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MATHEMATICS

On Automorphisms of a Distance-Regular Graph with Intersection Array {33, 30, 15; 1, 2, 15}

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We consider undirected graphs without loops or multiple edges. Given a vertex a in a graph Γ, let $\Gamma_i(a)$ denote the i-neighborhood of a, i.e., the subgraph induced by Γ on the set of all its vertices that are a distance of i away from a. Let $[a] = \Gamma_1(a)$ and $a^{\perp} = \{a\} \cup [a]$.

The degree of a vertex is defined as the number of vertices in its neighborhood. Γ is called a regular graph of degree k if the degree of any vertex in Γ is k is called an edge-regular graph with parameters (v, k, λ) if it is a regular graph of degree k on v vertices and each of its edges lies in λ triangles. Γ is called an amply regular graph with parameters (v, k, λ, μ) if it is an edge-regular graph with the corresponding parameters and $[a] \cap [b]$ contains μ vertices for any two vertices a, b separated by a distance of 2 in Γ. An amply regular graph of diameter 2 is said to be strongly regular.

If vertices u, w are separated by a distance of i in Γ, then $b_i(u, w)$ $(c_i(u, w))$ denotes the number of vertices in the intersection of $\Gamma_{i+1}(u)$ $(\Gamma_{i-1}(u))$ with $[w]$. A graph Γ of diameter d is called a distance-regular graph with an intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of the vertices u, w separated by the distance i in Γ for any $i = 0, 1, \ldots, d$. Let $a_i = k - b_i - c_i$. Note that, for a distance-regular graph, b_0 is the degree of the graph and $c_1 = 1$. Given a subset X of the automorphisms of Γ, let $\mathrm{Fix}(X)$ denote the set of all vertices of Γ that are fixed under any automorphism from X. Let $p^l_{ij}(x, y)$ denote the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$ for vertices x, y separated by a distance of l in Γ. In a distance-regular graph, the numbers $p^l_{ij}(x, y)$ are independent of the choice of x, y; they are denoted by p^l_{ij} and are known as the intersection numbers of Γ. A graph is said to be edge-symmetric if its automorphism group acts transitively on the set of its arcs (ordered edges).

The intersection arrays of distance-regular graphs with $\lambda = 2$ on at most 4096 vertices were found in [1].

Proposition 1. *Let Γ be a distance-regular graph of diameter larger than 2 on $v \le 4096$ vertices. If $\lambda = 2$, then one of the following assertions holds:*

(1) *Γ is a primitive graph with the intersection array* $\{6,3,3; 1,1,2\}$, $\{9,6,3; 1,2,3\}$, $\{12,9,9; 1,1,4\}$, $\{15,12,6; 1,2,10\}$, $\{18,15,9; 1,1,10\}$, $\{19,16,8; 1,2,8\}$, $\{24,21,3; 1,3,18\}$, $\{33, 30, 15; 1, 2, 15\}$, $\{35,32,8; 1,2,28\}$, $\{42,39,1; 1,1,42\}$, $\{51,48,8; 1,4,36\}$ $\{51,48,24; 1,2,24\}$, $\{55,52,34; 1,2,22\}$, $\{58,55,8; 1,2,44\}$, $\{60,57,16; 1,4,30\}$, $\{60,57,32; 1,4,18\}$, $\{63,60,10; 1,2,54\}$, $\{63,60,49; 1,4,15\}$, $\{68,65,32; 1,4,40\}$, $\{75,72,8; 1,2,60\}$, $\{75,72,42; 1,4,50\}$, $\{75,72,31; 1,8,45\}$, $\{80,77,61; 1,7,20\}$, $\{90, 87, 60; 1, 15, 18\}$, $\{99,96,12; 1,4,88\}$, $\{99,96,20; 1,4,72\}$, $\{99, 96, 6; 1, 6, 88\}$, $\{120,117,5; 1,5,108\}$, $\{143, 140, 34; 1, 7, 110\}$, $\{147,144,39; 1,12,117\}$, *or* $\{224,221,32; 1,16,208\}$.

(2) *Γ is an antipodal graph with $\mu = 2$, the intersection array* $\{2r + 1, 2r - 2, 1; 1, 2, 2r + 1\}$, $r \in \{2, 3, \ldots, 44\} - \{10, 16, 28, 34, 38\}$, *and* $v = 2r(r + 1)$.

(3) *Γ is an antipodal graph with $\mu \ge 3$ and the intersection array* $\{15,12,1; 1,4,15\}$, $\{18,15,1; 1,5,18\}$, $\{27, 24, 1; 1, 8, 27\}$, $\{35,32,1; 1,4,35\}$, $\{45,42,1; 1,6,45\}$, $\{42,39,1; 1,3,42\}$, $\{63,60,1; 1,4,63\}$, $\{75,72,1; 1,12,75\}$, $\{99, 96, 1; 1, 4, 99\}$, $\{108,105,1; 1,5,108\}$, $\{143, 140, 1; 1, 20, 143\}$, $\{147,144,1; 1,16,147\}$, *or* $\{171,168,1; 1,12,171\}$.

(4) *Γ is a primitive graph with the intersection array* $\{6,3,3,3; 1,1,1,2\}$, $\{12,9,6,3; 1,2,3,4\}$, $\{21, 18, 12, 4; 1, 1, 6, 21\}$, $\{15,12,9,6,3; 1,2,3,4,5\}$, $\{6,3,3,3,3,3; 1,1,1,1,1,2\}$, *or* $\{18,15,12,9,6,3; 1,2,3,4,5,6\}$.

The automorphisms of distance-regular graphs with intersection arrays $\{2r + 1, 2r - 2, 1; 1, 2, 2r + 1\}$, where $r \le 43$ and r is not a prime power were found in [2]. In this paper, we study the automorphisms of a distance-regular graph with intersection array $\{33, 30, 15; 1, 2, 15\}$. The number of vertices in such a graph is $v = 1 + 30 + 495 + 495 = 1024$, and it has the spectrum

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$33^1, 9^{198}, 1^{495}, -7^{330}$. The order of a clique in Γ is at most 4, since $\lambda = 2$.

Theorem. *Let Γ be a distance-regular graph with intersection array* {33, 30, 15; 1, 2, 15}, $G = \mathrm{Aut}(\Gamma)$, *g be an element of prime order p from G, and $\Omega = \mathrm{Fix}(g)$.*

Then $\pi(G) \subseteq \{2, 3, 5, 11\}$ and one of the following assertions holds:

(1) *Ω is an empty graph, $p = 2$, $\alpha_2(g) = 64l$, and $\alpha_1(g) = 64s - 16l$.*

(2) *Ω is an n-clique and either*

(i) *$n = 1$, $p = 11$, $\alpha_2(g) = 143 + 352l$, and $\alpha_1(g) = 121 + 352t - 88l$ or $p = 3$, $\alpha_2(g) = 15 + 96l$, and $\alpha_1(g) = 96t + 57 - 24l$ or*

(ii) *$n = 4$, $p = 5$, $\alpha_2(g) = 60 + 160l$, and $\alpha_1(g) = 100 + 640t - 40l$; $p = 3$, $\alpha_2(g) = 60 + 96l$, and $\alpha_1(g) = 384t + 228 - 24l$; or $p = 2$, $\alpha_2(g) = 60 + 64l$, and $\alpha_1(g) = 64t - 28 - 16l$.*

(3) *Ω is not an empty graph or a clique and either*

(i) *$p = 3$ and the degree of Ω is equal to 9 and Ω is a distance-regular graph with intersection array $\{9, 6, 1; 1, 2, 9\}$ or the degree of Ω is equal to 6 and Ω is a strongly regular graph with parameters $(16, 6, 2, 2)$ or*

(ii) *$p = 2$.*

Corollary. *The distance-regular graph with intersection array* {33, 30, 15; 1, 2, 15} *is not an edge-symmetric graph.*

Lemma 1. *Let Γ be a distance-regular graph with intersection array* {33, 30, 15; 1, 2, 15}. *Then the nonzero intersection numbers are*

(1) $p^1_{11} = 2$, $p^1_{12} = 30$, $p^1_{22} = 240$, $p^1_{23} = 225$, *and* $p^1_{33} = 270$;

(2) $p^2_{11} = 2$, $p^2_{12} = 16$, $p^2_{13} = 15$, $p^2_{22} = 238$, $p^2_{23} = 240$, *and* $p^2_{33} = 240$;

(3) $p^3_{12} = 15$, $p^3_{13} = 18$, $p^3_{22} = 240$, $p^3_{23} = 240$, *and* $p^3_{33} = 236$.

Proof. The lemma is proved by direct calculations.

The proof of the theorem relies on Higman's method for handling automorphisms of a distance-regular graph (see [3, Section 3]). The graph Γ is treated as a symmetric scheme of relations $(X, \mathfrak{R})$ with d classes, where X is the vertex set of the graph, R_0 is the equality relation on X, and the class R_i with $i \geq 1$ consists of pairs (u, w) such that $d(u, w) = i$. For $u \in \Gamma$, we set $k_i = |\Gamma_i(u)|$ and $v = |\Gamma|$. The class R_i is associated with a graph Γ_i on the vertex set X in which vertices u, w are adjacent if $(u, w) \in R_i$. Let A_i be the adjacency matrix of Γ_i for $i > 0$, and let $A_0 = I$ be the identity matrix. Then $A_i A_j = \sum p^l_{ij} A_l$ for the intersection numbers p^l_{ij}.

Let P_i be a matrix with p^l_{ij} placed at (j, l). Then the eigenvalues $p_1(0), \ldots, p_1(d)$ of P_1 are the eigenvalues of Γ with multiplicities $m_0 = 1, m_1, \ldots, m_d$. The matrices P and Q with $p_j(i)$ and $q_j(i) = \dfrac{m_j p_i(j)}{k_i}$, respectively, placed at (i, j) are called the first and second eigenvalue matrices of the scheme and are related by the equality $PQ = QP = vI$.

In the usual manner, the permutation representation of the group $G = \mathrm{Aut}(\Gamma)$ at the vertices of Γ gives a matrix representation ψ of G in $GL(v, \mathbf{C})$. The space $\mathbf{C}^v$ is the orthogonal direct sum of G-invariant eigenspaces $W_0, W_1, \ldots, W_d$ of the adjacency matrix $A = A_1$ of Γ. For any $g \in G$, the matrix $\psi(g)$ commutes with A. Therefore, the subspace W_i is $\psi(G)$-invariant. Let χ_i be a character of the representation ψ_{W_i}. Then, for $g \in G$, we obtain

$$\chi_i(g) = v^{-1} \sum_{j=0}^{d} Q_{ij} \alpha_j(g),$$

where $\alpha_j(g)$ is the number of points x in X such that $(x, x^g) \in R_j$ (see [3, Section 3.7]).

Lemma 2. *Let Γ be a distance-regular graph with the intersection array* {33, 30, 15; 1, 2, 15} *and $G = \mathrm{Aut}(\Gamma)$. If $g \in G$, χ_1 is a character of the projection of the representation ψ onto a subspace of dimension* 198, *and χ_2 is a character of the projection of ψ onto a subspace of dimension* 495, *then*

$$\chi_1(g) = \frac{13\alpha_0(g) + 4\alpha_1(g) + \alpha_2(g)}{128} - 10,$$

$$\chi_2(g) = \frac{15\alpha_0(g) - \alpha_2(g)}{32} + 15.$$

If $|g| = p$ is a prime number, then $\chi_1(g) - 198$ and $\chi_2(g) - 495$ are divided by p.

Proof. Let Γ be a distance-regular graph with an intersection array {33, 30, 15; 1, 2, 15}. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 198 & 54 & 6 & -10 \\ 495 & 15 & -17 & 15 \\ 330 & -70 & 10 & -6 \end{pmatrix}.$$

Therefore,

$$\chi_1(g) = \frac{99\alpha_0(g) + 27\alpha_1(g) + 3\alpha_2(g) - 5\alpha_3(g)}{512},$$

$$\chi_2(g) = \frac{495\alpha_0(g) + 15\alpha_1(g) - 17\alpha_2(g) + 15\alpha_3(g)}{1024}.$$

Taking into account the equality $\alpha_3(g) = 1024 - \alpha_0(g) - \alpha_1(g) - \alpha_2(g)$, we obtain $\chi_1(g) = \dfrac{13\alpha_0(g) + 4\alpha_1(g) + \alpha_2(g)}{128} - 10$.

In view of the equality $\alpha_1(g) + \alpha_3(g) = 1024 - \alpha_0(g) - \alpha_2(g)$, we have $\chi_2(g) = \frac{15\alpha_0(g) - \alpha_2(g)}{32} + 15$.

The other assertions of the lemma follow from Lemma 1 in [4].

Throughout the rest of this paper, we assume that Γ is a distance-regular graph with intersection array {33, 30, 15; 1, 2, 15}, $G = \mathrm{Aut}(\Gamma)$, g is an element of prime order p from G, and $\Omega = \mathrm{Fix}(g)$.

Lemma 3. *The following assertions hold:*

(1) *If* Ω *is an empty graph, then* $p = 2$, $\alpha_2(g) = 64l$, *and* $\alpha_1(g) = 64s - 16l$.

(2) *If* Ω *is an* n*-clique, then either*

(i) $n = 1$, $p = 11$, $\alpha_2(g) = 143 + 352l$, *and* $\alpha_1(g) = 121 + 352t - 88l$ *or* $p = 3$, $\alpha_2(g) = 15 + 96l$, $\alpha_1(g) = 96t + 57 - 24l$, *or*

(ii) $n = 4$, $p = 5$, $\alpha_2(g) = 60 + 160l$, $\alpha_1(g) = 100 + 640t - 40l$; $p = 3$, $\alpha_2(g) = 60 + 96l$, *and* $\alpha_1(g) = 384t + 228 - 24l$, *or* $p = 2$, $\alpha_2(g) = 60 + 64l$, *and* $\alpha_1(g) = 64t - 28 - 16l$.

(3) *If* Ω *is not an empty graph or a clique and* $p > 2$, *then* Ω *is an amply regular graph with* $\lambda_\Omega = \mu_\Omega = 2$ *and* $p = 3, 5$.

Proof. Let Ω be an empty graph. Since $v = 1024$, we have $p = 2$. Furthermore, the number $\chi_2(g) - 495$ is even. Therefore, $\alpha_2(g) = 64l$. From this, $\chi_1(g) = \frac{\alpha_1(g) + 16l}{32} - 10$ and $\alpha_1(g) = 64s - 16l$.

Let Ω be an n-clique. If $n = 1$, then p divides 33 and 495. Therefore, $p = 3, 11$. If $p = 11$, we have $\chi_2(g) = \frac{15 - \alpha_2(g)}{32} + 15$. Therefore, $\alpha_2(g) = 143 + 352l$, where $l \le 2$. Furthermore, $\chi_1(g) = \frac{13 + 4\alpha_1(g) + 143 + 352l}{128} - 10$. Therefore, $\alpha_1(g) = 121 + 352t - 88l$. If $p = 3$, we have $\chi_2(g) = \frac{15 - \alpha_2(g)}{32} + 15$. Therefore, $\alpha_2(g) = 15 + 96l$, where $l \le 10$. Furthermore, $\chi_1(g) = \frac{13 + 4\alpha_1(g) + 15 + 96l}{128} - 10$. Therefore, $\alpha_1(g) = 96t + 57 - 24l$.

If $n = 2$, then $p = 2$, a contradiction to the fact that 495 is not divided by 2.

If $n = 4$, then p divides 30. For $p = 5$, we have $\chi_2(g) = \frac{60 - \alpha_2(g)}{32} + 15$. Therefore, $\alpha_2(g) = 60 + 160l$, where $l \le 6$. Furthermore, $\chi_1(g) = \frac{52 + 4\alpha_1(g) + 60 + 160l}{128} - 10$. Therefore, $\alpha_1(g) = 100 + 640t - 40l$. If $p = 3$, we have $\chi_2(g) = \frac{60 - \alpha_2(g)}{32} + 15$. Therefore, $\alpha_2(g) = 60 + 96l$, where $l \le 10$. Furthermore, $\chi_1(g) = \frac{52 + 4\alpha_1(g) + 60 + 96l}{128} - 10$. Therefore, $\alpha_1(g) = 384t + 228 - 24l$. If $p = 2$, we have $\chi_2(g) = \frac{60 - \alpha_2(g)}{32} + 15$. Therefore, $\alpha_2(g) = 60 + 64l$, where $l \le 15$. Furthermore, $\chi_1(g) = \frac{52 + 4\alpha_1(g) + 60 + 64l}{128} - 10$. Therefore, $\alpha_1(g) = 64t - 28 - 16l$.

Let Ω be neither an empty graph nor a clique, and let $p > 2$. Then $\lambda_\Omega = \mu_\Omega = 2$. Therefore, Ω is an amply regular graph of degree k' on v' vertices.

If $p \ge 17$, then $k' = 33 - p$ and $v' \ge 1 + (33 - p) + \frac{(33 - p)(30 - p)}{2}$. If $p = 29$, we have $k' = 4$. However, a connected locally quadrilateral graph is an octahedron, a contradiction to $\mu_\Omega = 2$. If $p = 23$, we have $k' = 10$ and $v' \ge 1 + 10 + \frac{10 \cdot 7}{2} + 12$ (495 – 12 is divided by 23), a contradiction to $|\Gamma - \Omega| \ge 58 \cdot 23$. If $p = 19$, we have $k' = 14$ and $v' \ge 1 + 14 + \frac{14 \cdot 11}{2}$, a contradiction to $|\Gamma - \Omega| \ge 92 \cdot 19$. If $p = 17$, we have $k' = 16$ and $v' \ge 1 + 16 + \frac{16 \cdot 13}{2}$, a contradiction to $|\Gamma - \Omega| \ge 121 \cdot 17$.

If $p = 13$, then Ω is a regular graph of degree 7 or 20. Furthermore, 495 – 1 is divided by 13. Let $a, b \in \Omega$ and $d(a, b) = 3$. Since $p^3_{33} = 236$, $\Omega_3(a) \cap \Omega_3(b)$ contains at least two vertices. If Ω is a graph of degree 20, then $v' \ge 1 + 20 + \frac{20 \cdot 17}{2} + 14$ and the number of edges between Ω and $\Gamma - \Omega$ is at least 205 · 13, a contradiction. If Ω is a graph of degree 7, then the degree of $\Omega_3(a)$ is 5 and $c_3(\Omega) = b_2(\Omega) = 2$. Therefore, Ω is a distance-regular graph with the intersection array $\{7, 4, 2; 1, 2, 2\}$, a contradiction to [5, Proposition 1.9.1].

If $p = 11$, then Ω is a regular graph of degree 11 or 22. If Ω contains a vertex of degree 22, then $v' \ge 1 + 22 + \frac{22 \cdot 19}{2}$, a contradiction to $1024 \ge |\Gamma| \ge 132 \cdot 11$. Therefore, Ω is a regular graph of degree 11, $v' \ge 1 + 11 + 44$, and the number of edges between Ω and $\Gamma - \Omega$ is at least 56 · 22, a contradiction.

If $p = 7$, then Ω is a regular graph of degree 5, 12, 19, or 26. Furthermore, 495 – 5 is divided by 7. Let $a, b \in \Omega$ and $d(a, b) = 3$. Since $p^3_{33} = 236$, we conclude that $\Omega_3(a) \cap \Omega_3(b)$ contains at least five vertices and $|\Omega_3(a)| \ge 12$. If Ω contains a vertex a of degree 5, then

Ω is an icosahedron graph, a contradiction to the fact that 1024 − 12 is not divided by 7.

If Ω contains a vertex of degree 26, then $v' \geq 1 + 26 + \frac{26 \cdot 23}{2} + 12 = 338$, a contradiction to $1024 \geq |\Gamma| \geq 338 \cdot 8$.

If Ω contains a vertex of degree 19, then $v' \geq 1 + 19 + \frac{19 \cdot 9}{2} + 12$, a contradiction to $1024 \geq |\Gamma| \geq 118 \cdot 15$.

Therefore, Ω is a regular graph of degree 12, $v' \geq 1 + 12 + 54 + 12 = 79$, and the number of edges between Ω and $\Gamma - \Omega$ is at least 79 · 21, a contradiction.

The lemma and the theorem are proved.

Lemma 4. *Let Ω be an amply regular graph with parameters $(v', k', 2, 2)$. Then the following assertions hold:*

(1) *The number p is not equal to* 5.

(2) *If $p = 3$, then the degree of Ω is equal to* 9 *and Ω is a distance-regular graph with intersection array $\{9, 6, 1; 1, 2, 9\}$ or the degree of Ω is equal to 6 and Ω is a strongly regular graph with parameters* $(16, 6, 2, 2)$.

Proof. Let Ω be an amply regular graph with parameters $(v', k', 2, 2)$.

Let $p = 5$. Then Ω is a regular graph of degree 3, 8, 13, 18, 23, or 28. If Ω contains a vertex a of degree 3, then Ω is a 4-clique, a contradiction.

If Ω contains a vertex of degree 28, then $v' \geq 1 + 28 + \frac{28 \cdot 25}{2}$, a contradiction to $1024 \geq |\Gamma| \geq 379 \cdot 5$.

The cases when Ω contains a vertex of degree 23, 18, or 13 are treated in a similar manner.

Thus, Ω is a graph of degree 8. The neighborhood of a vertex in Ω is an octagon or the union of a triangle and a pentagon. If $d(\Omega) = 2$, then Ω is a strongly regular graph with parameters (29, 8, 2, 2), a contradiction. Let $a, b \in \Omega$ and $d(a, b) = 3$. Since $p^3_{33} = 236$, we conclude that $\Omega_3(a) \cap \Omega_3(b)$ contains a vertex b' and the degree of the graph $\Omega_3(a)$ is equal to 3. Now $v' > 1 + 8 + \frac{8 \cdot 5}{2} = 29$ and $|\Omega \cap \Gamma_3(a)| \geq 10$. If $|\Omega \cap \Gamma_3(a)| \geq 15$, then $1024 \geq |\Gamma| \geq 44 \cdot 25$, a contradiction. Therefore, $|\Omega \cap \Gamma_3(a)| = 10$ and each vertex in $\Omega_2(a)$ is adjacent to no or five vertices from $\Omega_3(a)$. Let A be the set of vertices in $\Omega_2(a)$ that are adjacent to five vertices in $\Omega_3(a)$, and let B be the set of vertices from $\Omega_2(a)$ that are adjacent to no vertex in $\Omega_3(a)$. Then $|A| = |B| = 5$ and the number of edges between A and B is at most 5, a contradiction to the fact that B is a clique.

Let $p = 3$. Then Ω is a regular graph of degree 3, 6, 9, 12, 15, 18, 21, 24, 27, or 30. In view of [5, Proposition 1.9.1], we have $c_3(a, b) \geq 3$ for any vertices $a, b \in \Omega$ with $d(a, b) = 3$. If Ω contains a vertex a of degree 3, then Ω is a 4-clique, a contradiction.

If Ω contains a vertex of degree 30, then $v' \geq 1 + 30 + \frac{30 \cdot 27}{2}$, a contradiction to $1024 \geq |\Gamma| \geq 436 \cdot 4$.

The cases when Ω contains a vertex of degree 27, ..., 12 are considered in a similar fashion.

If Ω contains a vertex of degree 9, then $v' \geq 1 + 9 + \frac{9 \cdot 6}{2} = 37$ and the neighborhood of a vertex in Ω is a nonagon, the union of a hexagon and a triangle, or the union of three triangles. If $d(\Omega) = 2$, then Ω is a strongly regular graph with parameters (37, 9, 2, 2), a contradiction. Let $a, b \in \Omega$ and $d(a, b) = 3$. Since $p^3_{33} = 236$, we conclude that $\Omega_3(a) \cap \Omega_3(b)$ contains two vertices.

If $|\Omega_3(a)| \geq 6$, we have $1024 \geq |\Gamma| \geq 42 \cdot 25$, a contradiction. Therefore, Ω is a distance-regular graph with the intersection array $\{9, 6, 1; 1, 2, 9\}$.

If Ω is a graph of degree 6, then $v' \geq 1 + 6 + \frac{6 \cdot 3}{2} = 16$ and the neighborhood of a vertex in Ω is a hexagon or the union of two triangles. If $d(\Omega) = 2$, then Ω is a strongly regular graph with parameters (16, 6, 2, 2). Let $c \in \Omega_2(a)$. Then the vertices from $[a] \cap [c]$ are adjacent to distinct vertices from $[c]$. Therefore, $[c]$ does not intersect $\Omega_3(a)$.

Lemma 5. *If Γ is an edge-symmetric graph, then the following assertions hold:*

(1) *Ω does not contain $[a]$ for any vertex $a \in \Omega$.*

(2) *The solvable radical of the group G coincide with $O_2(G)$; if $Q = O_2(G) \neq 1$, then each involution from Q acts on Γ without fixed points and either $|Q| = 2^{10}$ or $|Q| < 2^8$.*

(2) *The socle $\bar{T}$ of the group $\bar{G} = G/O_2(G)$ is isomorphic to $L_2(11)$, M_{11}, M_{12}, or $U_5(2)$.*

Proof. Let $[a] \subset \Omega$ for some vertex $a \in \Omega$. Then, for any vertex $u \in \Gamma - \Omega$, we have $[u] \cap \Omega = [u] \cap [a] = \{c, d\}$, a contradiction to $\|[c] \cap [d]\| \geq 3$.

Since $v = 2^{10}$, the solvable radical of G coincides with $O_2(G)$. Let $Q = O_2(G) \neq 1$. Then Q_a is a normal subgroup of G_a that fixes a vertex $b \in [a]$. Since G_a is transitive on $[a]$, we have $[a] \subset \Omega$ for any involution $g \in Q_a$. By assertion (1), $Q_a = 1$. Assume that $1 < |Q| < 2^{10}$. Then the orbit a^Q is a coclique for any vertex $a \in \Gamma$. In a regular graph with the least eigenvalue $-m$, the order of a coclique is at most $\frac{vm}{k+m}$. Therefore, $|Q| = |a^Q| \le \frac{128 \cdot 7}{5}$.

Recall that, by the theorem, $\pi(G) \subseteq \{2, 3, 5, 11\}$ and, by assumption, $|G|$ is divided by $2^{10} \cdot 33$. In view of [6, Table 1], the group $\bar{T}$ is isomorphic to $L_2(11)$, M_{11}, M_{12}, or $U_5(2)$.

Let us complete the proof of the corollary. In the case of $L_2(11)$, we have $|O_2(G)| = 2^{10}$ (otherwise, $|O_2(G)| = 2^7$, $G/O_2(G) = PGL_2(11)$, and G does not contain subgroups of index 2^{10}), $G_a = L_2(11), PGL_2(11)$ does not contain subgroups of index 33. The cases of M_{11} and M_{12} are treated in a similar manner. In the case of $U_5(2)$, we have $|O_2(G)| = 2^{10}$ (otherwise, G does not contain subgroups of index 2^{10}), $G_a = U_5(2), \mathrm{Aut}(U_5(2))$ does not contain subgroups of index 33. The corollary is proved.

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REFERENCES

1. A. A. Makhnev and M. S. Nirova, Vestn. Sib. Feder. Univ. **35**, 181–183 (2014).
2. I. N. Belousov and A. A. Makhnev, *Abstracts of Papers of the International Conference on Algebra and Mathematical Logic: Theory and Applications* (Kazan, 2014), p. 7.
3. P. J. Cameron, *Permutation Groups* (Cambridge Univ. Press, Cambridge, 1999).
4. A. L. Gavrilyuk and A. A. Makhnev, Dokl. Math. **81**, 439–442 (2010).
5. A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs* (Springer-Verlag, Berlin, 1989).
6. A. V. Zavarnitsine, Sib. Electron. Math. Rep. **6**, 1–12 (2009).

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