it is r-invertible for some $r \in \mathbb{N}_0$. We will use induction by r.

Induction base. If $r = 0$, then $\mathbf{u} = \mathbf{v}$, whence $\mathbf{A}'\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{A}'\{\varnothing\}$.

Induction step. Let $r > 0$. Suppose that the decompositions of the words $\mathbf{u}$ and $\mathbf{v}$ have the forms (2.8) and (2.9), respectively. Obviously, $\mathbf{u}_i \neq \mathbf{v}_i$ for some $i \in \{0, 1, \ldots, m\}$. Then the claim that the identity $\mathbf{u} \approx \mathbf{v}$ is linear-balanced implies that $\mathbf{u}_i = \mathbf{u}_i'yx\mathbf{u}_i''$ for some words $\mathbf{u}_i', \mathbf{u}_i''$ and letters x, y such that x precedes y in $\mathbf{v}_i$. We denote by $\mathbf{w}$ the word that is obtained from $\mathbf{u}$ by swapping of the occurrences of x and y in the block $\mathbf{u}_i$.

Suppose that $x, y \in \operatorname{con}(\mathbf{u}_j) = \operatorname{con}(\mathbf{v}_j)$ for some $j \neq i$. Lemma 3.12 shows that we can swap two adjacent occurrences of x and y if somewhere to the right of these occurrences there are two more occurrences of these letters lying in the same block. Thus, Lemma 3.12 or the statement dual to it implies that $\mathbf{A}'$ satisfies the identity $\mathbf{u} \approx \mathbf{w}$. The identity $\mathbf{w} \approx \mathbf{v}$ is $(r-1)$-invertible. By the induction assumption, $\mathbf{A}'\{\mathbf{w} \approx \mathbf{v}\} = \mathbf{A}'\Phi$ for some $\Phi \subseteq \Gamma$. Then $\mathbf{A}'\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{A}'\Phi$, and we are done.

Thus, we may assume that at most one of the letters x and y occurs in $\mathbf{u}_j$ for any $j \neq i$. Let s be the least number such that $\operatorname{con}(\mathbf{u}_s) \cap \{x, y\} \neq \varnothing$. Put

$$T_{x,y} = \{t_j \mid s < j \leq m,\ \operatorname{con}(\mathbf{u}_j) \cap \{x, y\} \neq \varnothing\}. \tag{3.12}$$

Since the identity $\mathbf{u} \approx \mathbf{v}$ is linear-balanced, the identity

$$\mathbf{u}(x, y, T_{x,y}) \approx \mathbf{v}(x, y, T_{x,y}) \tag{3.13}$$

coincides (up to renaming of letters) with an identity of the form (3.4) such that the equalities (3.3) hold. It is clear that $\mathbf{u} \stackrel{(3.13)}{\approx} \mathbf{w}$ and the identity $\mathbf{w} \approx \mathbf{v}$ is $(r-1)$-invertible. By the induction assumption, $\mathbf{A}'\{\mathbf{w} \approx \mathbf{v}\} = \mathbf{A}'\Phi$ for some $\Phi \subseteq \Gamma$. Then $\mathbf{A}'\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{A}'\{(3.13), \Phi\}$, and we are done. $\square$

4. Certain non-permutative fully invariant congruences

Here we find a number of pairs of non-permutative congruences of the type $(\theta_{\mathbf{X}}, \theta_{\mathbf{Y}})$ on the free monoid $\mathfrak{X}^*$. This will be very helpful for us in Section 6 because the variety $\mathbf{X} \vee \mathbf{Y}$ is not fi-permutable in this case by Lemma 2.1.

For any $n > 1$, we denote by $\mathbf{A}_n$ the variety of Abelian groups of exponent n. Put $\mathbf{Z}_1 = \operatorname{var} S(xysxtxhy)$, $\mathbf{Z}_2 = \operatorname{var} S(xysxtyhx)$ and $\mathbf{Z}_3 = \operatorname{var} S(xysytxhx)$.

Lemma 4.1. *The following congruences on the free monoid $\mathfrak{X}^*$ do not permute:*

(i) $\theta_{\mathbf{A}_n}$ *with* $n \geq 2$ *and* $\theta_{\mathbf{SL}}$*;*
(ii) $\theta_{\mathbf{A}_n \vee \mathbf{SL}}$ *with* $n \geq 2$ *and* $\theta_{\mathbf{C}_2}$*;*
(iii) $\theta_{\mathbf{C}_3}$ *and* $\theta_{\mathbf{D}_2}$*;*
(iv) $\theta_{\mathbf{D}_2}$ *and* $\theta_{\mathbf{E}}$*;*
(v) $\theta_{\mathbf{E}}$ *and* $\theta_{\mathbf{E}^\delta}$*;*

(vi) $\theta_{\mathbf{L}}$ *and* $\theta_{\mathbf{M}}$*;*
(vii) $\theta_{\mathbf{N}}$ *and* $\theta_{\mathbf{Z}_i}$ *with* $1 \le i \le 3$*;*
(viii) $\theta_{\mathbf{Z}_i}$ *and* $\theta_{\mathbf{Z}_j}$ *with* $1 \le i < j \le 3$.

Proof. (i) Suppose that the congruences $\theta_{\mathbf{A}_n}$ and $\theta_{\mathbf{SL}}$ permute. It is obvious that $x\,\theta_{\mathbf{A}_n}\,xy^n\,\theta_{\mathbf{SL}}\,xy$. Then $(x, xy) \in \theta_{\mathbf{A}_n}\theta_{\mathbf{SL}} = \theta_{\mathbf{SL}}\theta_{\mathbf{A}_n}$. Whence, there exists a word $\mathbf{w}$ such that $x\,\theta_{\mathbf{SL}}\,\mathbf{w}\,\theta_{\mathbf{A}_n}\,xy$. Now Lemma 2.3 applies with the conclusion that $\mathbf{w} = x^k$ for some $k \in \mathbb{N}$. But this is impossible because the variety $\mathbf{A}_n$ violates the identity $x^k \approx xy$.

(ii) Suppose that the congruences $\theta_{\mathbf{A}_n \vee \mathbf{SL}}$ and $\theta_{\mathbf{C}_2}$ permute. It is obvious that $x\,\theta_{\mathbf{A}_n\vee\mathbf{SL}}\,x^{n+1}\,\theta_{\mathbf{C}_2}\,x^2$. Then $(x, x^2) \in \theta_{\mathbf{A}_n\vee\mathbf{SL}}\theta_{\mathbf{C}_2} = \theta_{\mathbf{C}_2}\theta_{\mathbf{A}_n\vee\mathbf{SL}}$. Therefore, there exists a word $\mathbf{w}$ such that $x\,\theta_{\mathbf{C}_2}\,\mathbf{w}\,\theta_{\mathbf{A}_n\vee\mathbf{SL}}\,x^2$. Now Lemma 2.8 implies that $\mathbf{w} = x$. But this is not the case because the identity $x \approx x^2$ fails in $\mathbf{A}_n$.

(iii) Obviously, $xyx\,\theta_{\mathbf{C}_3}\,x^2y\,\theta_{\mathbf{D}_2}\,x^3y$. If the congruences $\theta_{\mathbf{C}_3}$ and $\theta_{\mathbf{D}_2}$ permute, then $(xyx, x^3y) \in \theta_{\mathbf{C}_3}\theta_{\mathbf{D}_2} = \theta_{\mathbf{D}_2}\theta_{\mathbf{C}_3}$. Therefore, $xyx\,\theta_{\mathbf{D}_2}\,\mathbf{w}\,\theta_{\mathbf{C}_3}\,x^3y$ for some word $\mathbf{w}$. Now Lemma 2.7 implies that $\mathbf{w} = xyx$. But then the variety $\mathbf{C}_3$ satisfies the identity (3.1), a contradiction with Lemma 2.8.

(iv) Obviously, $xyx\,\theta_{\mathbf{E}}\,x^2y\,\theta_{\mathbf{D}_2}\,yx^2$. Thus, $(xyx, yx^2) \in \theta_{\mathbf{E}}\theta_{\mathbf{D}_2}$. If the congruences $\theta_{\mathbf{E}}$ and $\theta_{\mathbf{D}_2}$ permute, then there is a word $\mathbf{w}$ such that $xyx\,\theta_{\mathbf{D}_2}\,\mathbf{w}\,\theta_{\mathbf{E}}\,yx^2$. Then $\mathbf{w} = xyx$ by Lemma 2.7 and, simultaneously, $\mathbf{w} = yx^\ell$ for some $\ell \ge 2$ by Lemma 2.14. We have a contradiction.

(v) Obviously, $x^2y\,\theta_{\mathbf{E}}\,xyx\,\theta_{\mathbf{E}^\delta}\,yx^2$, whence $(x^2y, yx^2) \in \theta_{\mathbf{E}}\theta_{\mathbf{E}^\delta}$. But Lemma 2.14 and the dual to it show that there is no word $\mathbf{w}$ such that the identities $x^2y \approx \mathbf{w}$ and $\mathbf{w} \approx yx^2$ hold in the varieties $\mathbf{E}^\delta$ and $\mathbf{E}$, respectively. Therefore, the congruences $\theta_{\mathbf{E}}$ and $\theta_{\mathbf{E}^\delta}$ do not permute.

(vi) Put $\mathbf{u} = xsxyztyhz$ and $\mathbf{v} = xszxytyhz$. Then $\mathbf{u}\,\theta_{\mathbf{L}}\,xsxzytyhz\,\theta_{\mathbf{M}}\,\mathbf{v}$, whence $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{L}}\theta_{\mathbf{M}}$. Suppose that the congruences $\theta_{\mathbf{L}}$ and $\theta_{\mathbf{M}}$ permute. Then $\mathbf{u}\,\theta_{\mathbf{M}}\,\mathbf{w}\,\theta_{\mathbf{L}}\,\mathbf{v}$ for some word $\mathbf{w}$. It is evident that the monoid $S(xyx)$ lies in the varieties $\mathbf{L}$ and $\mathbf{M}$. Then the word xyx is an isoterm for $\mathbf{L}$ and $\mathbf{M}$ by Lemma 2.7. Now we can apply Lemma 3.1 and conclude that $\mathbf{w} = xs\mathbf{a}tyhz$, where $\mathbf{a}$ is a linear word with $\mathrm{con}(\mathbf{a}) = \{x, y, z\}$. If $\mathbf{a} \in \{xzy, zxy, zyx\}$, then $\mathbf{u}(y, z, t, h) = yztyhz$ and $\mathbf{w}(y, z, t, h) \ne yztyhz$. This contradicts with the claim that the word $yztyhz$ is an isoterm for $\mathbf{M}$ by Lemma 2.7. If $\mathbf{a} \in \{xyz, yxz\}$, then $\mathbf{w}(x, z, s, t) = xsxztz$ and $\mathbf{v}(x, z, s, t) \ne xsxztz$. Finally, if $\mathbf{a} = yzx$, then $\mathbf{v}(x, y, s, t) = xsxyty$ and $\mathbf{w}(x, y, s, t) \ne xsxyty$. In both the cases we have a contradiction with the fact that the word $xsxyty$ is an isoterm for $\mathbf{L}$ by Lemma 2.7.

(vii) We consider the case when $i = 3$ only. The other cases can be considered quite analogously. Put $\mathbf{u} = yxzsyztxhx$ and $\mathbf{v} = xzysyztxhx$. Then $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{N}}\theta_{\mathbf{Z}_3}$ because $\mathbf{u}\,\theta_{\mathbf{N}}\,xyzsyztxhx\,\theta_{\mathbf{Z}_3}\,\mathbf{v}$. Suppose that the congruences $\theta_{\mathbf{N}}$ and $\theta_{\mathbf{Z}_3}$ permute. Then $\mathbf{u}\,\theta_{\mathbf{Z}_3}\,\mathbf{w}\,\theta_{\mathbf{N}}\,\mathbf{v}$ for some word $\mathbf{w}$. It is evident that the monoid $S(xyx)$ lies in the varieties $\mathbf{N}$ and $\mathbf{Z}_3$. Then the word xyx is an isoterm for $\mathbf{N}$ and $\mathbf{Z}_3$ by Lemma 2.7. Now Lemma 3.1 applies with the conclusion that $\mathbf{w} = \mathbf{a}s\mathbf{b}txhx$, where $\mathbf{a}$ is a linear word with $\mathrm{con}(\mathbf{a}) = \{x, y, z\}$ and $\mathbf{b} \in \{yz, zy\}$. If $\mathbf{a} \in \{xyz, xzy, zxy\}$, then $\mathbf{u}_z = yxsytxhx$ and $\mathbf{w}_z \ne yxsytxhx$. Further, if $\mathbf{a} \in \{yzx, zyx\}$, then $\mathbf{w}_y = zxsztxhx$ and $\mathbf{u}_y \ne zxsztxhx$. In both the cases we have a contradiction with the fact that the word $yxsytxhx$ is an isoterm for $\mathbf{Z}_3$ by Lemma 2.7. This means that $\mathbf{a} = yxz$ and $\mathbf{w} \in \{yxzsyztxhx, yxzszytxhx\}$. Then the identity $\mathbf{w}(y, z, s) \approx \mathbf{v}(y, z, s)$ coincides (up to renaming of letters) with one of the identities (3.8) or (3.11). But this is impossible because the identity $\mathbf{w} \approx \mathbf{v}$ holds in the variety $\mathbf{N}$ and the identities (3.8) and (3.11) fail in this variety.

(viii) We consider the case when $i = 1$ and $j = 2$. The other cases can be considered quite analogously. For brevity, put $\mathbf{p} = t_1xt_2yt_3xt_4z$. Let us consider the words $\mathbf{u} = xyz\mathbf{p}$, $\mathbf{v} = yzx\mathbf{p}$ and $\mathbf{w} = yxz\mathbf{p}$. It is a routine to verify that $\mathbf{u}\,\theta_{\mathbf{Z}_1}\,\mathbf{w}\,\theta_{\mathbf{Z}_2}\,\mathbf{v}$. Thus, $(\mathbf{u},\mathbf{v}) \in \theta_{\mathbf{Z}_1}\theta_{\mathbf{Z}_2}$. Suppose that the congruences $\theta_{\mathbf{Z}_1}$ and $\theta_{\mathbf{Z}_2}$ permute. Then $\mathbf{u}\,\theta_{\mathbf{Z}_2}\,\mathbf{w}'\,\theta_{\mathbf{Z}_1}\,\mathbf{v}$ for some word $\mathbf{w}'$. Lemma 2.7 implies that the word $xyt_1xt_2yt_3x$ is an isoterm for $\mathbf{Z}_2$. Therefore, the word xt_1xt_2x is an isoterm for $\mathbf{Z}_2$ as well. Now we can apply Lemma 3.1 and conclude that $\mathbf{w}' = \mathbf{ap}$, where $\mathbf{a}$ is a linear word with $\operatorname{con}(\mathbf{a}) = \{x, y, z\}$. If $\mathbf{a} \in \{yxz, yzx, zyx\}$, then the identity $\mathbf{u}_{\{z,t_4\}} \approx \mathbf{w}'_{\{z,t_4\}}$ is non-trivial and the left-hand side of it coincides with $xyt_1xt_2yt_3x$. Further, if $\mathbf{a} \in \{xzy, zxy\}$, then we substitute t_2zt_3 for t_2 in the identity $\mathbf{u}(y, z, t_2, t_4) \approx \mathbf{w}'(y, z, t_2, t_4)$ and obtain a non-trivial identity whose right-hand side is $zyt_2zt_3yt_4z$. Both the cases contradict the claim that $xysxtyhx$ is an isoterm for $\mathbf{Z}_2$ by Lemma 2.7. Therefore, $\mathbf{a} = xyz$. This means that the variety $\mathbf{Z}_1$ satisfies the identity $\mathbf{u} \approx \mathbf{v}$. But this is not the case because the identity $\mathbf{u}_{\{y,t_2\}} \approx \mathbf{v}_{\{y,t_2\}}$ is non-trivial and the left-hand side of it coincides with the word $xzt_1xt_3xt_4z$ which is an isoterm for $\mathbf{Z}_1$ by Lemma 2.7. □

We denote the first letter of a word $\mathbf{w}$ by $h(\mathbf{w})$.

Lemma 4.2. *Let $n \in \mathbb{N}$ and $\pi, \tau, \xi, \eta \in S_n$. Suppose that $\mathbf{w}_n[\pi,\tau] \neq \mathbf{w}_n[\xi,\eta]$ and put $\mathbf{X} = \operatorname{var} S(\mathbf{w}_n[\pi,\tau])$ and $\mathbf{Y} = \operatorname{var} S(\mathbf{w}_n[\xi,\eta])$. Then the congruences $\theta_{\mathbf{X}}$ and $\theta_{\mathbf{Y}}$ on the free monoid $\mathfrak{X}^*$ do not permute.*

Proof. Let $\mathbf{u} = \mathbf{p}_1x\mathbf{q}_1x\mathbf{q}_2\mathbf{p}_2$ and $\mathbf{v} = \mathbf{p}_1\mathbf{q}_1x\mathbf{q}_2x\mathbf{p}_2$, where

$$\mathbf{p}_1 = \Bigl(\prod_{i=1}^{n\pi-1} z_it_i\Bigr)\Bigl(\prod_{i=1}^{n} z_i't_i'\Bigr)\Bigl(\prod_{i=n\pi}^{n} z_it_i\Bigr), \qquad \mathbf{q}_1 = \prod_{i=1}^{n}(z_{i\pi}z_{n+i\tau}),$$
$$\mathbf{p}_2 = \Bigl(\prod_{i=n+1}^{n+n\tau-1} t_iz_i\Bigr)\Bigl(\prod_{i=n+1}^{2n} t_i'z_i'\Bigr)\Bigl(\prod_{i=n+n\tau}^{2n} t_iz_i\Bigr), \quad \mathbf{q}_2 = \prod_{i=1}^{n}(z_{i\xi}'z_{n+i\eta}').$$

It is clear that $\mathbf{X}$ and $\mathbf{Y}$ satisfy the identity (3.2). Lemma 3.7 implies that the variety $\mathbf{Y}$ satisfies the identity (3.5). The variety $\mathbf{Y}$ satisfies the identity $\mathbf{u} \approx \mathbf{p}_1\mathbf{q}_1x^2\mathbf{q}_2\mathbf{p}_2$ because

$$\mathbf{u} \stackrel{(3.5)}{\approx} \mathbf{p}_1x^2\mathbf{q}_1\mathbf{q}_2\mathbf{p}_2 \stackrel{(3.2)}{\approx} \mathbf{p}_1\mathbf{q}_1x^2\mathbf{q}_2\mathbf{p}_2.$$

Analogous arguments show that the identity $\mathbf{v} \approx \mathbf{p}_1\mathbf{q}_1x^2\mathbf{q}_2\mathbf{p}_2$ holds in $\mathbf{X}$. Then $(\mathbf{u},\mathbf{v}) \in \theta_{\mathbf{Y}}\theta_{\mathbf{X}}$. Suppose that the congruences $\theta_{\mathbf{X}}$ and $\theta_{\mathbf{Y}}$ permute. Then $\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{w}\,\theta_{\mathbf{Y}}\,\mathbf{v}$ for some word $\mathbf{w}$.

The variety $\mathbf{X}$ satisfies the identity $\mathbf{u}_x \approx \mathbf{w}_x$. Lemma 3.1 implies that $\mathbf{w}_x = \mathbf{p}_1\mathbf{q}'\mathbf{p}_2$, where $\mathbf{q}'$ is a linear word with $\operatorname{con}(\mathbf{q}') = \{z_i, z_i' \mid 1 \le i \le 2n\}$. One can verify that $\mathbf{q}' = \mathbf{q}_1\mathbf{q}_2$. It is proved in Gusev and Vernikov [8, p. 28] that the word $xzxyty$ is an isoterm for $\mathbf{X}$ and $\mathbf{Y}$. It can be checked directly that if $h(\mathbf{q}') \neq h(\mathbf{q}_1\mathbf{q}_2)$, then we can deduce from $\mathbf{u}_x \approx \mathbf{w}_x$ a non-trivial identity of the form $xzxyty \approx \mathbf{a}$, resulting in a contradiction. Indeed, suppose that $h(\mathbf{q}') = z_{n+i\tau}$. Then

$$\begin{aligned} &\mathbf{u}(z_{1\pi}, t_{1\pi}, z_{n+i\tau}, t_{n+i\tau}) = z_{1\pi}t_{1\pi}z_{1\pi}z_{n+i\tau}t_{n+i\tau}z_{n+i\tau} \\ \text{and}\quad &\mathbf{w}(z_{1\pi}, t_{1\pi}, z_{n+i\tau}, t_{n+i\tau}) = z_{1\pi}t_{1\pi}z_{n+i\tau}z_{1\pi}t_{n+i\tau}z_{n+i\tau}, \end{aligned}$$

whence $\mathbf{X}$ satisfies the identity $xzxyty \approx xzyxty$. Analogous arguments show that $h(\mathbf{q}')$ differs from the letters $z_{i\pi}$ with $i > 1$, $z'_{i\xi}$ and $z'_{1+i\eta}$. Thus, $h(\mathbf{q}') = h(\mathbf{q}_1\mathbf{q}_2)$. Arguing in a similar way and moving step by step from left to right by the word $\mathbf{q}'$, we can establish that $\mathbf{q}' = \mathbf{q}_1\mathbf{q}_2$, whence $\mathbf{u}_x = \mathbf{w}_x = \mathbf{p}_1\mathbf{q}_1\mathbf{q}_2\mathbf{p}_2$. Lemma 2.7 implies that

$$\begin{aligned}&\mathbf{w}(x, t_1, z_1, t_2, z_2, \dots, t_n, z_n) = \mathbf{w}_n[\pi, \tau] \quad \text{and}\\ &\mathbf{w}(x, t'_1, z'_1, t'_2, z'_2, \dots, t'_n, z'_n) = \biggl(\prod_{i=1}^{n} z'_i t'_i\biggr) x \biggl(\prod_{i=1}^{n} z'_{i\xi} z'_{n+i\eta}\biggr) x \biggl(\prod_{i=n+1}^{2n} t'_i z'_i\biggr).\end{aligned}$$

These two equalities together with the claim that $\mathbf{u}_x = \mathbf{w}_x = \mathbf{p}_1\mathbf{q}_1\mathbf{q}_2\mathbf{p}_2$ are possible only in the case when $\mathbf{w} = \mathbf{p}_1 x \mathbf{q}_1\mathbf{q}_2 x \mathbf{p}_2$.

Put

$$X = \{x, z_i, t_i, z'_{n\xi}, z'_{n+n\eta}, t'_{n\xi}, t'_{n+n\eta} \mid 1 \le i \le 2n, i \ne n\pi, i \ne n + n\tau\}.$$

Then $\mathbf{X}$ satisfies the identity $\mathbf{u}(X) \approx \mathbf{w}(X)$, that is,

$$\mathbf{p}_1(X) x \mathbf{q}_1(X) x \mathbf{q}_2(X) \mathbf{p}_2(X) \approx \mathbf{p}_1(X) x \mathbf{q}_1(X) \mathbf{q}_2(X) x \mathbf{p}_2(X).$$

The right-hand side of this identity, that is, the word

$$\begin{aligned}&\biggl(\prod_{i=1}^{n\pi-1} z_i t_i\biggr) z'_{n\xi} t'_{n\xi} \biggl(\prod_{i=n\pi+1}^{n} z_i t_i\biggr) x \biggl(\prod_{i=1}^{n-1} z_{i\pi} z_{n+i\tau}\biggr) z'_{n\xi} z'_{n+n\eta} x\\ &\cdot \biggl(\prod_{i=n+1}^{n+n\tau-1} t_i z_i\biggr) t'_{n+n\eta} z'_{n+n\eta} \biggl(\prod_{i=n+n\tau+1}^{2n} t_i z_i\biggr),\end{aligned}$$

coincides (up to renaming of letters) with the word $\mathbf{w}_n[\pi, \tau]$ which is an isoterm for $\mathbf{X}$ by Lemma 2.7. We have a contradiction. □

Lemma 4.3. *Let $n, m \in \mathbb{N}_0$, $n + m > 0$, $\pi \in S_{n+m}$ and $\mathbf{V} = \operatorname{var} S(\mathbf{c}_{n,m}[\pi])$. Then there are permutations $\rho, \tau \in S_{n+m+1}$ such that the varieties $\mathbf{X} = \operatorname{var} S(\mathbf{c}_{n+1,m}[\rho])$ and $\mathbf{Y} = \operatorname{var} S(\mathbf{c}_{n,m+1}[\tau])$ are contained in $\mathbf{V}$ and the congruences $\theta_{\mathbf{X}}$ and $\theta_{\mathbf{Y}}$ on the free monoid $\mathfrak{X}^*$ do not permute.*

Proof. Let $\mathbf{X} = \operatorname{var} S(\mathbf{c}_{n+1,m}[\rho])$ and $\mathbf{Y} = \operatorname{var} S(\mathbf{c}_{n,m+1}[\tau])$, where

$$\begin{aligned}\rho &= \begin{pmatrix} 1 & 2 & \dots & n+m & n+m+1 \\ 1\pi & 2\pi & \dots & (n+m)\pi & n+m+1 \end{pmatrix}\\ \text{and}\quad \tau &= \begin{pmatrix} 1 & 2 & 3 & \dots & n+m+1 \\ 1 & 1\pi+1 & 2\pi+1 & \dots & (n+m)\pi+1 \end{pmatrix}.\end{aligned}$$

For any $k, \ell \in \mathbb{N}_0$ and $\xi \in S_{k+\ell}$, the identity σ_2 implies a non-trivial identity of the form $\mathbf{c}_{k,\ell}[\xi] \approx \mathbf{a}$. Then Lemmas 2.7 and 2.15(ii) imply that all the varieties $\mathbf{V}$, $\mathbf{X}$ and $\mathbf{Y}$ contain $\mathbf{M}^\delta$, whence the word $xzytxy$ is an isoterm for $\mathbf{V}$, $\mathbf{X}$ and $\mathbf{Y}$ by Lemma 2.7. If $\mathbf{X} \nsubseteq \mathbf{V}$, then $\mathbf{V}$ satisfies the identity $\mathbf{c}_{n+1,m}[\rho] \approx \mathbf{c}'_{n+1,m}[\rho]$ by Lemma 3.13. Since

$$\begin{aligned}&(\mathbf{c}_{n+1,m}[\rho])_{\{t_{n+m+1}, z_{n+m+1}\}} = \mathbf{c}_{n,m}[\pi]\\ \text{and}\quad &(\mathbf{c}'_{n+1,m}[\rho])_{\{t_{n+m+1}, z_{n+m+1}\}} = \mathbf{c}'_{n,m}[\pi],\end{aligned}$$

we have a contradiction with Lemma 2.7. Therefore, $\mathbf{X} \subseteq \mathbf{V}$. It can be verified similarly that $\mathbf{Y} \subseteq \mathbf{V}$.

Arguments similar to ones from the proof of Lemma 3.7 show that the variety $\mathbf{X}$ satisfies the identity

$$\mathbf{c}_{n,m+1}[\tau] \approx \mathbf{c}'_{n,m+1}[\tau], \tag{4.1}$$

while the variety $\mathbf{Y}$ satisfies the identity

$$\mathbf{c}_{n+1,m}[\rho] \approx \mathbf{c}'_{n+1,m}[\rho]. \tag{4.2}$$

Let $\mathbf{u} = \mathbf{p}yxz\mathbf{q}$ and $\mathbf{v} = \mathbf{p}xzy\mathbf{q}$, where

$$\begin{aligned}\mathbf{p} &= \biggl(\prod_{i=1}^{n} z_i t_i\biggr)\biggl(\prod_{i=1}^{n+1} z'_i t'_i\biggr) \quad \text{and}\\ \mathbf{q} &= t\biggl(\prod_{i=n+1}^{n+m+1} z_i t_i\biggr)\biggl(\prod_{i=n+2}^{n+m+1} z'_i t'_i\biggr)x\biggl(\prod_{i=1}^{n+m+1} z_{i\tau}\biggr)y\biggl(\prod_{i=1}^{n+m+1} z'_{i\rho}\biggr)z.\end{aligned}$$

It is easy to see that $\mathbf{u} \stackrel{(4.1)}{\approx} \mathbf{p}xyz\mathbf{q} \stackrel{(4.2)}{\approx} \mathbf{v}$. Thus, $\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{p}xyz\mathbf{q}\,\theta_{\mathbf{Y}}\,\mathbf{v}$, whence $(\mathbf{u},\mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. Suppose that the congruences $\theta_{\mathbf{X}}$ and $\theta_{\mathbf{Y}}$ permute. Then there is a word $\mathbf{w}$ such that $\mathbf{u}\,\theta_{\mathbf{Y}}\,\mathbf{w}\,\theta_{\mathbf{X}}\,\mathbf{v}$. Since $xzytxy$ and therefore, xtx are isoterms for $\mathbf{X}$ and $\mathbf{Y}$, Lemma 3.1 and the assertion dual to Lemma 2.5 in Gusev [6] imply that $\mathbf{w} = \mathbf{p}\mathbf{a}\mathbf{q}$, where $\mathbf{a}$ is a linear word with $\mathrm{con}(\mathbf{a}) = \{x,y,z\}$. It is easy to see that:

(i) if $\mathbf{a} \in \{xyz, xzy, zxy\}$, then $\mathbf{Y}$ satisfies the identity (4.1);

(ii) if $\mathbf{a} \in \{yzx, zyx\}$, then $\mathbf{Y}$ satisfies the identity

$$\begin{aligned}&\biggl(\prod_{i=1}^{n} z_i t_i\biggr)xzt\biggl(\prod_{i=n+1}^{n+m+1} z_i t_i\biggr)x\biggl(\prod_{i=1}^{n+m+1} z_{i\tau}\biggr)z\\ \approx &\biggl(\prod_{i=1}^{n} z_i t_i\biggr)zxt\biggl(\prod_{i=n+1}^{n+m+1} z_i t_i\biggr)x\biggl(\prod_{i=1}^{n+m+1} z_{i\tau}\biggr)z\end{aligned}$$

which coincides (up to renaming of letters) with the identity (4.1);

(iii) if $\mathbf{a} = yxz$, then $\mathbf{X}$ satisfies the identity

$$\begin{aligned}&\biggl(\prod_{i=1}^{n+1} z'_i t'_i\biggr)yzt\biggl(\prod_{i=n+2}^{n+m+1} z'_i t'_i\biggr)y\biggl(\prod_{i=1}^{n+m+1} z'_{i\rho}\biggr)z\\ \approx &\biggl(\prod_{i=1}^{n+1} z'_i t'_i\biggr)zyt\biggl(\prod_{i=n+2}^{n+m+1} z'_i t'_i\biggr)y\biggl(\prod_{i=1}^{n+m+1} z'_{i\rho}\biggr)z\end{aligned}$$

which coincides (up to renaming of letters) with the identity (4.2).

In either case we have a contradiction with Lemma 2.7. □

5. Certain fi-permutable varieties

In view of Lemma 2.1, to verify that a monoid variety $\mathbf{V}$ is fi-permutable, it suffices to prove that if $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{V}$ and $(\mathbf{u},\mathbf{v}) \in \theta_{\mathbf{X}} \vee \theta_{\mathbf{Y}}$ (equivalently, an identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{X} \wedge \mathbf{Y}$), then $(\mathbf{u},\mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. We will use this argument many times throughout the rest of the article without explicitly mentioning it.

The section is divided into four subsections each of which is devoted to proving the fi-permutability of one of the varieties $\mathbf{D}_\infty \vee \mathbf{N}$, $\mathbf{P}_n$, $\mathbf{Q}_{r,s}$ or $\mathbf{R}$.

5.1. **The variety $\mathbf{D}_\infty \vee \mathbf{N}$.** The lattice $L(\mathbf{D}_\infty \vee \mathbf{N})$ has a very simple structure and admits of an exhaustive description, which will be obtained below (see Corollary 5.3).

Lemma 5.1. *Any variety from the interval $[\mathbf{D}_2, \mathbf{D}_\infty \vee \mathbf{N}]$ may be given within the variety $\mathbf{D}_\infty \vee \mathbf{N}$ by the identities (3.8), σ_1 or δ_k with $k \ge 2$.*

Proof. It is evident that $\mathbf{D}_\infty \vee \mathbf{N} \subseteq \mathbf{A} \wedge \mathbf{O}\{\alpha_1,\, \alpha_2,\, \alpha_3\}$. Lemmas 2.16 and 3.2 imply that any variety from the interval $[\mathbf{D}_2, \mathbf{D}_\infty \vee \mathbf{N}]$ may be given within the variety $\mathbf{D}_\infty \vee \mathbf{N}$ by the identity δ_k with $k \ge 2$ and the identities of the form (2.7) with $r \in \mathbb{N}_0$, $e_0, f_0 \in \mathbb{N}$, $e_1, f_1, \dots, e_r, f_r \in \mathbb{N}_0$, $\sum_{i=0}^r e_i \ge 2$ and $\sum_{i=0}^r f_i \ge 2$. If either at least one of the numbers $e_0, f_0, e_1, f_1, \dots, e_r, f_r$ is greater than 1 or $\sum_{i=0}^r e_i > 2$ or $\sum_{i=0}^r f_i > 2$, then the identity (2.7) holds in $\mathbf{D}_\infty \vee \mathbf{N}$. Therefore, if an identity of the form (2.7) fails in $\mathbf{D}_\infty \vee \mathbf{N}$, then this identity coincides (up to renaming of letters) with either σ_1 or (3.8). $\square$

Since the identities δ_ℓ with $\ell < k$, σ_1 and (3.8) fail in the varieties $\mathbf{D}_k$, $\mathbf{M}$ and $\mathbf{N}$, respectively, and σ_1 implies (3.8), Lemma 5.1 implies the following assertion.

Corollary 5.2. *If $2 \le k \le \infty$, then $\mathbf{D}_k \vee \mathbf{N} = \mathbf{A} \wedge \mathbf{O}\{\alpha_1,\, \alpha_2,\, \alpha_3,\, \delta_k\}$ and $\mathbf{D}_k \vee \mathbf{M} = \mathbf{A} \wedge \mathbf{O}\{\alpha_1,\, \alpha_2,\, \alpha_3,\, \delta_k,\, (3.8)\}$.* $\square$

Corollary 5.3. *The lattice $L(\mathbf{D}_\infty \vee \mathbf{N})$ has the form shown in Fig. 5.1.*

Proof. In view of Lemma 3.3(i), it suffices to check that the interval $[\mathbf{D}_2, \mathbf{D}_\infty \vee \mathbf{N}]$ is as in Fig. 5.1. According to Lemma 3.3(ii), the interval $[\mathbf{D}_2, \mathbf{D}_\infty \vee \mathbf{N}]$ is a disjoint union of the intervals of the form $[\mathbf{D}_k, (\mathbf{D}_\infty \vee \mathbf{N})\{\delta_k\}]$, where $2 \le k \le \infty$. Now Lemmas 5.1 and 2.11 and Corollary 5.2 apply and we conclude that the interval $[\mathbf{D}_k, (\mathbf{D}_\infty \vee \mathbf{N})\{\delta_k\}]$ is the chain $\mathbf{D}_k \subseteq \mathbf{D}_k \vee \mathbf{M} \subseteq \mathbf{D}_k \vee \mathbf{N}$. All the varieties in this chain are distinct because σ_1 holds in $\mathbf{D}_k$ but fails in $\mathbf{M}$, while (3.8) holds in $\mathbf{D}_k \vee \mathbf{M}$ but fails in $\mathbf{N}$. $\square$

Proposition 5.4. *The variety $\mathbf{D}_\infty \vee \mathbf{N}$ is fi-permutable.*

Proof. Let $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{D}_\infty \vee \mathbf{N}$ and an identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{X} \wedge \mathbf{Y}$. We need to check that $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$.

In view of Lemma 2.5 and Corollary 5.3, we may assume without loss of generality that $\mathbf{X} = \mathbf{D}_k \vee \mathbf{U}$ and $\mathbf{Y} = \mathbf{D}_\ell \vee \mathbf{W}$, where $2 \le \ell < k \le \infty$, $\mathbf{U}, \mathbf{W} \in \{\mathbf{D}_2, \mathbf{M}, \mathbf{N}\}$ and $\mathbf{U} \subset \mathbf{W}$. Lemma 3.11 implies that $\mathbf{D}_\infty \vee \mathbf{N} \subseteq \mathbf{A}^*$. Now Lemmas 3.3(ii) and 3.9 apply and we conclude that there are a linear-balanced identity $\mathbf{u}' \approx \mathbf{v}'$ that holds in $\mathbf{X} \wedge \mathbf{Y}$ and a word $\mathbf{p}$ such that $\mathbf{u} \,\theta_{\mathbf{X}}\, \mathbf{p}\mathbf{u}'$ and $\mathbf{p}\mathbf{v}' \,\theta_{\mathbf{Y}}\, \mathbf{v}$. It follows from Corollary 5.3 that $\mathbf{X} \wedge \mathbf{Y} = \mathbf{D}_\ell \vee \mathbf{U}$. It can be easily verified that an arbitrary linear-balanced identity holds in $\mathbf{D}_\infty$ and therefore, in $\mathbf{D}_k$. Therefore, the identity $\mathbf{u}' \approx \mathbf{v}'$ holds in $\mathbf{D}_k \vee (\mathbf{D}_\ell \vee \mathbf{U}) = \mathbf{D}_k \vee \mathbf{U} = \mathbf{X}$, whence $\mathbf{u}' \,\theta_{\mathbf{X}}\, \mathbf{v}'$. Then $\mathbf{u} \,\theta_{\mathbf{X}}\, \mathbf{p}\mathbf{u}' \,\theta_{\mathbf{X}}\, \mathbf{p}\mathbf{v}' \,\theta_{\mathbf{Y}}\, \mathbf{v}$, whence $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$, and we are done. $\square$

5.2. **The variety $\mathbf{P}_n$.** We put $\mathbf{P} = \mathrm{var}\{\delta_1\}$.

Lemma 5.5. *Any subvariety of the variety $\mathbf{P}$ can be given within $\mathbf{P}$ by a finite number of the following identities:*

$$x^k \approx x^\ell, \tag{5.1}$$

$$x^k y^\ell \approx y^\ell x^k, \tag{5.2}$$

where $k, \ell \in \mathbb{N}$.

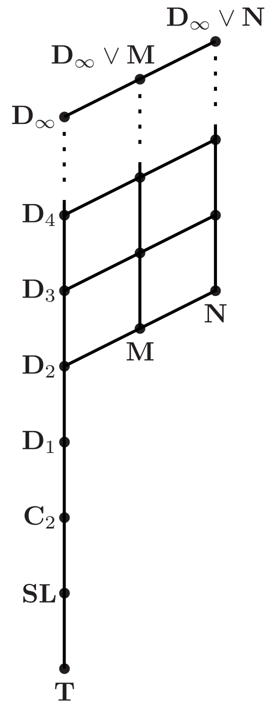

FIGURE 5.1. The lattice $L(\mathbf{D}_\infty \vee \mathbf{N})$

Proof. By results of Head [9], every commutative monoid variety can be given by the identities $xy \approx yx$ and (5.1) for some $k, \ell \in \mathbb{N}$. It remains to prove the required assertion for non-commutative subvarieties of the variety $\mathbf{P}$. To achieve this goal, it suffices to verify that if an identity $\mathbf{u} \approx \mathbf{v}$ does not imply the commutative law, then it is equivalent in the variety $\mathbf{P}$ to some system of identities of the form either (5.1) or (5.2). In view of Lemma 2.16 and the inclusion $\mathbf{P} \subseteq \mathbf{O}$, we may assume that one of the following two statements holds:

(a) the identity $\mathbf{u} \approx \mathbf{v}$ coincides with an efficient identity of the form (2.6) with $r, e_0, f_0, e_1, f_1, \dots, e_r, f_r \in \mathbb{N}_0$;
(b) the identity $\mathbf{u} \approx \mathbf{v}$ coincides with an efficient identity of the form (2.7) with $r \in \mathbb{N}_0$, $e_0, f_0 \in \mathbb{N}$, $e_1, f_1, \dots, e_r, f_r \in \mathbb{N}_0$, $\sum_{i=0}^r e_i \geq 2$ and $\sum_{i=0}^r f_i \geq 2$.

Put $e = \sum_{i=0}^r e_i$ and $f = \sum_{i=0}^r f_i$.

Suppose that the claim (a) holds. Since the identity (2.6) is efficient, we may assume without loss of generality that $e_0 > 0$. The variety $\mathbf{P}\{(2.6)\}$ satisfies the identity $x^e \approx x^f$. If $f_0 > 0$, then $\mathbf{P}\{(2.6)\} = \mathbf{P}\{x^e \approx x^f\}$, and we are done. Finally, if $f_0 = 0$, then $\mathbf{P}\{(2.6)\}$ satisfies the identity $x^e t \approx t x^f$. It is easy to see that in this case $\mathbf{P}\{(2.6)\} = \mathbf{P}_n\{x^e \approx x^f,\ x^e t \approx t x^f\}$, and we are done again.

Suppose now that the claim (b) holds. Since

$$x^e y^f \stackrel{\delta_1}{\approx} x^{e_0} y^{f_0} \left(\prod_{i=1}^r x^{e_i} y^{f_i} \right) \stackrel{(2.7)}{\approx} y^{f_0} x^{e_0} \left(\prod_{i=1}^r x^{e_i} y^{f_i} \right) \stackrel{\delta_1}{\approx} y^f x^e,$$

the variety $\mathbf{P}\{(2.7)\}$ satisfies the identity $x^e y^f \approx y^f x^e$. On the other hand, the identity (2.7) holds in $\mathbf{P}\{x^e y^f \approx y^f x^e\}$ because this variety satisfies the identities

$$x^{e_0} y^{f_0} \left(\prod_{i=1}^{r} t_i x^{e_i} y^{f_i}\right) \stackrel{\delta_1}{\approx} x^e y^f \cdot \prod_{i=1}^{r} t_i \approx y^f x^e \cdot \prod_{i=1}^{r} t_i \stackrel{\delta_1}{\approx} y^{f_0} x^{e_0} \left(\prod_{i=1}^{r} t_i x^{e_i} y^{f_i}\right).$$

Therefore, $\mathbf{P}\{(2.7)\} = \mathbf{P}\{x^e y^f \approx y^f x^e\}$, and we are done. $\square$

Lemma 5.6. *If $k \le \ell < n$ and $p \le q < n$, then $\mathbf{P}_n\{(5.2)\} \subseteq \mathbf{P}_n\{x^p y^q \approx y^q x^p\}$ if and only if $k \le p$ and $\ell \le q$.*

Proof. Necessity. Suppose that $\mathbf{P}_n\{(5.2)\} \subseteq \mathbf{P}_n\{x^p y^q \approx y^q x^p\}$. Consider an identity $x^p y^q \approx \mathbf{w}$ of $\mathbf{P}_n$. Clearly, $\mathrm{con}(\mathbf{w}) = \{x, y\}$, $\mathrm{occ}_x(\mathbf{w}) = p$, $\mathrm{occ}_y(\mathbf{w}) = q$ and $h(\mathbf{w}) = x$. It follows that $\mathrm{var}\{(5.2)\}$ satisfies a non-trivial identity $\mathbf{w} \approx \mathbf{w}'$. But it is easy to see that if either $p < k$ or $q < \ell$, then $\mathbf{w}$ is an isoterm for $\mathrm{var}\{(5.2)\}$. Therefore, $k \le p$ and $\ell \le q$.

Sufficiency follows from the fact that the identities

$$x^p y^q \stackrel{\delta_1}{\approx} x^k y^\ell x^{p-k} y^{q-\ell} \stackrel{(5.2)}{\approx} y^\ell x^k x^{p-k} y^{q-\ell} \stackrel{\delta_1}{\approx} y^q x^p$$

hold in $\mathbf{P}_n\{(5.2)\}$. $\square$

Lemma 5.7. *Let $\mathbf{X}$ and $\mathbf{Y}$ be subvarieties of the variety $\mathbf{P}_n$. If the variety $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identity* (5.2) *for some $1 \le k, \ell < n$, then this identity holds in either $\mathbf{X}$ or $\mathbf{Y}$.*

Proof. Suppose that x^k is not an isoterm for $\mathbf{X}$. Then Lemma 2.8 applies with the conclusion that $\mathbf{X}$ satisfies the identity $x^k \approx x^r$ for some $r > k$. Since $\mathbf{X} \subseteq \mathbf{P}_n$, this implies that $\mathbf{X}$ satisfies the identities $x^k y^\ell \approx x^n y^\ell \approx y^\ell x^n \approx y^\ell x^k$, and we are done. Thus, we may assume that x^k is an isoterm for $\mathbf{X}$. Analogously, we may assume that y^ℓ is an isoterm for $\mathbf{X}$ as well. By symmetry, x^k and y^ℓ are isoterms for $\mathbf{Y}$ too.

Since the identity (5.2) holds in $\mathbf{X} \wedge \mathbf{Y}$, there is a sequence of words $\mathbf{w}_0$, $\mathbf{w}_1$, …, $\mathbf{w}_m$ such that $\mathbf{w}_0 = x^k y^\ell$, $\mathbf{w}_m = y^\ell x^k$ and, for each $i = 0, 1, \dots, m-1$, the identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in either $\mathbf{X}$ or $\mathbf{Y}$. The claim that x^k and y^ℓ are isoterms for $\mathbf{X}$ and $\mathbf{Y}$ imply that $\mathrm{con}(\mathbf{w}_i) = \{x, y\}$, $\mathrm{occ}_x(\mathbf{w}_i) = k$ and $\mathrm{occ}_y(\mathbf{w}_i) = \ell$ for any $i = 0, 1, \dots, m$. Evidently, there is $j \in \{0, 1, \dots, m-1\}$ such that $h(\mathbf{w}_j) = x$ but $h(\mathbf{w}_{j+1}) = y$. The identity $\mathbf{w}_j \approx \mathbf{w}_{j+1}$ holds in either $\mathbf{X}$ or $\mathbf{Y}$. Taking into account that the varieties $\mathbf{X}$ and $\mathbf{Y}$ satisfy the identity δ_1, we have that the identity $\mathbf{w}_j \approx \mathbf{w}_{j+1}$ is equivalent to (5.2) in one of the varieties $\mathbf{X}$ or $\mathbf{Y}$. $\square$

Corollary 5.8. *For any $n \in \mathbb{N}$, the lattice $L(\mathbf{P}_n)$ is distributive.*

Proof. Lemmas 5.5, 2.10 and 5.7 show that it suffices to refer to Lemma 2.19 with $\mathbf{V} = \mathbf{P}_n$, $\mathbf{W} = \mathbf{T}$ and the identity system Σ that consists of the identities (5.1) and (5.2) for all $k, \ell \in \mathbb{N}$. $\square$

Proposition 5.9. *For any $n \in \mathbb{N}$, the variety $\mathbf{P}_n$ is fi-permutable.*

Proof. Let $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{P}_n$ and an identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{X} \wedge \mathbf{Y}$. We need to check that $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}} \theta_{\mathbf{Y}}$.

We may assume that both $\mathbf{X}$ and $\mathbf{Y}$ are non-trivial because the required conclusion is evident otherwise. Lemma 2.3 and the fact that $\mathbf{P}_n$ is aperiodic imply that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. Then, since $\mathbf{P}_n$ satisfies the identity δ_1, we may assume that $\mathbf{u} = x_1^{s_1} x_2^{s_2} \cdots x_m^{s_m}$ and $\mathbf{v} = x_{1\pi}^{t_{1\pi}} x_{2\pi}^{t_{2\pi}} \cdots x_{m\pi}^{t_{m\pi}}$ for some $m \in \mathbb{N}$ and some permutation $\pi \in S_m$. The variety $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identity $x_i^{s_i} \approx x_i^{t_i}$ for each $i = 1, 2, \dots, m$. Lemma 2.10 implies that this identity holds in either $\mathbf{X}$ or $\mathbf{Y}$. We put $\mathbf{u}' = x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}$ and $\mathbf{v}' = x_{1\pi}^{r_{1\pi}} x_{2\pi}^{r_{2\pi}} \cdots x_{m\pi}^{r_{m\pi}}$, where

$$r_i = \begin{cases} s_i & \text{if the identity } x^{s_i} \approx x^{t_i} \text{ holds in } \mathbf{Y}, \\ t_i & \text{if the identity } x^{s_i} \approx x^{t_i} \text{ holds in } \mathbf{X} \text{ but does not hold in } \mathbf{Y} \end{cases}$$

for any $i = 1, 2, \dots, m$. Then $\mathbf{u} \,\theta_{\mathbf{X}}\, \mathbf{u}'$ and $\mathbf{v}' \,\theta_{\mathbf{Y}}\, \mathbf{v}$. It remains to prove that $(\mathbf{u}', \mathbf{v}') \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. Indeed, in this case $\mathbf{u} \,\theta_{\mathbf{X}}\, \mathbf{u}' \,\theta_{\mathbf{X}}\, \mathbf{w} \,\theta_{\mathbf{Y}} \mathbf{v}' \,\theta_{\mathbf{Y}}\, \mathbf{v}$ for some word $\mathbf{w}$ and therefore, $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$.

By Lemma 5.7, for any $i < j$, the identity $\mathbf{u}'(x_i, x_j) \approx \mathbf{v}'(x_i, x_j)$ holds in either $\mathbf{X}$ or $\mathbf{Y}$. Let us step by step as long as possible apply to the word $\mathbf{v}'$ non-trivial identities of the kind $\mathbf{u}'(x_i, x_j) \approx \mathbf{v}'(x_i, x_j)$ that hold in $\mathbf{Y}$ as follows: at each step, we will replace some subword of the form $x_j^{r_j} x_i^{r_i}$ of the word $\mathbf{v}'$ to the subword $x_i^{r_i} x_j^{r_j}$ whenever $i < j$ and the identity $x_j^{r_j} x_i^{r_i} \approx x_i^{r_i} x_j^{r_j}$ holds in $\mathbf{Y}$. As a result, we obtain the word $\mathbf{w}_1 = x_{1\tau}^{r_{1\tau}} x_{2\tau}^{r_{2\tau}} \cdots x_{m\tau}^{r_{m\tau}}$ for some permutation $\tau \in S_m$. Clearly, $\mathbf{w}_1 \,\theta_{\mathbf{Y}}\, \mathbf{v}'$. If $\mathbf{w}_1 = \mathbf{u}'$, then $(\mathbf{u}', \mathbf{v}') \in \theta_{\mathbf{Y}} \subseteq \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. Suppose now that $\mathbf{w}_1 \neq \mathbf{u}'$.

Now we will step by step as long as possible apply to the word $\mathbf{w}_1$ non-trivial identities of the kind $\mathbf{u}'(x_i, x_j) \approx \mathbf{v}'(x_i, x_j)$ that hold in $\mathbf{X}$ by the same way as above: at each step, we will replace some subword of the form $x_j^{r_j} x_i^{r_i}$ of the word $\mathbf{w}_1$ to the subword $x_i^{r_i} x_j^{r_j}$ whenever $i < j$ and the identity $x_j^{r_j} x_i^{r_i} \approx x_i^{r_i} x_j^{r_j}$ holds in $\mathbf{X}$. As a result, we obtain some word $\mathbf{w}_2$. It suffices to verify that $\mathbf{w}_2 = \mathbf{u}'$ because $\mathbf{u}' = \mathbf{w}_2 \,\theta_{\mathbf{X}}\, \mathbf{w}_1 \,\theta_{\mathbf{Y}}\, \mathbf{v}'$ and therefore, $(\mathbf{u}', \mathbf{v}') \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$ in this case.

Arguing by contradiction, we suppose that $\mathbf{w}_2 \neq \mathbf{u}'$. Then there are indexes a and b such that $a < b$ and $x_b^{r_b} x_a^{r_a}$ is a subword of the word $\mathbf{w}_2$. Then $b = p\tau$ and $a = q\tau$ for some $p < q$. Therefore,

$$\mathbf{w}_1 = \left(\prod_{i=1}^{p-1} x_{i\tau}^{r_{i\tau}}\right) x_b^{r_b} \left(\prod_{i=p+1}^{q-1} x_{i\tau}^{r_{i\tau}}\right) x_a^{r_a} \left(\prod_{i=q+1}^{m} x_{i\tau}^{r_{i\tau}}\right).$$

The identity $x^{r_a} y^{r_b} \approx y^{r_b} x^{r_a}$ fails in $\mathbf{X}$ by the definition of the word $\mathbf{w}_2$. Then it holds in $\mathbf{Y}$ by Lemma 5.7. Suppose that $q = p + 1$. Then we can apply the identity $x^{r_a} y^{r_b} \approx y^{r_b} x^{r_a}$ to the word $\mathbf{w}_1$ and replace the subword $x_b^{r_b} x_a^{r_a}$ to the subword $x_a^{r_a} x_b^{r_b}$. As a result, we obtain the word

$$\mathbf{w}_1' = \left(\prod_{i=1}^{p-1} x_{i\tau}^{r_{i\tau}}\right) x_a^{r_a} x_b^{r_b} \left(\prod_{i=q+1}^{m} x_{i\tau}^{r_{i\tau}}\right)$$

such that the identity $\mathbf{v}' \approx \mathbf{w}_1'$ holds in $\mathbf{Y}$. But this is impossible by the definition of the word $\mathbf{w}_1$. Therefore, $p + 1 < q$.

We will assume without loss of generality that $r_b \leq r_a$ (the case when $r_a < r_b$ can be considered quite analogously). For each $i = p+1, p+2, \dots, q-1$, the variety $\mathbf{X}$ satisfies either the identity $x^{r_{i\tau}} y^{r_b} \approx y^{r_b} x^{r_{i\tau}}$ or the identity $x^{r_a} y^{r_{i\tau}} \approx y^{r_{i\tau}} x^{r_a}$. This claim, Lemma 5.6 and the inequality $r_b \leq r_a$ imply that $\mathbf{X}$ satisfies the identity $x^{r_a} y^{r_{i\tau}} \approx y^{r_{i\tau}} x^{r_a}$ for each $i = p+1, p+2, \dots, q-1$. Since the identity $x^{r_a} y^{r_b} \approx y^{r_b} x^{r_a}$ fails in $\mathbf{X}$, we can apply Lemma 5.6 again and obtain that $r_{i\tau} > r_b$ for each $i = p+1, p+2, \dots, q-1$. Since the identity $x^{r_a} y^{r_b} \approx y^{r_b} x^{r_a}$ holds in the variety $\mathbf{Y}$, Lemma 5.6 implies that this variety satisfies also the identity $x^{r_a} y^{r_{i\tau}} \approx y^{r_{i\tau}} x^{r_a}$ for each $i = p+1, p+2, \dots, q-1$. This implies that $(q-1)\tau < a$ because we have a contradiction with the definition of the word $\mathbf{w}_1$ and the fact that $\mathbf{Y}$ satisfies the identity $x^{r_a}y^{r_{(q-1)\tau}} \approx y^{r_{(q-1)\tau}}x^{r_a}$ otherwise.

Suppose that $i\tau < a$ for each $i = p+1, p+2, \dots, q-1$. Then $(p+1)\tau < a < b$. By the definition of the word $\mathbf{w}_1$, this implies that $\mathbf{Y}$ violates the identity $x^{r_{(p+1)\tau}}y^{r_b} \approx y^{r_b}x^{r_{(p+1)\tau}}$. Then $r_{(p+1)\tau} < r_a$ by Lemma 5.6. In view of Lemma 5.7, the identity $x^{r_{(p+1)\tau}}y^{r_b} \approx y^{r_b}x^{r_{(p+1)\tau}}$ holds in $\mathbf{X}$. But this is impossible because this identity implies $x^{r_a}y^{r_b} \approx y^{r_b}x^{r_a}$ by Lemma 5.6.

Finally, suppose that $j\tau > a$ for some $j \in \{p+1, p+2, \dots, q-1\}$. Let j be the largest number with such a property. Put $d = j\tau$. Then there is $c \le a$ such that $x_d^{r_d}x_c^{r_c}$ is a subword of the word $\mathbf{w}_1$. The definition of the word $\mathbf{w}_1$ implies that the variety $\mathbf{Y}$ does not satisfy the identity $x^{r_d}y^{r_c} \approx y^{r_c}x^{r_d}$. Since $r_d \ge r_b$ and the identity $x^{r_a}y^{r_b} \approx y^{r_b}x^{r_a}$ is true in $\mathbf{Y}$, Lemma 5.6 implies that $r_c < r_a$. Therefore, $c < a$. Then the definition of the word $\mathbf{w}_2$ and the inequalities $c < a < b$ imply that the variety $\mathbf{X}$ satisfies the identity $x^{r_c}y^{r_b} \approx y^{r_b}x^{r_c}$. Now Lemma 5.6 applies and we obtain a contradiction with the inequality $r_c < r_a$ and the fact that the identity $x^{r_a}y^{r_b} \approx y^{r_b}x^{r_a}$ fails in $\mathbf{X}$. This contradiction completes the proof. □

5.3. The variety $\mathbf{Q}_{r,s}$.

Proposition 5.10. *For any $1 \le r, s \le 3$, the variety $\mathbf{Q}_{r,s}$ is fi-permutable.*

Proof. Let $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{Q}_{r,s}$ and an identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{X} \wedge \mathbf{Y}$. We need to check that $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$.

In view of Lemmas 2.5 and 3.3(i), we may assume that $\mathbf{X}, \mathbf{Y} \in [\mathbf{D}_2, \mathbf{Q}_{r,s}]$. Lemma 3.11 implies that $\mathbf{Q}_{r,s} \subseteq \mathbf{A}^*$. Then, by Lemmas 3.3(ii) and 3.9, there are a linear-balanced identity $\mathbf{u}' \approx \mathbf{v}'$ that holds in $\mathbf{X} \wedge \mathbf{Y}$ and a word $\mathbf{p}$ such that $\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{p}\mathbf{u}'$ and $\mathbf{p}\mathbf{v}'\,\theta_{\mathbf{Y}}\,\mathbf{v}$. It suffices to verify that $(\mathbf{u}', \mathbf{v}') \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. Indeed, in this case there is a word $\mathbf{w}$ such that $\mathbf{u}'\,\theta_{\mathbf{X}}\,\mathbf{w}\,\theta_{\mathbf{Y}}\,\mathbf{v}'$. Then $\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{p}\mathbf{u}'\,\theta_{\mathbf{X}}\,\mathbf{p}\mathbf{w}\,\theta_{\mathbf{Y}}\,\mathbf{p}\mathbf{v}'\,\theta_{\mathbf{Y}}\,\mathbf{v}$, whence $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$, and we are done. This allows us to suppose below that the identity $\mathbf{u} \approx \mathbf{v}$ is linear-balanced.

Clearly, we may assume that $\mathbf{u} \ne \mathbf{v}$ because the required conclusion is evident otherwise. Let (2.8) and (2.9) be the decompositions of the words $\mathbf{u}$ and $\mathbf{v}$, respectively. The varieties $\mathbf{X}$ and $\mathbf{Y}$ satisfy the identity σ_3. Therefore, we may assume that, for each $i = 0, 1, \dots, m$, $\mathbf{u}_i = \mathbf{p}_i\mathbf{q}_i\mathbf{r}_i$ and $\mathbf{v}_i = \mathbf{p}_i'\mathbf{q}_i'\mathbf{r}_i'$, where $\mathbf{p}_i$ and $\mathbf{p}_i'$ consist of the first occurrences of letters in the words $\mathbf{u}$ and $\mathbf{v}$, respectively; $\mathbf{q}_i$ and $\mathbf{q}_i'$ consist of non-first and non-last occurrences of letters in the words $\mathbf{u}$ and $\mathbf{v}$, respectively; finally, $\mathbf{r}_i$ and $\mathbf{r}_i'$ consist of the last occurrences of letters in the words $\mathbf{u}$ and $\mathbf{v}$, respectively.

Let $i \in \{0, 1, \dots, m\}$. Clearly, $\mathrm{con}(\mathbf{p}_i) = \mathrm{con}(\mathbf{p}_i')$. Suppose that $\mathbf{p}_i \ne \mathbf{p}_i'$. Then there are letters $x, y \in \mathrm{con}(\mathbf{p}_i)$ such that x precedes y in $\mathbf{p}_i$ but y precedes x in $\mathbf{p}_i'$. Let

$$T_{x,y} = \{t_j \mid 1 \le j \le m,\ \mathrm{con}(\mathbf{u}_j) \cap \{x, y\} \ne \varnothing\}.$$

Clearly, the variety $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identity (3.13). Suppose that $\mathbf{a}_0t_{i_1}\mathbf{a}_1t_{i_2}\mathbf{a}_2\cdots t_{i_k}\mathbf{a}_k$ and $\mathbf{b}_0t_{i_1}\mathbf{b}_1t_{i_2}\mathbf{b}_2\cdots t_{i_k}\mathbf{b}_k$ are decompositions of the words $\mathbf{u}(x, y, T_{x,y})$ and $\mathbf{v}(x, y, T_{x,y})$, respectively. Then $\mathrm{con}(\mathbf{a}_i) = \mathrm{con}(\mathbf{b}_i)$ for all $i = 0, 1, \dots, k$. Further, $x, y \in \mathrm{con}(\mathbf{a}_0)$ and $x, y \in \mathrm{con}(\mathbf{b}_0)$ by the choice of letters x and y. We may assume without loss of generality that $\mathbf{a}_0 = xy$. Then $\mathbf{b}_0 = yx$ by the choice of x and y. The variety $\mathbf{Q}_{r,s}$ satisfies the identity $\mathbf{d}_{0,0}[\pi] \approx \mathbf{d}_{0,0}'[\pi]$, that is, the identity $xytxy \approx xytyx$. Thus, if $x, y \in \mathrm{con}(\mathbf{a}_i) = \mathrm{con}(\mathbf{b}_i)$ for some $i > 0$, then we may assume that $\mathbf{a}_i = \mathbf{b}_i = xy$. Therefore, the identity (3.13) coincides with an identity of the form $xyt_1\mathbf{a}_1t_2\mathbf{a}_2\cdots t_k\mathbf{a}_k \approx yxt_1\mathbf{a}_1t_2\mathbf{a}_2\cdots t_k\mathbf{a}_k$, where $\mathbf{a}_i \in \{x, y, xy\}$ for all $i = 1, 2, \dots, k$.

For any $r = 1, 2, 3$, we put $\Phi_r = \{(3.8),\ \alpha_i \mid i = 1, 2, 3,\ i \ne r\}$. The identity (3.8) is nothing but the identity $\mathbf{c}_{0,0}[\pi] \approx \mathbf{c}'_{0,0}[\pi]$. Hence the variety $\mathbf{Q}_{r,s}$ satisfies the identity system Φ_r. Put

$$\Sigma_r = \begin{cases} \{xyt_1xt_2x\cdots t_nxty \approx yxt_1xt_2x\cdots t_nxty \mid n \in \mathbb{N}\} & \text{if } r = 1,\\ \{\sigma_1,\ \alpha_2\} & \text{if } r = 2,\\ \{xytyt_1xt_2x\cdots t_nx \approx yxtyt_1xt_2x\cdots t_nx \mid n \in \mathbb{N}\} & \text{if } r = 3. \end{cases}$$

It is easy to see that, for each $r = 1, 2, 3$, an identity of the form (3.13) either follows from the identity system Φ_r (and therefore, holds in $\mathbf{Q}_{r,s}$) or coincides (up to renaming of letters) with some identity from the identity system Σ_r.

The set of all identities of the form (3.13) implies the identities

$$\mathbf{u} \approx \mathbf{p}'_0\mathbf{q}_0\mathbf{r}_0\Bigl(\prod_{i=1}^{m} t_i\mathbf{p}'_i\mathbf{q}_i\mathbf{r}_i\Bigr) \quad\text{and}\quad \mathbf{v} \approx \mathbf{p}_0\mathbf{q}'_0\mathbf{r}'_0\Bigl(\prod_{i=1}^{m} t_i\mathbf{p}_i\mathbf{q}'_i\mathbf{r}'_i\Bigr). \tag{5.3}$$

Thus, these two identities follow from the identities that hold in the variety $\mathbf{Q}_{r,s}$ and some (possibly empty) subsystem Γ of the system Σ_r. Suppose that $\Gamma \ne \varnothing$. Then the definition of Σ_r implies that the set of all varieties of the form $\operatorname{var}\{\gamma\}$ with $\gamma \in \Gamma$ forms a chain with the least element. Therefore, Γ is equivalent to a single identity $\gamma \in \Gamma$, whence the identities (5.3) hold in the variety $\mathbf{Q}_{r,s}\{\gamma\}$.

The identity γ coincides with some identity of the form (3.13), whence it holds in $\mathbf{X} \wedge \mathbf{Y}$. Now Corollary 3.6 applies and we conclude that the identity γ holds in either $\mathbf{X}$ or $\mathbf{Y}$, say, in $\mathbf{X}$. Then $\mathbf{X} \subseteq \mathbf{Q}_{r,s}\{\gamma\}$, whence $\mathbf{X}$ satisfies (5.3). The same is evidently true whenever $\Gamma = \varnothing$. Thus, $\mathbf{X}$ satisfies (5.3) in either case. Analogous arguments show that one of the varieties $\mathbf{X}$ or $\mathbf{Y}$ satisfies the identity

$$\mathbf{p}'_0\mathbf{q}_0\mathbf{r}_0\Bigl(\prod_{i=1}^{m} t_i\mathbf{p}'_i\mathbf{q}_i\mathbf{r}_i\Bigr) \approx \mathbf{p}'_0\mathbf{q}_0\mathbf{r}'_0\Bigl(\prod_{i=1}^{m} t_i\mathbf{p}'_i\mathbf{q}_i\mathbf{r}'_i\Bigr).$$

Finally, since the variety $\mathbf{Y}$ satisfies the identity σ_3, this variety satisfies also the identity

$$\mathbf{p}'_0\mathbf{q}_0\mathbf{r}'_0\Bigl(\prod_{i=1}^{m} t_i\mathbf{p}'_i\mathbf{q}_i\mathbf{r}'_i\Bigr) \approx \mathbf{v}.$$

Therefore,

$$\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{p}'_0\mathbf{q}_0\mathbf{r}_0\Bigl(\prod_{i=1}^{m} t_i\mathbf{p}'_i\mathbf{q}_i\mathbf{r}_i\Bigr)\,\theta_{\mathbf{Z}}\,\mathbf{p}'_0\mathbf{q}_0\mathbf{r}'_0\Bigl(\prod_{i=1}^{m} t_i\mathbf{p}'_i\mathbf{q}_i\mathbf{r}'_i\Bigr)\,\theta_{\mathbf{Y}}\,\mathbf{v},$$

where $\mathbf{Z} \in \{\mathbf{X}, \mathbf{Y}\}$. Hence $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$, and we are done. □

5.4. **The variety R.** For any $k, \ell \in \mathbb{N}$, we fix the following notation for an identity:

$$\gamma_{k,\ell}:\quad \Bigl(\prod_{i=1}^{k} xt_i\Bigr)xy\Bigl(\prod_{i=k+1}^{k+\ell} t_iy\Bigr) \approx \Bigl(\prod_{i=1}^{k} xt_i\Bigr)yx\Bigl(\prod_{i=k+1}^{k+\ell} t_iy\Bigr).$$

The following evident observation will be very useful.

Lemma 5.11. *Let $\mathbf{p}$ and $\mathbf{q}$ be words, $x \in \operatorname{con}(\mathbf{p})$ and $y \in \operatorname{con}(\mathbf{q})$. If $e \le \operatorname{occ}_x(\mathbf{p})$ and $f \le \operatorname{occ}_y(\mathbf{q})$, then the identity $\gamma_{e,f}$ implies the identity $\mathbf{p}xy\mathbf{q} \approx \mathbf{p}yx\mathbf{q}$.* □

Lemma 5.12. *For any $k, \ell, p, q \in \mathbb{N}$, the inclusion $\mathbf{R}\{\gamma_{k,\ell}\} \subseteq \mathbf{R}\{\gamma_{p,q}\}$ holds if and only if $k \le p$ and $\ell \le q$.*

Proof. *Necessity* follows from the fact that if either $p < k$ or $q < \ell$, then the word

$$\biggl(\prod_{i=1}^{p} xt_i\biggr) xy \biggl(\prod_{i=p+1}^{p+q} t_i y\biggr)$$

is an isoterm for the variety $\mathbf{R}\{\gamma_{k,\ell}\}$.

Sufficiency follows from Lemma 5.11. □

Throughout Lemmas 5.13–5.15 and their proofs, $\mathbf{u} \approx \mathbf{v}$ is a fixed linear-balanced identity, (2.8) *and* (2.9) *are decompositions of the words $\mathbf{u}$ and $\mathbf{v}$, respectively, i is a fixed number with $1 \le i \le m$,*

$$\mathbf{u}' = \prod_{j=0}^{i-1} \mathbf{u}_j,\ \mathbf{u}'' = \prod_{j=i+1}^{m} \mathbf{u}_j,\ \mathbf{v}' = \prod_{j=0}^{i-1} \mathbf{v}_j \ \text{ and }\ \mathbf{v}'' = \prod_{j=i+1}^{m} \mathbf{v}_j. \tag{5.4}$$

Thus,

$$\mathbf{u} = \mathbf{u}'\mathbf{u}_i\mathbf{u}'' \quad \text{and} \quad \mathbf{v} = \mathbf{v}'\mathbf{v}_i\mathbf{v}''. \tag{5.5}$$

Lemma 5.13. *Let $\mathbf{X}$ be a monoid variety and the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{X}$. Suppose that a letter x precedes a letter y in the block $\mathbf{u}_i$ but y precedes x in the block $\mathbf{v}_i$. If $x \in \mathrm{con}(\mathbf{u}') \setminus \mathrm{con}(\mathbf{u}'')$ and $y \in \mathrm{con}(\mathbf{u}'') \setminus \mathrm{con}(\mathbf{u}')$, then $\mathbf{X}$ satisfies the identity $\gamma_{e,f}$, where $e = \mathrm{occ}_x(\mathbf{u}) - 1$ and $f = \mathrm{occ}_y(\mathbf{u}) - 1$.*

Proof. Let s be the least number such that $\mathrm{con}(\mathbf{u}_s) \cap \{x, y\} \ne \varnothing$ and $T_{x,y}$ be the set of letters defined by the equality (3.12). Clearly, the variety $\mathbf{X}$ satisfies the identity (3.13). Since the identity $\mathbf{u} \approx \mathbf{v}$ is linear-balanced, the identity (3.13) coincides (up to renaming of letters) with the identity $\gamma_{e,f}$, where $e = \mathrm{occ}_x(\mathbf{u}) - 1$ and $f = \mathrm{occ}_y(\mathbf{u}) - 1$, and we are done. □

Lemma 5.14. *Let $\mathbf{X}, \mathbf{Y} \in [\mathbf{D}_2, \mathbf{R}]$ and the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{X} \wedge \mathbf{Y}$. Suppose that $\mathbf{v}_i = \mathbf{r}_1\mathbf{r}_2x\mathbf{r}_3$ for some words $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and a letter x. If the letter x precedes each letter from $\mathrm{con}(\mathbf{r}_2)$ in the block $\mathbf{u}_i$, then one of the varieties $\mathbf{X}$ or $\mathbf{Y}$ satisfies the identity $\mathbf{v} \approx \mathbf{v}'\mathbf{r}_1x\mathbf{r}_2\mathbf{r}_3\mathbf{v}''$.*

Proof. If $\mathbf{r}_2 = \lambda$, then the required conclusion is evident. Let now $\mathbf{r}_2 \ne \lambda$. Suppose that $x \in \mathrm{con}(\mathbf{v}') \cap \mathrm{con}(\mathbf{v}'')$. Since letters from $\mathbf{r}_2$ are multiple in $\mathbf{v}$, $\mathrm{con}(\mathbf{r}_2) \subseteq \mathrm{con}(\mathbf{v}'\mathbf{v}'')$. Then the identity $\mathbf{v} \approx \mathbf{v}'\mathbf{r}_1x\mathbf{r}_2\mathbf{r}_3\mathbf{v}''$ follows from σ_1 and σ_2, whence it holds in $\mathbf{X}$. Therefore, we may assume without loss of generality that $x \in \mathrm{con}(\mathbf{v}') \setminus \mathrm{con}(\mathbf{v}'')$. We can write the word $\mathbf{r}_2$ in the form $\mathbf{r}_2 = \mathbf{p}_1\mathbf{q}_1\mathbf{p}_2\mathbf{q}_2 \cdots \mathbf{p}_n\mathbf{q}_n$ for some words $\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2, \dots, \mathbf{p}_n, \mathbf{q}_n$ such that $\mathrm{con}(\mathbf{p}_1\mathbf{p}_2 \cdots \mathbf{p}_n) \subseteq \mathrm{con}(\mathbf{v}')$ and $\mathrm{con}(\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_n) \cap \mathrm{con}(\mathbf{v}') = \varnothing$. Let y be a letter from $\mathrm{con}(\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_n)$ with the least number of occurrences in the word $\mathbf{v}''$ among all letters from $\mathrm{con}(\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_n)$. According to Lemma 5.13, $\mathbf{X} \wedge \mathbf{Y}$ satisfies $\gamma_{e,f}$, where $e = \mathrm{occ}_x(\mathbf{v}) - 1$ and $f = \mathrm{occ}_y(\mathbf{v}) - 1$. In view of Corollary 3.6, the identity $\gamma_{e,f}$ holds in either $\mathbf{X}$ or $\mathbf{Y}$, say, in $\mathbf{X}$. In view of Lemma 5.11, the choice of the letter y allows us, using the identity $\gamma_{e,f}$, to swap the letter x and the letter immediately to the left of x whenever that adjacent letter lies in $\mathrm{con}(\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_n)$. The identity σ_2 allows us to do the same whenever the letter immediately to the left of x lies in $\mathrm{con}(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n)$. Therefore, $\mathbf{X}$ satisfies the identities

$$\begin{aligned}\mathbf{v} &\stackrel{\gamma_{e,f}}{\approx} \mathbf{v}'\mathbf{r}_1\Bigl(\prod_{j=1}^{n-1}\mathbf{p}_j\mathbf{q}_j\Bigr)\mathbf{p}_n x\mathbf{q}_n\mathbf{r}_3\mathbf{v}'' \stackrel{\sigma_2}{\approx} \mathbf{v}'\mathbf{r}_1\Bigl(\prod_{j=1}^{n-1}\mathbf{p}_j\mathbf{q}_j\Bigr)x\mathbf{p}_n\mathbf{q}_n\mathbf{r}_3\mathbf{v}''\\ &\stackrel{\gamma_{e,f}}{\approx}\cdots\stackrel{\gamma_{e,f}}{\approx} \mathbf{v}'\mathbf{r}_1\mathbf{p}_1x\mathbf{q}_1\Bigl(\prod_{j=2}^{n}\mathbf{p}_j\mathbf{q}_j\Bigr)\mathbf{r}_3\mathbf{v}'' \stackrel{\sigma_2}{\approx} \mathbf{v}'\mathbf{r}_1x\mathbf{r}_2\mathbf{r}_3\mathbf{v}''.\end{aligned}$$

This completes the proof. □

Let $\mathbf{X}$ be a monoid variety. A pair of letters (a,b) is called $\mathbf{X}$-*invertible in the block* $\mathbf{u}_i$ *of the word* $\mathbf{u}$ *in the identity* $\mathbf{u}\approx\mathbf{v}$ (or $(\mathbf{X},\mathbf{u},\mathbf{v},i)$-*invertible*, for short) if $\mathbf{u}_i=\mathbf{c}ab\mathbf{d}$ for some words $\mathbf{c}$ and $\mathbf{d}$, the letter b precedes the letter a in the block $\mathbf{v}_i$ and $\mathbf{X}$ satisfies the identity $\mathbf{u}\approx\mathbf{u}'\mathbf{c}ba\mathbf{d}\mathbf{u}''$.

Lemma 5.15. *Let* $\mathbf{X}$ *be a subvariety of the variety* $\mathbf{R}$. *Suppose that, for some letters* x *and* y, *the word* xy *is a subword of* $\mathbf{u}_i$, *while* y *precedes* x *in* $\mathbf{v}_i$. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be linear words with* $\mathrm{con}(\mathbf{a})=\mathrm{con}(\mathbf{b})=\mathrm{con}(\mathbf{u}_i)$. *If* x *precedes* y *in* $\mathbf{a}$ *but* y *precedes* x *in* $\mathbf{b}$ *and the identity* $\mathbf{v}'\mathbf{a}\mathbf{v}''\approx\mathbf{v}'\mathbf{b}\mathbf{v}''$ *holds in* $\mathbf{X}$, *then the pair* (x,y) *is* $(\mathbf{X},\mathbf{u},\mathbf{v},i)$-*invertible.*

Proof. By the hypothesis, $\mathbf{u}_i=\mathbf{p}_1xy\mathbf{p}_2$ for some words $\mathbf{p}_1$ and $\mathbf{p}_2$. If $x,y\in\mathrm{con}(\mathbf{u}')$ [respectively, $x,y\in\mathrm{con}(\mathbf{u}'')$], then the identity $\mathbf{u}\approx\mathbf{u}'\mathbf{p}_1yx\mathbf{p}_2\mathbf{u}''$ follows from σ_2 [respectively, σ_1], whence it holds in $\mathbf{X}$ and, therefore, the pair (x,y) is $(\mathbf{X},\mathbf{u},\mathbf{v},i)$-invertible. Thus, we may assume without loss of generality that $x\in\mathrm{con}(\mathbf{u}')\setminus\mathrm{con}(\mathbf{u}'')$ and $y\in\mathrm{con}(\mathbf{u}'')\setminus\mathrm{con}(\mathbf{u}')$. Then $x\in\mathrm{con}(\mathbf{v}')\setminus\mathrm{con}(\mathbf{v}'')$ and $y\in\mathrm{con}(\mathbf{v}'')\setminus\mathrm{con}(\mathbf{v}')$ because the identity $\mathbf{u}\approx\mathbf{v}$ is linear-balanced. Since the identity $\mathbf{v}'\mathbf{a}\mathbf{v}''\approx\mathbf{v}'\mathbf{b}\mathbf{v}''$ is linear-balanced and holds in $\mathbf{X}$, Lemma 5.13 implies that $\mathbf{X}$ satisfies the identity $\gamma_{e,f}$, where $e=\mathrm{occ}_x(\mathbf{u})-1$ and $f=\mathrm{occ}_y(\mathbf{u})-1$. Then $\mathbf{X}$ satisfies $\mathbf{u}\stackrel{\gamma_{e,f}}{\approx}\mathbf{u}'\mathbf{p}_1yx\mathbf{p}_2\mathbf{u}''$ by Lemma 5.11, whence the pair (x,y) is $(\mathbf{X},\mathbf{u},\mathbf{v},i)$-invertible. □

Proposition 5.16. *The variety* $\mathbf{R}$ *is* fi-*permutable.*

Proof. Let $\mathbf{X},\mathbf{Y}\subseteq\mathbf{R}$ and an identity $\mathbf{u}\approx\mathbf{v}$ holds in $\mathbf{X}\wedge\mathbf{Y}$. We need to check that $(\mathbf{u},\mathbf{v})\in\theta_{\mathbf{X}}\theta_{\mathbf{Y}}$.

Since $\mathbf{R}\subseteq\mathbf{A}^*$, we can repeat literally arguments from the second paragraph of the proof of Proposition 5.10 and reduce our considerations to the case when $\mathbf{X},\mathbf{Y}\in[\mathbf{D}_2,\mathbf{R}]$ and the identity $\mathbf{u}\approx\mathbf{v}$ is linear-balanced.

Let (2.8) be the decomposition of the word $\mathbf{u}$. Since the identity $\mathbf{u}\approx\mathbf{v}$ is linear-balanced, the decomposition of the word $\mathbf{v}$ has the form (2.9) and the identity $\mathbf{u}\approx\mathbf{v}$ is n-invertible for some $n\in\mathbb{N}_0$ (the notion of n-invertible identity was introduced in Subsection 3.5). We will use induction on n.

Induction base. If $n=0$, then $\mathbf{u}=\mathbf{v}$, and we are done.

Induction step. Let now $n>0$. Since $\mathbf{u}\ne\mathbf{v}$, there is $i\in\{0,1,\dots,m\}$ with $\mathbf{u}_i\ne\mathbf{v}_i$. Let us fix a number i with such a property. Let $\mathbf{u}'$, $\mathbf{u}''$, $\mathbf{v}'$ and $\mathbf{v}''$ be words defined by the equalities (5.4). Thus, the equalities (5.5) are valid. Suppose that the block $\mathbf{u}_i$ of the word $\mathbf{u}$ contains an $(\mathbf{X},\mathbf{u},\mathbf{v},i)$-invertible pair of letters (a,b). This means that $\mathbf{u}_i=\mathbf{c}ab\mathbf{d}$ for some words $\mathbf{c}$ and $\mathbf{d}$, the identity $\mathbf{u}\approx\mathbf{u}'\mathbf{c}ba\mathbf{d}\mathbf{u}''$ holds in $\mathbf{X}$ and the letter b precedes the letter a in the block $\mathbf{v}_i$. The identity $\mathbf{u}'\mathbf{c}ba\mathbf{d}\mathbf{u}''\approx\mathbf{v}$ is $(n-1)$-invertible and holds in $\mathbf{X}\wedge\mathbf{Y}$. By the induction assumption,

$(\mathbf{u}'cba\mathbf{d}\mathbf{u}'', \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. This implies that $\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{u}'cba\mathbf{d}\mathbf{u}''\,\theta_{\mathbf{X}}\,\mathbf{w}\,\theta_{\mathbf{Y}}\,\mathbf{v}$ for some word $\mathbf{w}$, whence $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$.

Thus, we may assume that

(a) the block $\mathbf{u}_i$ does not contain $(\mathbf{X}, \mathbf{u}, \mathbf{v}, i)$-invertible pairs of letters.

Analogous arguments allows us to assume that

(b) the block $\mathbf{v}_i$ does not contain $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, i)$-invertible pairs of letters.

In order to facilitate understanding of further considerations, we outline their general scheme. We want to find the words $\mathbf{u}^\sharp$ and $\mathbf{v}^\sharp$ such that the identity $\mathbf{u}^\sharp \approx \mathbf{v}^\sharp$ is t-invertible for some $t < n$ and holds in the variety $\mathbf{X} \wedge \mathbf{Y}$, while the varieties $\mathbf{X}$ and $\mathbf{Y}$ satisfy the identities $\mathbf{u} \approx \mathbf{u}^\sharp$ and $\mathbf{v} \approx \mathbf{v}^\sharp$, respectively. This is sufficient for our aims. Indeed, if such words $\mathbf{u}^\sharp$ and $\mathbf{v}^\sharp$ exist, then $(\mathbf{u}^\sharp, \mathbf{v}^\sharp) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$ by the induction assumption. Then $\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{u}^\sharp\,\theta_{\mathbf{X}}\,\mathbf{w}\,\theta_{\mathbf{Y}}\,\mathbf{v}^\sharp\,\theta_{\mathbf{Y}}\,\mathbf{v}$ for some word $\mathbf{w}$, whence $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}$, and we are done.

To understand, how to construct the sought words $\mathbf{u}^\sharp$ and $\mathbf{v}^\sharp$ from the words $\mathbf{u}$ and $\mathbf{v}$, we need to clarify the structure of the words $\mathbf{u}$ and $\mathbf{v}$ whenever the claims (a) and (b) hold. As we will see, these two claims impose rather strict restrictions on the words $\mathbf{u}$ and $\mathbf{v}$. After we find that the claims (a) and (b) make many cases impossible, we will eventually come to the conclusion that the words $\mathbf{u}$ and $\mathbf{v}$ must have some well-defined form, which will allows us to define the words $\mathbf{u}^\sharp$ and $\mathbf{v}^\sharp$ with the desired properties.

Let $\mathbf{q}$ be the largest common prefix of the words $\mathbf{u}_i$ and $\mathbf{v}_i$. Then $\mathbf{u}_i = \mathbf{q}x_0\mathbf{u}_i'y_0\mathbf{u}_i''$ and $\mathbf{v}_i = \mathbf{q}y_0\mathbf{v}_i'x_0\mathbf{v}_i''$ for some different letters x_0, y_0 and some words $\mathbf{u}_i', \mathbf{u}_i'', \mathbf{v}_i'$ and $\mathbf{v}_i''$. Lemma 5.14 implies that one of the varieties $\mathbf{X}$ and $\mathbf{Y}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{u}'\mathbf{q}y_0x_0\mathbf{u}_i'\mathbf{u}_i''\mathbf{u}''. \tag{5.6}$$

Suppose that this identity holds in $\mathbf{X}$. Let $z = x_0$ whenever $\mathbf{u}_i' = \lambda$ and z be the last letter of the word $\mathbf{u}_i'$ otherwise. Then the pair (y_0, z) is $(\mathbf{X}, \mathbf{u}, \mathbf{v}, i)$-invertible. This is impossible by the claim (a). Therefore, the identity (5.6) holds in $\mathbf{Y}$. If $\mathbf{v}_i' = \lambda$, then Lemma 5.15 implies that the pair (x_0, y_0) is $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, i)$-invertible, contradicting with the claim (b). Therefore, $\mathbf{v}_i' \ne \lambda$. Analogous arguments show that $\mathbf{u}_i' \ne \lambda$. Let $\mathbf{u}_i' = x_1x_2\cdots x_p$ and $\mathbf{v}_i' = y_1y_2\cdots y_q$. Thus,

$$\mathbf{u} = \mathbf{u}'\mathbf{q}x_0x_1\cdots x_py_0\mathbf{u}_i''\mathbf{u}'' \quad \text{and} \quad \mathbf{v} = \mathbf{v}'\mathbf{q}y_0y_1\cdots y_qx_0\mathbf{v}_i''\mathbf{v}''.$$

Put $X_j = \{x_1, x_2, \dots, x_j\}$ for all $j = 1, 2, \dots, p$ and $Y_j = \{y_1, y_2, \dots, y_j\}$ for all $j = 1, 2, \dots, q$. We are going to verify that either y_j precedes y_{j+1} in $\mathbf{u}_i$ for any $j = 0, 1, \dots, q-1$ or x_j precedes x_{j+1} in $\mathbf{v}_i$ for any $j = 0, 1, \dots, p-1$. Suppose that this is not the case. Then there are $r \in \{0, 1, \dots, p-1\}$ and $s \in \{0, 1, \dots, q-1\}$ such that x_{r+1} precedes x_r in $\mathbf{v}_i$ and y_{s+1} precedes y_s in $\mathbf{u}_i$. We may assume without any loss that r and s are the least numbers with such properties. It follows that $X_r \subseteq \mathrm{con}(\mathbf{v}_i'')$ and $Y_s \subseteq \mathrm{con}(\mathbf{u}_i'')$. Then we apply Lemma 5.14 $s + r + 2$ times and obtain that the variety $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identities

$$\begin{aligned}
\mathbf{u} &\approx \mathbf{u}'\mathbf{q}y_0x_0\mathbf{u}_i'\mathbf{u}_i''\mathbf{u}'' \approx \mathbf{u}'\mathbf{q}y_0y_1x_0\mathbf{u}_i'(\mathbf{u}_i'')_{Y_1}\mathbf{u}'' \approx \mathbf{u}'\mathbf{q}y_0y_1y_2x_0\mathbf{u}_i'(\mathbf{u}_i'')_{Y_2}\mathbf{u}''\\
&\approx \cdots \approx \mathbf{u}'\mathbf{q}y_0y_1y_2\cdots y_sx_0\mathbf{u}_i'(\mathbf{u}_i'')_{Y_s}\mathbf{u}'' \quad \text{and}\\
\mathbf{v} &\approx \mathbf{v}'\mathbf{q}x_0y_0\mathbf{v}_i'\mathbf{v}_i''\mathbf{v}'' \approx \mathbf{v}'\mathbf{q}x_0x_1y_0\mathbf{v}_i'(\mathbf{v}_i'')_{X_1}\mathbf{v}'' \approx \mathbf{v}'\mathbf{q}x_0x_1x_2y_0\mathbf{v}_i'(\mathbf{v}_i'')_{X_2}\mathbf{v}''\\
&\approx \cdots \approx \mathbf{v}'\mathbf{q}x_0x_1x_2\cdots x_ry_0\mathbf{v}_i'(\mathbf{v}_i'')_{X_r}\mathbf{v}''.
\end{aligned}$$

In particular, the variety $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identities

$$\mathbf{u}'\mathbf{q}y_0y_1\cdots y_{s-1}x_0\mathbf{u}_i'(\mathbf{u}_i'')_{Y_{s-1}}\mathbf{u}'' \approx \mathbf{u}'\mathbf{q}y_0y_1\cdots y_sx_0\mathbf{u}_i'(\mathbf{u}_i'')_{Y_s}\mathbf{u}'', \tag{5.7}$$

$$\mathbf{v}'\mathbf{q}x_0x_1\cdots x_{r-1}y_0\mathbf{v}_i'(\mathbf{v}_i'')_{X_{r-1}}\mathbf{v}'' \approx \mathbf{v}'\mathbf{q}x_0x_1\cdots x_ry_0\mathbf{v}_i'(\mathbf{v}_i'')_{X_r}\mathbf{v}''. \tag{5.8}$$

If the identity (5.7) holds in the variety $\mathbf{Y}$, then the pair (y_s, y_{s+1}) is $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, i)$-invertible by Lemma 5.15, contradicting with the claim (b). Therefore, the identity (5.7) fails in $\mathbf{Y}$ and so holds in $\mathbf{X}$ by Lemma 5.14. Analogously, the identity (5.8) holds in $\mathbf{Y}$ but fails in $\mathbf{X}$.

Further, if $y_s, y_{s+1} \in \mathrm{con}(\mathbf{v}')$ or $y_s, y_{s+1} \in \mathrm{con}(\mathbf{v}'')$, then, since $\mathbf{Y}$ satisfies σ_1 and σ_2, the pair (y_s, y_{s+1}) is $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, i)$-invertible, contradicting with the claim (b). Therefore, we may assume without loss of generality that $y_s \in \mathrm{con}(\mathbf{v}') \setminus \mathrm{con}(\mathbf{v}'')$ and $y_{s+1} \in \mathrm{con}(\mathbf{v}'') \setminus \mathrm{con}(\mathbf{v}')$.

In view of Lemma 5.13, $\mathbf{X}$ satisfies the identity $\gamma_{f_s,f_{s+1}}$. If this identity holds in the variety $\mathbf{Y}$, then, by Lemma 5.11, the pair (y_s, y_{s+1}) is $(\mathbf{Y}, \mathbf{v}, \mathbf{u}, i)$-invertible contradicting with the claim (b). Therefore, the identity $\gamma_{f_s,f_{s+1}}$ fails in $\mathbf{Y}$.

Analogously, we can verify that one of the following two claims holds:

(c) $x_r \in \mathrm{con}(\mathbf{v}') \setminus \mathrm{con}(\mathbf{v}'')$, $x_{r+1} \in \mathrm{con}(\mathbf{v}'') \setminus \mathrm{con}(\mathbf{v}')$ and the identity $\gamma_{e_r,e_{r+1}}$ fails in $\mathbf{X}$ and holds in $\mathbf{Y}$;

(d) $x_{r+1} \in \mathrm{con}(\mathbf{v}') \setminus \mathrm{con}(\mathbf{v}'')$, $x_r \in \mathrm{con}(\mathbf{v}'') \setminus \mathrm{con}(\mathbf{v}')$ and the identity γ_{e_{r+1},e_r} fails in $\mathbf{X}$ and holds in $\mathbf{Y}$.

First, suppose that the claim (c) is true. Since $\mathbf{X}$ satisfies the identity (5.7), the identity $\gamma_{f_s,e_{r+1}}$ holds in $\mathbf{X}$ by Lemma 5.13. Then Lemma 5.12 and the fact that $\mathbf{X}$ violates $\gamma_{e_r,e_{r+1}}$ imply that $e_r < f_s$. Since $\mathbf{Y}$ satisfies the identity (5.8), the identity $\gamma_{e_r,f_{s+1}}$ holds in $\mathbf{Y}$ by Lemma 5.13. Then Lemma 5.12 and the fact that $\mathbf{Y}$ violates $\gamma_{f_s,f_{s+1}}$ imply that $f_s < e_r$, a contradiction. So, the claim (c) is false.

Suppose now that the claim (d) is true. Since $\mathbf{X}$ satisfies the identity (5.7), the identity γ_{f_s,e_r} holds in $\mathbf{X}$ by Lemma 5.13. Then Lemma 5.12 and the fact that $\mathbf{X}$ violates γ_{e_{r+1},e_r} imply that $e_{r+1} < f_s$. Since $\mathbf{Y}$ satisfies the identity (5.8), the identity γ_{f_s,e_r} holds in $\mathbf{Y}$ by Lemma 5.13. Then Lemma 5.12 and the fact that $\mathbf{Y}$ violates $\gamma_{f_s,f_{s+1}}$ imply that $f_{s+1} < e_r$.

Further, if $x_{r+1} \in \mathrm{con}(\mathbf{v}_i')$, then $x_{r+1} = y_c$ for some $c \in \{1, 2, \dots, q\}$. This means that $y_c \in \mathrm{con}(\mathbf{u}_i')$. Then $s + 1 < c$ because $Y_s \subseteq \mathrm{con}(\mathbf{u}_i'')$ and $x_{r+1} \neq y_{s+1}$. It follows that y_{s+1} precedes x_{r+1} in $\mathbf{v}_i$. If $x_{r+1} \notin \mathrm{con}(\mathbf{v}_i')$, then, evidently, y_{s+1} precedes x_{r+1} in $\mathbf{v}_i$. So, y_{s+1} precedes x_{r+1} in $\mathbf{v}_i$ in either case. Analogously, we can verify that x_{r+1} precedes y_{s+1} in $\mathbf{u}_i$. Then $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identity $\gamma_{e_{r+1},f_{s+1}}$ by Lemma 5.13. According to Corollary 3.6, either $\mathbf{X}$ or $\mathbf{Y}$ satisfies this identity. If $\gamma_{e_{r+1},f_{s+1}}$ holds in $\mathbf{X}$, then γ_{e_{r+1},e_r} holds in $\mathbf{X}$ too by Lemma 5.12, a contradiction. If $\gamma_{e_{r+1},f_{s+1}}$ holds in $\mathbf{Y}$, then $\gamma_{f_s,f_{s+1}}$ holds in $\mathbf{Y}$ too by Lemma 5.12, a contradiction again. So, the claim (d) is false too.

Thus, the hypothesis that there are $r \in \{0, 1, \dots, p-1\}$ and $s \in \{0, 1, \dots, q-1\}$ such that x_{r+1} precedes x_r in $\mathbf{v}_i$ and y_{s+1} precedes y_s in $\mathbf{u}_i$ is false in either case. Then either y_j precedes y_{j+1} in $\mathbf{u}_i$ for any $j = 0, 1, \dots, q-1$ or x_j precedes x_{j+1} in $\mathbf{v}_i$ for any $j = 0, 1, \dots, p-1$. By symmetry, we may assume that y_j precedes y_{j+1} in $\mathbf{u}_i$ for any $j = 0, 1, \dots, q-1$. In particular, $X_p \cap Y_q = \varnothing$. It follows that $X_p \subseteq \mathrm{con}(\mathbf{v}_i'')$ and $Y_q \subseteq \mathrm{con}(\mathbf{u}_i'')$. Considerations similar to the deduction of the identities (5.7) and (5.8) allow us to conclude that the variety $\mathbf{X} \wedge \mathbf{Y}$ satisfies the identities

$$\mathbf{u}'\mathbf{q}y_0y_1\cdots y_{q-1}x_0\mathbf{u}_i'(\mathbf{u}_i'')_{Y_{q-1}}\mathbf{u}''\approx\mathbf{u}'\mathbf{q}y_0y_1\cdots y_qx_0\mathbf{u}_i'(\mathbf{u}_i'')_{Y_q}\mathbf{u}'',\tag{5.9}$$

$$\mathbf{v}'\mathbf{q}x_0x_1\cdots x_{p-1}y_0\mathbf{v}_i'(\mathbf{v}_i'')_{X_{p-1}}\mathbf{v}''\approx\mathbf{v}'\mathbf{q}x_0x_1\cdots x_py_0\mathbf{v}_i'(\mathbf{v}_i'')_{X_p}\mathbf{v}''.\tag{5.10}$$

If the identity (5.9) holds in the variety $\mathbf{Y}$, then the pair (y_{q-1},y_q) is $(\mathbf{Y},\mathbf{v},\mathbf{u},i)$-invertible by Lemma 5.15, contradicting with the claim (b). Therefore, the identity (5.9) fails in $\mathbf{Y}$ and so holds in $\mathbf{X}$ by Lemma 5.14. Analogously, the identity (5.10) holds in $\mathbf{Y}$ but fails in $\mathbf{X}$.

If either $x_p,y_0\in\mathrm{con}(\mathbf{u}')$ or $x_p,y_0\in\mathrm{con}(\mathbf{u}'')$, then, since $\mathbf{X}$ satisfies σ_1 and σ_2, the pair (y_0,x_p) is $(\mathbf{X},\mathbf{u},\mathbf{v},i)$-invertible. This contradicts the claim (a). Therefore, we may assume without loss of generality that $y_0\in\mathrm{con}(\mathbf{u}')\setminus\mathrm{con}(\mathbf{u}'')$ and $x_p\in\mathrm{con}(\mathbf{u}'')\setminus\mathrm{con}(\mathbf{u}')$.

Analogously, either $y_q\in\mathrm{con}(\mathbf{v}')\setminus\mathrm{con}(\mathbf{v}'')$ and $x_0\in\mathrm{con}(\mathbf{v}'')\setminus\mathrm{con}(\mathbf{v}')$ or $x_0\in\mathrm{con}(\mathbf{v}')\setminus\mathrm{con}(\mathbf{v}'')$ and $y_q\in\mathrm{con}(\mathbf{v}'')\setminus\mathrm{con}(\mathbf{v}')$. Further considerations are divided into two cases.

Case 1: $y_q\in\mathrm{con}(\mathbf{v}')\setminus\mathrm{con}(\mathbf{v}'')$ and $x_0\in\mathrm{con}(\mathbf{v}'')\setminus\mathrm{con}(\mathbf{v}')$. If the identity γ_{f_0,e_p} holds in $\mathbf{X}$, then Lemma 5.11 implies that the pair (x_p,y_0) is $(\mathbf{X},\mathbf{u},\mathbf{v},i)$-invertible, contradicting with the claim (a). Therefore, γ_{f_0,e_p} fails in $\mathbf{X}$. But, since the identity (5.9) holds in the variety $\mathbf{X}$, this variety satisfies the identity γ_{f_q,e_p} by Lemma 5.13. This fact and Lemma 5.12 imply that $f_0<f_q$. Further, since $\mathbf{Y}$ satisfies the identity (5.6), $\mathbf{Y}$ satisfies also the identity γ_{f_0,e_0} by Lemma 5.13. Now Lemma 5.12 applies with the conclusion that the identity γ_{f_q,e_0} holds in $\mathbf{Y}$. According to Lemma 5.11, the pair (y_q,x_0) is $(\mathbf{Y},\mathbf{v},\mathbf{u},i)$-invertible. But this is not the case by the claim (b). So, Case 1 is impossible.

Case 2: $x_0\in\mathrm{con}(\mathbf{v}')\setminus\mathrm{con}(\mathbf{v}'')$ and $y_q\in\mathrm{con}(\mathbf{v}'')\setminus\mathrm{con}(\mathbf{v}')$. Further considerations are divided into three subcases.

Subcase 2.1: there are $k\in\{0,1,\dots,p\}$ and $\ell\in\{0,1,\dots,q\}$ such that $x_k,y_\ell\in\mathrm{con}(\mathbf{v}')\setminus\mathrm{con}(\mathbf{v}'')$ and the identities γ_{f_ℓ,f_q} and γ_{e_k,e_p} fail in $\mathbf{X}$ and $\mathbf{Y}$, respectively. Since the identities (5.9) and (5.10) hold in the varieties $\mathbf{X}$ and $\mathbf{Y}$, respectively, Lemma 5.13 implies that $\mathbf{X}$ satisfies γ_{e_k,f_q} and $\mathbf{Y}$ satisfies γ_{f_ℓ,e_p}. Then $f_\ell<e_k<f_\ell$ by Lemma 5.12, a contradiction. So, this subcase is impossible.

Subcase 2.2: $\mathbf{X}$ satisfies the identity γ_{f_j,f_q} for any j such that $y_j\in\mathrm{con}(\mathbf{v}')\setminus\mathrm{con}(\mathbf{v}'')$. Clearly, $\mathbf{u}_i''=\mathbf{a}_iy_q\mathbf{b}_i$ for some words $\mathbf{a}_i$ and $\mathbf{b}_i$ and $y_1,y_2,\dots,y_{q-1}\in\mathrm{con}(\mathbf{a}_i)$. Let $\mathbf{w}=x_0\mathbf{u}_i'y_0\mathbf{a}_i$. Then $\mathbf{u}=\mathbf{u}'\mathbf{q}\mathbf{w}y_q\mathbf{b}_i\mathbf{u}''$. For convenience, we rename letters from $\mathrm{con}(\mathbf{w})$ and put $\mathbf{w}=z_1z_2\cdots z_a$. We are going to check that the variety $\mathbf{X}$ satisfies the identities

$$\begin{cases}\mathbf{u}=\mathbf{u}'\mathbf{q}z_1z_2\cdots z_ay_q\mathbf{b}_i\mathbf{u}''\approx\mathbf{u}'\mathbf{q}z_1z_2\cdots z_{a-1}y_qz_a\mathbf{b}_i\mathbf{u}''\\ \quad\approx\mathbf{u}'\mathbf{q}z_1z_2\cdots z_{a-2}y_qz_{a-1}z_a\mathbf{b}_i\mathbf{u}''\approx\cdots\approx\mathbf{u}'\mathbf{q}y_qz_1z_2\cdots z_a\mathbf{b}_i\mathbf{u}''\\ \quad=\mathbf{u}'\mathbf{q}y_qx_0\mathbf{u}_i'y_0\mathbf{a}_i\mathbf{b}_i\mathbf{u}''=\mathbf{u}'\mathbf{q}y_qx_0\mathbf{u}_i'y_0(\mathbf{u}_i'')_{y_q}\mathbf{u}''.\end{cases}\tag{5.11}$$

Let $r\in\{1,2,\dots,a\}$. If $z_r\in\mathrm{con}(\mathbf{v}'')$, then we can swap y_q with z_r using the identity σ_2. Let now $z_r\notin\mathrm{con}(\mathbf{v}'')$ and therefore, $z_r\in\mathrm{con}(\mathbf{v}')$. If $z_r=y_j$ for some $j\in\{1,2,\dots,q-1\}$, then we can swap y_q with z_r using the identity γ_{f_j,f_q} that holds in $\mathbf{X}$ by the hypothesis of Subcase 2.2. Finally, suppose that $z_r\notin\{y_1,y_2,\dots,y_{q-1}\}$. Since $\mathbf{X}$ satisfies the identity (5.9), we can apply Lemma 5.13 and conclude that $\mathbf{X}$ satisfies the identity γ_{h,f_q}, where $h=\mathrm{occ}_{z_r}(\mathbf{u})-1$. Then we can swap y_q with z_r by Lemma 5.11. Thus, we really can step by step swap y_q with z_a, z_{a-1}, $\dots$,

z_1 using identities that hold in $\mathbf{X}$ on each step. This implies that $\mathbf{X}$ satisfies the identities (5.11) and in particular, the identity $\mathbf{u} \approx \mathbf{u}^\sharp$, where

$$\mathbf{u}^\sharp = \mathbf{u}'\mathbf{q}y_q x_0 \mathbf{u}_i' y_0 (\mathbf{u}_i'')_{y_q} \mathbf{u}''.$$

Further, $\mathbf{X}$ violates γ_{f_0,e_p} because the pair (y_0, x_p) is $(\mathbf{X}, \mathbf{u}, \mathbf{v}, i)$-invertible by Lemma 5.11 otherwise. Recall that $y_0 \in \operatorname{con}(\mathbf{u}') \setminus \operatorname{con}(\mathbf{u}'')$. Since the identity $\mathbf{u} \approx \mathbf{v}$ is linear-balanced, $y_0 \in \operatorname{con}(\mathbf{v}') \setminus \operatorname{con}(\mathbf{v}'')$. Then $\mathbf{X}$ satisfies γ_{f_0,f_q} by the hypothesis, whence Lemma 5.12 implies that $e_p < f_q$. Since the identity (5.10) holds in the variety $\mathbf{Y}$, this variety satisfies γ_{f_j,e_p} for any j such that $y_j \in \operatorname{con}(\mathbf{v}') \setminus \operatorname{con}(\mathbf{v}'')$ by Lemma 5.13. Taking into account Lemma 5.12, we get that γ_{f_j,f_q} holds in $\mathbf{Y}$ for any j such that $y_j \in \operatorname{con}(\mathbf{v}') \setminus \operatorname{con}(\mathbf{v}'')$. Then arguments similar to ones from the previous paragraph imply that $\mathbf{Y}$ satisfies the identity $\mathbf{v} \approx \mathbf{v}^\sharp$, where

$$\mathbf{v}^\sharp = \mathbf{v}'\mathbf{q}y_q y_0 (\mathbf{v}_i')_{y_q} x_0 \mathbf{v}_i'' \mathbf{v}''.$$

To obtain $\mathbf{v}$ from $\mathbf{u}$ by swapping of adjacent occurrences of multiple letters, we need to transform the word $x_0\mathbf{u}_i'y_0\mathbf{u}_i''$ to the word $y_0\mathbf{v}_i'x_0\mathbf{v}_i''$. By the hypothesis, this can be done in n steps. Further, to obtain $\mathbf{v}^\sharp$ from $\mathbf{u}^\sharp$ by the same way, we need to transform the word $x_0\mathbf{u}_i'y_0(\mathbf{u}_i'')_{y_q}$ to the word $y_0(\mathbf{v}_i')_{y_q}x_0\mathbf{v}_i''$. We see that in the second case we need to replace one letter less than in the first one and the mutual location of all letters in the word $x_0\mathbf{u}_i'y_0(\mathbf{u}_i'')_{y_q}$ [respectively, $y_0(\mathbf{v}_i')_{y_q}x_0\mathbf{v}_i''$] coincides with the mutual location of all letters except y_q in the word $x_0\mathbf{u}_i'y_0\mathbf{u}_i''$ [respectively, $y_0\mathbf{v}_i'x_0\mathbf{v}_i''$]. Therefore, the identity $\mathbf{u}^\sharp \approx \mathbf{v}^\sharp$ is t-invertible for some $t < n$. It is clear that this identity holds in $\mathbf{X} \wedge \mathbf{Y}$. Thus, the words $\mathbf{u}^\sharp$ and $\mathbf{v}^\sharp$ have the properties indicated in the paragraph after the claim (b). As we have seen there, this implies the desired conclusion.

Subcase 2.3: $\mathbf{Y}$ satisfies the identity γ_{e_j,e_p} for any j such that $x_j \in \operatorname{con}(\mathbf{v}') \setminus \operatorname{con}(\mathbf{v}'')$. This subcase is similar to Subcase 2.2.

Proposition 5.16 is proved. □

6. Proof of main results

Proof of Theorem 1.1. Throughout the proof, we will use Lemma 2.1 many times without explicitly specifying this.

Necessity. Let $\mathbf{V}$ be a non-group fi-permutable variety of monoids. Then $\mathbf{SL} \subseteq \mathbf{V}$ by Lemma 2.3. Now Lemma 4.1(i) applies and we conclude that $\mathbf{A}_n \nsubseteq \mathbf{V}$ for any $n > 1$. In other words, the variety $\mathbf{V}$ is aperiodic, whence it satisfies the identity (2.3) for some $n \in \mathbb{N}$. Let n be the least number with such a property.

If $n = 1$, then the variety $\mathbf{V}$ is completely regular. But every completely regular aperiodic variety consists of idempotent monoids, and we are done.

Suppose now that $n > 2$. Then Lemma 2.8 implies that $\mathbf{C}_3 \subseteq \mathbf{V}$. It is verified by Gusev [7, Lemma 5] that the lattice $L(\mathbf{C}_3 \vee \mathbf{E})$ is non-modular. Clearly, the lattice $L(\mathbf{C}_3 \vee \mathbf{E}^\delta)$ is non-modular too. Now Lemma 2.2 applies and we conclude that $\mathbf{E}, \mathbf{E}^\delta \nsubseteq \mathbf{V}$. Then Lemma 2.13 and the statement dual to it imply that $\mathbf{V}$ satisfies the identities (2.5) and $x^nyx^n \approx x^ny$. Then the identity $x^ny \approx yx^n$ holds in $\mathbf{V}$. Further, $\mathbf{D}_2 \nsubseteq \mathbf{V}$ by Lemma 4.1(iii). Then Lemmas 2.8 and 2.12 imply that one of the identities δ_1 or $yx^2 \approx xyx$ is true in $\mathbf{V}$. This means that either $\mathbf{V} \subseteq \mathbf{P}_n$ or $\mathbf{V} \subseteq \mathbf{P}_n^\delta$, and we are done.

Finally, suppose that $n = 2$. Then Lemma 2.8 implies that $\mathbf{C}_2 \subseteq \mathbf{V}$. Suppose that $\mathbf{E} \subseteq \mathbf{V}$. Then Lemma 4.1(v) implies that $\mathbf{E}^\delta \nsubseteq \mathbf{V}$. Now the statement dual to

Lemma 2.13 shows that $\mathbf{V}$ satisfies the identity

$$x^2yx^2 \approx x^2y. \tag{6.1}$$

Put $\mathbf{LRB} = \mathrm{var}\{xy \approx xyx\}$. By Lee [16, Proposition 4.1(i) and Lemma 3.3(iv)], the lattice $L(\mathbf{C}_2 \vee \mathbf{LRB})$ is non-modular. Then Lemma 2.2 implies that $\mathbf{LRB} \nsubseteq \mathbf{V}$. Hence there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{V}$ but fails in $\mathbf{LRB}$. For any word $\mathbf{w}$, we denote by $\mathrm{ini}(\mathbf{w})$ the word obtained from $\mathbf{w}$ by retaining only the first occurrence of each letter. It is evident that an identity $\mathbf{a} \approx \mathbf{b}$ holds in the variety $\mathbf{LRB}$ if and only if $\mathrm{ini}(\mathbf{a}) = \mathrm{ini}(\mathbf{b})$. Thus, $\mathrm{ini}(\mathbf{u}) \neq \mathrm{ini}(\mathbf{v})$. Lemma 2.3 implies that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. Therefore, we may assume that there are letters $x, y \in \mathrm{con}(\mathbf{u})$ such that $\mathbf{u}(x, y) = x^sy\mathbf{w}_1$ and $\mathbf{v}(x, y) = y^tx\mathbf{w}_2$, where $s, t > 0$ and $\mathrm{con}(\mathbf{w}_1) = \mathrm{con}(\mathbf{w}_2) = \{x, y\}$. Let us substitute x^2 and y^2 for x and y, respectively, in the identity $\mathbf{u}(x, y) \approx \mathbf{v}(x, y)$. After that we apply the identities (3.1) and (6.1), resulting in the identity $x^2y^2 \approx y^2x^2$.

Further, $\mathbf{D}_2 \nsubseteq \mathbf{V}$ by Lemma 4.1(iv). Then Lemma 2.12 implies that $\mathbf{V}$ satisfies the identity

$$xyx \approx x^qyx^r, \tag{6.2}$$

where either $q > 1$ or $r > 1$. If $q > 1$, then $\mathbf{V}$ satisfies the identities

$$xyx \stackrel{(6.2)}{\approx} x^qyx^r \stackrel{(6.1)}{\approx} x^qyx^{r+2} \stackrel{(3.1)}{\approx} x^2yx^2 \stackrel{(6.1)}{\approx} x^2y,$$

whence $\mathbf{V} \subseteq \mathbf{E} \subset \mathbf{K}$. Let now $q \le 1$. Then $r > 1$. If $q = 0$, then the claim that $\mathbf{V}$ satisfies the identity (3.1) implies that the identity $xyx \approx yx^2$ holds in $\mathbf{V}$. Besides that, $\mathbf{V}$ satisfies the identity (6.1). Now Lemma 2.11 implies that $\mathbf{V} \subseteq \mathbf{D}_1 \subset \mathbf{K}$. Finally, let $q = 1$. Since the variety $\mathbf{V}$ satisfies the identity (3.1), it satisfies also the identity

$$xyx \approx xyx^2. \tag{6.3}$$

Besides that, the identities $x^2yx \stackrel{(6.3)}{\approx} x^2yx^2 \stackrel{(6.1)}{\approx} x^2y$ hold in $\mathbf{V}$. Therefore, $\mathbf{V} \subseteq \mathbf{K}$.

Thus, if $\mathbf{E} \subseteq \mathbf{V}$, then $\mathbf{V} \subseteq \mathbf{K}$. By symmetry, if $\mathbf{E}^\delta \subseteq \mathbf{V}$, then $\mathbf{V} \subseteq \mathbf{K}^\delta$. We are done in both the cases.

Below we assume that $\mathbf{E}, \mathbf{E}^\delta \nsubseteq \mathbf{V}$. Then Lemma 2.13 and the dual statement imply that $\mathbf{V} \subseteq \mathbf{A}$. In view of Lemmas 2.5 and 3.3(i), we may assume that $\mathbf{D}_2 \subseteq \mathbf{V}$. If $\mathbf{V}$ does not contain the varieties $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{M}^\delta$, then Lemmas 2.11 and 2.15 imply that $\mathbf{V} \subseteq \mathbf{D}_\infty \subset \mathbf{D}_\infty \vee \mathbf{N}$, and we are done. In view of symmetry, we may assume that either $\mathbf{M} \subseteq \mathbf{V}$ or $\mathbf{L} \subseteq \mathbf{V}$.

Suppose at first that $\mathbf{M} \subseteq \mathbf{V}$. Then it follows from Lemma 4.1(vi) that $\mathbf{L} \nsubseteq \mathbf{V}$. Then Lemma 2.15(iii) implies that $\mathbf{V}$ satisfies the identity σ_3.

Suppose that $\mathbf{N} \subseteq \mathbf{V}$. It is verified by Gusev [6, Theorem 1.1 and Fig. 1] that the lattice $L(\mathbf{N} \vee \mathbf{M}^\delta)$ is non-modular. Now Lemma 2.2 applies and we have that $\mathbf{M}^\delta \nsubseteq \mathbf{V}$. Further, $\mathbf{Z}_i \nsubseteq \mathbf{V}$ for each $i = 1, 2, 3$ by Lemma 4.1(vii). In view of the above and Corollary 3.6, we have that $\mathbf{V}$ satisfies the identities σ_2 and α_i with $i = 1, 2, 3$. Then Corollary 5.2 implies that $\mathbf{V} \subseteq \mathbf{D}_\infty \vee \mathbf{N}$, and we are done.

Whence, we may assume that $\mathbf{N} \nsubseteq \mathbf{V}$. Taking into account Lemma 4.3, we have that $S(\mathbf{c}_{n,m}[\pi]) \notin \mathbf{V}$ for all $n, m \in \mathbb{N}_0$ and $\pi \in S_{n+m}$. Then Lemma 3.14 applies with the conclusion that $\mathbf{V}$ satisfies the identity (3.9) for all $n, m \in \mathbb{N}_0$ and $\rho \in S_{n+m}$. The lattice $L(\mathbf{M} \vee \mathbf{N}^\delta)$ is non-modular because the lattice $L(\mathbf{N} \vee \mathbf{M}^\delta)$ is so. Now Lemma 2.2 applies and we have that $\mathbf{N}^\delta \nsubseteq \mathbf{V}$. Then the dual to Lemma 3.14

implies that $\mathbf{V}$ satisfies the identity $\mathbf{d}_{n,m}[\rho] \approx \mathbf{d}'_{n,m}[\rho]$ for all $n, m \in \mathbb{N}_0$ and $\rho \in S_{n+m}$.

Finally, Lemma 4.1(viii) implies that there are $r, s \in \{1,2,3\}$ such that $\mathbf{Z}_i, \mathbf{Z}_j \nsubseteq \mathbf{V}$ for all $i \in \{1,2,3\} \setminus \{r\}$ and $j \in \{1,2,3\} \setminus \{s\}$. Then $\mathbf{V}$ satisfies the identities α_i and β_j with $i \in \{1,2,3\} \setminus \{r\}$ and $j \in \{1,2,3\} \setminus \{s\}$ by Lemma 3.5. Summarizing all we say above, we have that $\mathbf{V} \subseteq \mathbf{Q}_{r,s}$, and we are done.

Finally, suppose that $\mathbf{L} \subseteq \mathbf{V}$. Then Lemma 4.1(vi) and the dual to it imply that $\mathbf{M}, \mathbf{M}^\delta \nsubseteq \mathbf{V}$. Then Lemma 2.15(i),(ii) implies that $\mathbf{V}$ satisfies the identities σ_1 and σ_2. It is proved in Gusev and Vernikov [8, p. 32] that every variety of the form $\operatorname{var} S(\mathbf{w}_n[\pi,\tau])$ contains two non-comparable subvarieties of the same form. This claim and Lemma 4.2 imply that $S(\mathbf{w}_n[\pi,\tau]) \notin \mathbf{V}$ for all $n \in \mathbb{N}$ and $\pi, \tau \in S_n$.

It is checked in the paragraph starting on p. 32 and ending on p. 33 in [8] that if $\mathbf{L} \subseteq \mathbf{X} \subseteq \mathbf{A}$, $\mathbf{X}$ satisfies the identities σ_1, σ_2 and δ_2 and $S(\mathbf{w}_n[\pi,\tau]) \notin \mathbf{X}$ for all $n \in \mathbb{N}$ and $\pi, \tau \in S_n$, then $\mathbf{X}$ satisfies the identity (3.5) for all $n \in \mathbb{N}$ and $\pi, \tau \in S_n$. Repeating arguments from that paragraph in [8] but referring to Lemma 3.10 rather than to [8, Lemma 4.10], we can verify that the same conclusion is true without the hypothesis that $\mathbf{X}$ satisfies the identity δ_2. Together with the saying in the previous paragraph, we see that the identity (3.5) holds in $\mathbf{V}$ for any $n \in \mathbb{N}$ and $\pi, \tau \in S_n$. Therefore, $\mathbf{V} \subseteq \mathbf{R}$, and we are done.

Sufficiency. An arbitrary group variety is fi-permutable because it is congruence permutable. Lemmas 2.3 and 2.6 imply that an arbitrary variety of idempotent monoids is fi-permutable too. By symmetry, it remains to consider the varieties $\mathbf{K}$, $\mathbf{D}_\infty \vee \mathbf{N}$, $\mathbf{P}_n$, $\mathbf{Q}_{r,s}$ and $\mathbf{R}$. The variety $\mathbf{K}$ is fi-permutable by Lemmas 2.5 and 2.18. The same conclusion for the varieties $\mathbf{D}_\infty \vee \mathbf{N}$, $\mathbf{P}_n$, $\mathbf{Q}_{r,s}$ and $\mathbf{R}$ follows from Propositions 5.4, 5.9, 5.10 and 5.16, respectively.

Theorem 1.1 is proved. □

Proof of Theorem 1.2. Necessity. Let $\mathbf{V}$ be a non-completely regular almost fi-permutable variety of monoids. Then $\mathbf{C}_2 \subseteq \mathbf{V}$ by Corollary 2.9. Now the inclusion $\mathbf{SL} \subset \mathbf{C}_2$ and Lemma 4.1(ii) imply that $\mathbf{A}_n \nsubseteq \mathbf{V}$ for any $n > 1$. Therefore, $\mathbf{V}$ is an aperiodic variety. Lemma 2.3 shows that for aperiodic varieties the properties to be fi-permutable and almost fi-permutable are equivalent. Now Theorem 1.1 implies that $\mathbf{V}$ is contained in one of the varieties listed in the item (iii) of this theorem.

Sufficiency immediately follows from Lemma 2.6 and Theorem 1.1.

Theorem 1.2 is proved. □

Proof of Corollaries 1.3 and 1.4. The lattice of all varieties of idempotent monoids is completely described by Wismath [30]. In particular, it turns out to be distributive. Note that, in actual fact, this claim follows from Lemma 2.4 and the assertion that the lattice of all varieties of idempotent semigroups is distributive; this fact was independently discovered by Biryukov, Fennemore and Gerhard in early 1970s (see Evans [2, Section XI], for instance).

Theorems 1.1 and 1.2 show that, up to duality, it remains to check that the varieties $\mathbf{K}$, $\mathbf{D}_\infty \vee \mathbf{N}$, $\mathbf{P}_n$, $\mathbf{Q}_{r,s}$ and $\mathbf{R}$ have distributive subvariety lattices. The lattices $L(\mathbf{K})$, $L(\mathbf{D}_\infty \vee \mathbf{N})$ and $L(\mathbf{P}_n)$ with any $n \in \mathbb{N}$ are distributive by Lemma 2.18 and Corollaries 5.3 and 5.8, respectively. In view of Proposition 3.15, to complete the proof, it suffices to verify that $\mathbf{Q}_{r,s} \subseteq \mathbf{A}'$ and $\mathbf{R} \subseteq \mathbf{A}'$. The second inclusion is evident because the identities (3.7) and $\mathbf{d}_{n,m,k}[\pi] \approx \mathbf{d}'_{n,m,k}[\pi]$ follow from the identities σ_1 and σ_2, respectively. Lemma 3.11 implies that the variety $\mathbf{Q}_{r,s}$ satisfies the identity (3.5) for any $n \in \mathbb{N}_0$ and $\pi, \tau \in S_n$. By symmetry, it remains to verify that $\mathbf{Q}_{r,s}$ satisfies also the identity (3.7) for any $n, m, k \in \mathbb{N}_0$ and $\rho \in S_{n+m+k}$. If $k = 0$, then this claim is evident. Finally, let $k \geq 1$. Put

$$X = \{t_i, z_i \mid n + m + 1 \leq i \leq n + m + k\}.$$

Clearly, the identity $(\mathbf{c}_{n,m,k}[\rho])_X \approx (\mathbf{c}'_{n,m,k}[\rho])_X$ coincides with

$$\mathbf{c}_{n,m}[\rho'] \approx \mathbf{c}'_{n,m}[\rho'] \tag{6.4}$$

for some $\rho' \in S_{n+m}$. Then $\mathbf{Q}_{r,s}$ satisfies the identities

$$\mathbf{c}_{n,m,k}[\rho] \stackrel{\sigma_3}{\approx} \mathbf{p}xy\mathbf{q}x\mathbf{r}y\mathbf{s} \stackrel{(6.4)}{\approx} \mathbf{p}yx\mathbf{q}x\mathbf{r}y\mathbf{s} \stackrel{\sigma_3}{\approx} \mathbf{c}'_{n,m,k}[\rho],$$

where

$$\mathbf{p} = \prod_{i=1}^{n}(z_i t_i), \ \mathbf{q} = t\biggl(\prod_{i=n+1}^{n+m} z_i t_i\biggr), \ \mathbf{r} = \prod_{i\rho \leq n+m} z_{i\rho} \quad \text{and}$$
$$\mathbf{s} = \biggl(\prod_{i\rho > n+m} z_{i\rho}\biggr)\biggl(\prod_{i=n+m+1}^{n+m+k} t_i z_i\biggr).$$

Corollaries 1.3 and 1.4 are proved. □

7. Generalizations of fi-permutability

Let α and β be congruences on an algebra A. For any n, we put

$$\alpha \circ_n \beta = \underbrace{\alpha\beta\alpha\beta\cdots}_{n\ \text{letters}}.$$

Congruences α and β are said to *n-permute* if $\alpha \circ_n \beta = \beta \circ_n \alpha$. Clearly, 2-permutative congruences are nothing but simply permutative ones. 3-permutative congruences, that is, congruences α and β such that $\alpha\beta\alpha = \beta\alpha\beta$ are usually called *weakly permutative*. It is evident that if congruences α and β on some algebra n-permute then $\alpha \vee \beta = \alpha \circ_n \beta$. A variety of algebras $\mathbf{V}$ is called *congruence n-permutable* [*weakly congruence permutable*] if, on any member of $\mathbf{V}$, every two congruences n-permute [respectively, weakly permute].

As well as congruence permutability, the property to be congruence n-permutable for some n is very rigid for semigroup or monoid varieties. According to Lipparini [19, Corollary 0], a semigroup variety is congruence n-permutable for some n if and only if it is a periodic group variety (this fact with $n = 3$ follows also from Jones [13, Theorem 1.2(iii)]). The same is true for monoid varieties. This fact can be easily deduced from Lemma 2.3 and results by Freese and Nation [3] and Lipparini [19], as well as from Lemma 2.3 and [10, Lemma 9.13].

For arbitrary n, we call a variety of algebras $\mathbf{V}$ *fi-n-permutable* if any two fully invariant congruences on every $\mathbf{V}$-free object n-permute. Clearly, fi-2-permutable varieties is simply fi-permutable ones; fi-3-permutable varieties are called *weakly fi-permutable*. It is proved by Vernikov [25] that every weakly fi-permutable semigroup variety is either completely regular or a nil-variety. We provide here an analogous, in a sense, fact concerning monoid varieties.

Lemma 7.1. *If $\mathbf{V}$ is a weakly fi-permutable variety of monoids, then $\mathbf{V}$ is either completely regular or aperiodic.*

Proof. Suppose that $\mathbf{V}$ is neither completely regular nor aperiodic. Then $\mathbf{C}_2 \subseteq \mathbf{V}$ by Corollary 2.9 and $\mathbf{A}_n \subseteq \mathbf{V}$ for some $n > 1$. We denote by ∇ the universal relation on the free monoid $\mathfrak{X}^*$. Then $\theta_{\mathbf{A}_n} \vee \theta_{\mathbf{C}_2} = \nabla$ because $\mathbf{A}_n \wedge \mathbf{C}_2 = \mathbf{T}$. By the analog of Lemma 2.1 for weakly permutative equivalences, the congruences $\theta_{\mathbf{A}_n}$ and $\theta_{\mathbf{C}_2}$ weakly permute, whence $\nabla = \theta_{\mathbf{A}_n} \vee \theta_{\mathbf{C}_2} = \theta_{\mathbf{C}_2}\theta_{\mathbf{A}_n}\theta_{\mathbf{C}_2}$. Therefore, $(x, y) \in \theta_{\mathbf{C}_2}\theta_{\mathbf{A}_n}\theta_{\mathbf{C}_2}$ for any letters x and y. Thus, $x\,\theta_{\mathbf{C}_2}\,\mathbf{w}_1\,\theta_{\mathbf{A}_n}\,\mathbf{w}_2\,\theta_{\mathbf{C}_2}\,y$ for some words $\mathbf{w}_1$ and $\mathbf{w}_2$. Now Lemma 2.8 applies with the conclusion that $\mathbf{w}_1 = x$ and $\mathbf{w}_2 = y$. Therefore, $x\,\theta_{\mathbf{A}_n}\,y$ that is evidently not the case. □

Weakly fi-permutable completely regular semigroup varieties are completely determined in Vernikov [25]. The following problem naturally arises.

Problem 7.2. Describe

(i) weakly fi-permutable completely regular monoid varieties;
(ii) weakly fi-permutable aperiodic monoid varieties.

We do not know even whether there exists a completely regular monoid variety that is not weakly fi-permutable.

It follows from Lipparini [19, Theorem 1] that if a variety of algebras $\mathbf{V}$ is fi-n-permutable for some n, then the lattice $L(\mathbf{V})$ satisfies some non-trivial lattice identity. There are no any additional information about fi-n-permutable semigroup or monoid varieties with $n > 3$ so far. The following observation shows that the analog of Lemma 7.1 for fi-n-permutable monoid varieties with $n > 3$ is false.

Remark 7.3. *If p is a prime number, then the variety $\mathbf{A}_p \vee \mathbf{C}_2$ is fi-4-permutable.*

Proof. Let $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{A}_p \vee \mathbf{C}_2$ and $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}} \vee \theta_{\mathbf{Y}}$. By the analog of Lemma 2.1 for 4-permutative equivalences, it suffices to verify that $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}\theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. Results by Head [9] imply that the lattice $L(\mathbf{A}_p \vee \mathbf{C}_2)$ has the form shown in Fig. 7.1. In view of Lemma 2.5, we may assume without loss of generality that either one of the varieties $\mathbf{X}$ or $\mathbf{Y}$ coincides with $\mathbf{A}_p$, while another one lies in the set $\{\mathbf{SL}, \mathbf{C}_2\}$ or one of these two varieties coincides with $\mathbf{A}_p \vee \mathbf{SL}$, while another one equals $\mathbf{C}_2$. Suppose that $\mathbf{X} = \mathbf{A}_p$ and $\mathbf{Y} \in \{\mathbf{SL}, \mathbf{C}_2\}$. Then

$$\mathbf{u}\,\theta_{\mathbf{X}}\,\mathbf{u}^{p+1}\mathbf{v}^{p}\,\theta_{\mathbf{Y}}\,\mathbf{u}^{p}\mathbf{v}^{p+1}\,\theta_{\mathbf{X}}\,\mathbf{v}, \tag{7.1}$$

whence $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}\theta_{\mathbf{X}}$. Further, if $\mathbf{X} = \mathbf{A}_p \vee \mathbf{SL}$ and $\mathbf{Y} = \mathbf{C}_2$ then the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{X} \wedge \mathbf{Y} = \mathbf{SL}$. Then Lemma 2.3 implies that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. Therefore, (7.1) holds, whence $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}\theta_{\mathbf{X}}$ again. Finally, if either $\mathbf{Y} = \mathbf{A}_p$ and $\mathbf{X} \in \{\mathbf{SL}, \mathbf{C}_2\}$ or $\mathbf{Y} = \mathbf{A}_p \vee \mathbf{SL}$ and $\mathbf{X} = \mathbf{C}_2$, then the same arguments as above show that $(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{Y}}\theta_{\mathbf{X}}\theta_{\mathbf{Y}}$. Thus,

$$(\mathbf{u}, \mathbf{v}) \in \theta_{\mathbf{X}}\theta_{\mathbf{Y}}\theta_{\mathbf{X}} \cup \theta_{\mathbf{Y}}\theta_{\mathbf{X}}\theta_{\mathbf{Y}} \subseteq \theta_{\mathbf{X}}\theta_{\mathbf{Y}}\theta_{\mathbf{X}}\theta_{\mathbf{Y}}$$

in either case, and we are done. □

It is natural to define *almost fi-n-permutable* [*almost weakly fi-permutable*] varieties of semigroups or monoids as varieties on whose free objects any two fully invariant congruences contained in the least semilattice congruence n-permute [respectively, weakly permute]. A classification of almost weakly fi-permutable semigroup varieties in some wide partial case was provided in Vernikov [24]. A minor inaccuracy in this result is fixed by Vernikov and Shaprynskiĭ [26]. Almost fi-n-permutable semigroup varieties with $n > 3$ as well as almost fi-n-permutable monoid varieties with $n > 2$ are not examined so far.

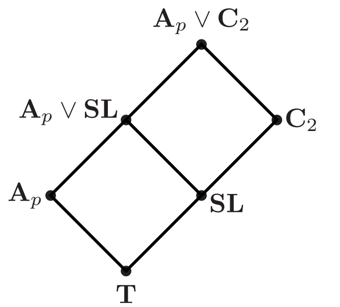

Figure 7.1. The lattice $L(\mathbf{A}_p \vee \mathbf{C}_2)$

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Institute of Natural Sciences and Mathematics, Ural Federal University, Lenina str. 51, 620000 Ekaterinburg, Russia

E-mail address: sergey.gusb@gmail.com, bvernikov@gmail.com