On Automorphisms of a Distance-Regular Graph with Intersection Array \(\{33, 30, 15; 1, 2, 15\}\)

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We consider undirected graphs without loops or multiple edges. Given a vertex \(a\) in a graph \(\Gamma\), let \(\Gamma_i(a)\) denote the \(i\)-neighborhood of \(a\), i.e., the subgraph induced by \(\Gamma\) on the set of all its vertices that are a distance of \(i\) away from \(a \). Let \(\{a\} = \Gamma_1(a)\) and \(a^+ = \{a\} \cup \{a\}\).

The degree of a vertex is defined as the number of vertices in its neighborhood. \(\Gamma\) is called a regular graph of degree \(d\) if the degree of any vertex in \(\Gamma\) is \(d\) is called a regular graph of degree \(d\). An amply regular graph of diameter 2 is said to be strongly regular.

Proposition 1. Let \(\Gamma\) be a distance-regular graph of diameter larger than 2 on \(v \leq 4096\) vertices. If \(\lambda = 2\), then one of the following assertions holds:

1. \(\Gamma\) is a primitive graph with the intersection array \(\{6, 3, 3; 1, 1, 2\}, \{9, 6, 3; 1, 2, 3\}, \{12, 9, 9; 1, 1, 4\}, \{15, 12, 6; 1, 1, 10\}, \{18, 15, 9; 1, 1, 10\}, \{19, 16, 8; 1, 2, 8\}, \{24, 21, 3; 1, 1, 18\}, \{33, 30, 15; 1, 2, 15\}, \{35, 32, 8; 1, 2, 28\}, \{42, 39, 1; 1, 1, 42\}, \{51, 48, 8; 4, 36\}, \{51, 48, 24; 1, 2, 24\}, \{55, 52, 34; 1, 1, 22\}, \{58, 55, 8; 1, 2, 44\}, \{60, 57, 16; 1, 4, 30\}, \{60, 57, 32; 1, 4, 18\}, \{63, 60, 10; 1, 2, 54\}, \{63, 60, 49; 1, 4, 15\}, \{68, 65, 32; 1, 4, 40\}, \{75, 72, 8; 1, 2, 60\}, \{75, 72, 42; 1, 4, 50\}, \{75, 72, 31; 1, 8, 45\}, \{80, 77, 61; 1, 7, 20\}, \{90, 87, 60; 1, 1, 18\}, \{99, 96, 12; 1, 4, 88\}, \{99, 96, 20; 1, 4, 72\}, \{99, 96, 6; 1, 6, 88\}, \{120, 117, 5; 1, 5, 108\}, \{143, 140, 34; 1, 7, 110\}, \{147, 144, 39; 1, 12, 117\}, or \(\{224, 221, 32; 1, 16, 208\}\).

2. \(\Gamma\) is an antipodal graph with \(\mu = 2\), the intersection array \(\{2 + r, 1, 2; 1, 2, 2r + 1\}, \ r \in \{2, 3, \ldots, 44\} - \{10, 16, 28, 34, 38\}, \text{ and } v = 2(2r + 1)\).

3. \(\Gamma\) is an antipodal graph with \(\mu \geq 3\) and the intersection array \(\{1, 5, 12; 1, 4, 15\}, \{18, 15, 1; 1, 5, 18\}, \{27, 24, 1; 1, 8, 27\}, \{35, 32, 1; 1, 4, 35\}, \{45, 42, 1; 1, 6, 45\}, \{42, 39, 1; 1, 3, 42\}, \{63, 60, 1; 1, 4, 63\}, \{75, 72, 1; 1, 12, 75\}, \{99, 96, 1; 1, 4, 99\}, \{108, 105, 1; 1, 5, 108\}, \{143, 140, 1; 1, 20, 143\}, \{147, 144, 1; 1, 16, 147\}, or \{171, 168, 1; 1, 12, 171\}.

4. \(\Gamma\) is a primitive graph with the intersection array \(\{6, 3, 3; 1, 1, 2\}, \{12, 9, 6; 3, 1, 2, 4, 5\}, \{21, 18, 12, 4; 1, 1, 6, 21\}, \{15, 12, 9, 6, 3; 1, 2, 3, 4, 5\}, \{6, 3, 3, 3, 3; 1, 1, 1, 1, 2\}, \text{ or } \{18, 15, 12, 9, 6, 3; 1, 2, 3, 4, 5\}.

The automorphisms of distance-regular graphs with intersection arrays \(\{2r + 1, 2r - 1, 2r + 1\}\), where \(r \leq 43\) and \(r\) is not a prime power were found in [2]. In this paper, we study the automorphisms of a distance-regular graph with intersection array \(\{33, 30, 15; 1, 2, 15\}\). The number of vertices in such a graph is \(v = 1 + 30 + 495 + 495 = 1024\), and it has the spectrum...
Let $P_i$ be a matrix with $p_{ij}^i$ placed at $(j,i)$. Then the eigenvalues $p_i(0), \ldots, p_i(d)$ of $P_i$ are the eigenvalues of $\Gamma$ with multiplicities $m_0 = 1, m_1, \ldots, m_d$. The matrices $P$ and $Q$ with $p_{ij}(i)$ and $q_{ij}(i)$ placed at $(i,j)$ are called the first and second eigenvalue matrices of the group and are related by the equality $PQ = QP = vI$.

In the usual manner, the permutation representation of the group $G = \text{Aut}(\Gamma)$ at the vertices of $\Gamma$ gives a matrix representation $\psi$ of $G$ in $GL(v, \mathbb{C})$. The space $\mathbb{C}^v$ is the orthogonal direct sum of $G$-invariant eigenspaces $W_0, W_1, \ldots, W_d$ of the adjacency matrix $A = A_1$ of $\Gamma$. For any $g \in G$, the matrix $\psi(g)$ commutes with $A$. Therefore, the subspace $W_i$ is $\psi(G)$-invariant. Let $\chi_i$ be a character of the representation $\psi_{|W_i}$. Then, for $g \in G$, we obtain

$$\chi_i(g) = \psi^{-1}(x) \sum_{j=0}^d Q_{ij} \chi_j(g),$$

where $\alpha_j(g)$ is the number of points $x$ in $X$ such that $(x,x^g) \in R_i$ (see [3, Section 3.7]).

**Lemma 2.** Let $\Gamma$ be a distance-regular graph with an intersection array $(33, 30, 15; 1, 2, 15)$ and $G = \text{Aut}(\Gamma)$. If $g \in G$, $\chi_1$ is a character of the projection of the representation $\psi$ onto a subspace of dimension 495, then

$$\chi_1(g) = \frac{13\alpha_9(g) + 4\alpha_1(g) + \alpha_2(g)}{128} - 10,$$

$$\chi_2(g) = \frac{15\alpha_9(g) - \alpha_3(g)}{32} + 15.$$

If $|g| = p$ is a prime number, then $\chi_1(g) - 198$ and $\chi_2(g) - 495$ are divided by $p$.

**Proof.** Let $\Gamma$ be a distance-regular graph with an intersection array $(33, 30, 15; 1, 2, 15)$. We have

$$Q = \begin{pmatrix}
1 & 1 & 1 \\
198 & 54 & 6 \\
495 & 15 & -17 \\
330 & -70 & 10 & -6
\end{pmatrix}.$$
In view of the equality \( \alpha(g) + \alpha(\gamma) = 1024 - \alpha(g) - \alpha(\gamma) \), we have \( \chi_2(g) = \frac{15\alpha(g) - \alpha(\gamma)}{32} + 15 \).

The other assertions of the lemma follow from Lemma 1 in [4].

Throughout the rest of this paper, we assume that \( \Gamma \) is a distance-regular graph with intersection array \( \{33, 30, 15; 1, 2, 15\} \), \( G = \text{Aut}(\Gamma) \), \( g \) is an element of prime order \( p \) from \( G \), and \( \Omega = \text{Fix}(g) \).

**Lemma 3.** The following assertions hold:

1. If \( \Omega \) is an empty graph, then \( p = 2 \), \( \alpha(g) = 64l \), and \( \alpha(\gamma) = 64s - 16l \).

2. If \( \Omega \) is an \( n \)-clique, then either
   
   (i) \( n = 1 \), \( p = 11 \), \( \alpha(g) = 143 + 352l \), and \( \alpha(\gamma) = 121 + 352l - 88l \) or \( p = 3 \), \( \alpha(g) = 15 + 96l \), and \( \alpha(\gamma) = 96l + 57 - 24l \) or
   
   (ii) \( n = 4 \), \( p = 5 \), \( \alpha(g) = 60 + 160l \), \( \alpha(\gamma) = 100 + 640l - 40l \); or \( p = 3 \), \( \alpha(g) = 60 + 96l \), and \( \alpha(\gamma) = 384l + 228 - 24l \) or \( p = 2 \), \( \alpha(g) = 60 + 64l \), and \( \alpha(\gamma) = 64l + 28 - 16l \).

3. If \( \Omega \) is not an empty graph or a clique and \( p > 2 \), then \( \Omega \) is an amply regular graph with \( \lambda_{\Omega} = \mu_{\Omega} = 2 \) and \( p = 3, 5 \).

**Proof.** Let \( \Omega \) be an empty graph. Since \( v = 1024 \), we have \( p = 2 \). Furthermore, the number \( \chi_2(g) - 495 \) is even. Therefore, \( \alpha(g) = 64l \). From this, \( \chi_2(g) = \frac{\alpha(g) + 16l}{32} - 10 \) and \( \alpha(\gamma) = 64s - 16l \).

Let \( \Omega \) be an \( n \)-clique. If \( n = 1 \), then \( p \) divides 33 and 495. Therefore, \( p = 3, 11 \). If \( p = 11 \), we have \( \chi_2(g) = \frac{15 - \alpha\gamma(\gamma)}{32} + 15 \). Therefore, \( \alpha(g) = 143 + 352l \), where \( l \leq 2 \). Furthermore, \( \chi_2(g) = \frac{13 + 4\alpha\gamma(\gamma) + 143 + 352l}{128} \) - 10. Therefore, \( \alpha(g) = 121 + 352l - 88l \).

If \( p = 3 \), we have \( \chi_2(g) = \frac{15 - \alpha\gamma(\gamma)}{32} + 15 \).

Therefore, \( \alpha(g) = 15 + 96l \), where \( l \leq 10 \). Furthermore, \( \chi_2(g) = \frac{13 + 4\alpha\gamma(\gamma) + 15 + 96l}{128} \) - 10. Therefore, \( \alpha(g) = 96l + 57 - 24l \).

If \( n = 2 \), then \( p = 2 \), a contradiction to the fact that 495 is not divided by 2.

If \( n = 4 \), then \( p \) divides 30. For \( p = 5 \), we have \( \chi_2(g) = \frac{60 - \alpha\gamma(\gamma)}{32} + 15 \). Therefore, \( \alpha(g) = 60 + 160l \), where \( l \leq 6 \). Furthermore, \( \chi_2(g) = \frac{52 + 4\alpha\gamma(\gamma) + 60 + 160l}{128} \) - 10. Therefore, \( \alpha(g) = 100 + 640l - 40l \). If \( p = 3 \), we have \( \chi_2(g) = \frac{60 - \alpha\gamma(\gamma)}{32} + 15 \). Therefore, \( \alpha(g) = 60 + 96l \), where \( \frac{32}{l} \leq 10 \). Furthermore, \( \chi_2(g) = \frac{52 + 4\alpha\gamma(\gamma) + 60 + 96l}{128} \) - 10. Therefore, \( \alpha(g) = 72 + 192l \) and \( \alpha(g) = 72 + 192l \).

Let \( \Omega \) be neither an empty graph nor a clique and let \( p > 2 \). Then \( \lambda_{\Omega} = \mu_{\Omega} = 2 \). Therefore, \( \Omega \) is an amply regular graph of degree \( k' \) on \( v' \) vertices.

If \( p \geq 17 \), then \( k' = 33 - p \) and \( v' \geq 1 + (33 - p) + (33 - p)(30 - p) \). If \( p = 29 \), we have \( k' = 4 \). However, a connected locally quadrilateral graph is an octahedron, a contradiction to \( \mu_{\Omega} = 2 \). If \( p = 23 \), we have \( k' = 10 \) and \( v' \geq 1 + 70 + 10 \cdot \frac{7}{2} (495 - 12) = 23, 24 \), a contradiction to \( |\Gamma - \Omega| \geq 58 \cdot 23 \). If \( p = 19 \), we have \( k' = 4 \) and \( v' \geq 1 + 14 + 14 \cdot \frac{11}{2} \), a contradiction to \( |\Gamma - \Omega| \geq 26 \cdot 12 \). If \( p = 13 \), then \( \Omega \) is a regular graph of degree 7 or 20. Furthermore, 495 – 1 is divided by 13. Let \( a, b \in \Omega \) and \( d(a, b) = 3 \). Since \( p^3 = 236 \), \( \Omega(a) \cap \Omega(b) \) contains at least two vertices. If \( \Omega \) is a graph of degree 20, then \( v' \geq 1 + 19 + \frac{20 \cdot 2}{113} \), a contradiction to \( |\Gamma - \Omega| \geq 26 \cdot 12 \). If \( p = 11 \), then \( \Omega \) is a regular graph of degree 11 or 22. If \( \Omega \) contains a vertex of degree 22, then \( v' \geq 1 + 33 + \frac{22 \cdot 19}{2} \), a contradiction to \( |\Gamma - \Omega| \geq 26 \cdot 12 \). Therefore, \( \Omega \) is a regular graph of degree 11, \( v' \geq 1 + 11 + 44 \), and the number of edges between \( \Omega \) and \( \Gamma - \Omega \) is at least 56 · 22, a contradiction.

If \( p = 7 \), then \( \Omega \) is a regular graph of degree 5, 12, 19, or 26. Furthermore, 495 – 5 is divided by 7. Let \( a, b \in \Omega \) and \( d(a, b) = 3 \). Since \( p^5 = 236 \), we conclude that \( \Omega(a) \cap \Omega(b) \) contains at least five vertices and \( |\Omega(a) \cap \Omega(b)| \geq 12 \). If \( \Omega \) contains a vertex \( a \) of degree 5, then...
Ω is an icosahedron graph, a contradiction to the fact that $1024 - 12$ is not divided by 7.

If Ω contains a vertex of degree 26, then $\nu' = 1 + 26 + \frac{26 \cdot 23}{2} + 12 = 338$, a contradiction to $1024 \geq |\Gamma| \geq 338 - 8$.

If Ω contains a vertex of degree 19, then $\nu' = 1 + 19 + \frac{19 \cdot 9}{2} + 12$, a contradiction to $1024 \geq |\Gamma| \geq 118 \cdot 15$.

Therefore, Ω is a regular graph of degree 12, $\nu' \geq 1 + 12 + 54 + 12 = 79$, and the number of edges between Ω and Γ − Ω is at least 79 · 21, a contradiction.

The lemma and the theorem are proved.

**Lemma 4.** Let Ω be an amply regular graph with parameters $(\nu', k', 2, 2)$. Then the following assertions hold:

1. The number $p$ is not equal to 5.

2. If $p = 3$, then the degree of Ω is equal to 9 and Ω is a distance-regular graph with intersection array $(9, 6, 1, 1, 2, 9)$ or the degree of Ω is equal to 6 and Ω is a strongly regular graph with parameters $(16, 6, 2, 2)$.

**Proof.** Let Ω be an amply regular graph with parameters $(\nu', k', 2, 2)$.

Let $p = 5$. Then Ω is a regular graph of degree 3, 8, 13, 18, 23, or 28. If Ω contains a vertex $a$ of degree 3, then Ω is a 4-clique, a contradiction.

If Ω contains a vertex of degree 28, then $\nu' \geq 1 + 28 + \frac{28 \cdot 25}{2}$, a contradiction to $1024 \geq |\Gamma| \geq 379 \cdot 5$.

The cases when Ω contains a vertex of degree 23, 18, or 13 are treated in a similar manner.

Thus, Ω is a graph of degree 8. The neighborhood of a vertex in Ω is an octagon or the union of a triangle and a pentagon. If $d(\Omega) = 2$, then Ω is a strongly regular graph with parameters (29, 8, 2, 2), a contradiction. Let $a, b \in \Omega$ and $d(a, b) = 3$. Since $p_{33}^3 = 236$, we conclude that $\Omega_2(a) \cap \Omega_2(b)$ contains a vertex $b'$ and the degree of the graph $\Omega_2(a)$ is equal to 3.

Now $\nu' > 1 + 8 + \frac{8 \cdot 5}{2} = 29$ and $|\Omega \cap \Gamma_3(a)| \geq 10$. If $|\Omega \cap \Gamma_3(a)| > 15$, then $1024 \geq |\Gamma| \geq 44 \cdot 25$, a contradiction. Therefore, $|\Omega \cap \Gamma_3(a)| = 10$ and each vertex in $\Omega_2(a)$ is adjacent to no or five vertices from $\Omega_3(a)$. Let $A$ be the set of vertices in $\Omega_2(a)$ that are adjacent to five vertices in $\Omega_3(a)$, and let $B$ be the set of vertices from $\Omega_2(a)$ that are adjacent to no vertex in $\Omega_3(a)$. Then $|A| = |B| = 5$ and the number of edges between $A$ and $B$ is at most 5, a contradiction to the fact that $B$ is a clique.

Let $p = 3$. Then Ω is a regular graph of degree 3, 6, 9, 12, 15, 18, 21, 24, 27, or 30. In view of [5, Proposition 1.9.1], we have $c_{2}(a, b) \geq 3$ for any vertices $a, b \in \Omega$ with $d(a, b) = 3$. If Ω contains a vertex $a$ of degree 3, then Ω is a 4-clique, a contradiction.

If Ω contains a vertex of degree 30, then $\nu' \geq 1 + 30 + \frac{30 \cdot 27}{2}$, a contradiction to $1024 \geq |\Gamma| \geq 436 \cdot 4$.

The cases when Ω contains a vertex of degree 27, ..., 12 are considered in a similar fashion.

If Ω contains a vertex of degree 9, then $\nu' \geq 1 + 9 + \frac{9 \cdot 6}{2} = 37$ and the neighborhood of a vertex in Ω is a nonagon, the union of a hexagon and a triangle, or the union of three triangles. If $d(\Omega) = 2$, then Ω is a strongly regular graph with parameters $(37, 9, 2, 2)$, a contradiction. Let $a, b \in \Omega$ and $d(a, b) = 3$. Since $p^{33}_{33} = 236$, we conclude that $\Omega_3(a) \cap \Omega_3(b)$ contains two vertices.

If $|\Omega_3(a)| \geq 6$, we have $1024 \geq |\Gamma| \geq 42 \cdot 25$, a contradiction. Therefore, Ω is a distance-regular graph with the intersection array $(9, 6, 1, 1, 2, 9)$.

If Ω is a graph of degree 6, then $\nu' \geq 1 + 6 + \frac{6 \cdot 3}{2} = 16$ and the neighborhood of a vertex in Ω is a hexagon or the union of two triangles. If $d(\Omega) = 2$, then Ω is a strongly regular graph with parameters $(16, 6, 2, 2)$. Let $c \in \Omega_3(a)$. Then the vertices from $[a] \cap [c]$ are adjacent to distinct vertices from [c]. Therefore, [c] does not intersect $\Omega_3(a)$.

**Lemma 5.** If $\Gamma$ is an edge-symmetric graph, then the following assertions hold:

1. $\Omega$ does not contain [a] for any vertex $a \in \Omega$.

2. The solvable radical of the group $G$ coincides with $O_3(G)$; if $Q = O_3(G) \neq 1$, then each involution from $Q$ acts on $\Gamma$ without fixed points and either $|Q| = 2^{10}$ or $|Q| < 2^5$.

2. The socle $\overline{G}$ of the group $G = G/O_3(G)$ is isomorphic to $L_2(11), M_{11}, M_{12}$, or $U_4(2)$.

**Proof.** Let $[a] \subset \Omega$ for some vertex $a \in \Omega$. Then, for any vertex $u \in \Gamma - \Omega$, we have $[a] \cap \Omega = [a] \cap [u] = \{c, d\}$, a contradiction to $|c] \cap [d| \geq 3$.

Since $\nu = 2^{10}$, the solvable radical of $G$ coincides with $O_2(G)$. Let $Q = O_2(G) \neq 1$. Then $Q_a$ is a normal subgroup of $G_a$ that fixes a vertex $b \in [a]$. Since $G_a$ is transitive on $[a]$, we have $[a] \subset \Omega$ for any involution $g \in Q_a$. By assertion (1), $Q_a = 1$. Assume that $1 < |Q_a| < 2^{10}$. Then the orbit $dQ$ is a coclique for any ver-
tex $a \in \Gamma$. In a regular graph with the least eigenvalue $-m$, the order of a coclique is at most $\frac{\sqrt{m}}{k + m}$. Therefore, $|Q| = |a^0| \leq \frac{128 \cdot 7}{5}$.

Recall that, by the theorem, $\pi(G) \subseteq \{2, 3, 5, 11\}$ and, by assumption, $|G|$ is divided by $2^{10} \cdot 33$. In view of [6, Table 1], the group $\overline{T}$ is isomorphic to $L_2(11)$, $M_{11}$, $M_{12}$, or $U_3(2)$.

Let us complete the proof of the corollary. In the case of $L_2(11)$, we have $|O_2(G)| = 2^{10}$ (otherwise, $|O_2(G)| = 2^7$, $G/O_2(G) = PGL_2(11)$, and $G$ does not contain subgroups of index $2^{10}$), $G_\alpha = L_2(11)$, $PGL_2(11)$ does not contain subgroups of index 33. The cases of $M_{11}$ and $M_{12}$ are treated in a similar manner. In the case of $U_3(2)$, we have $|O_2(G)| = 2^{10}$ (otherwise, $G$ does not contain subgroups of index $2^{10}$), $G_\alpha = U_3(2)$, $\text{Aut}(U_3(2))$ does not contain subgroups of index 33. The corollary is proved.

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REFERENCES


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