Autoresonant excitation of dark solitons

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Continuouslyphase-locked (autoresonant) dark solitons of the defocusing nonlinear Schrodinger equation are excited and controlled by driving the system by a slowly chirped wavelike perturbation. The theory of these excitations is developed using Whitham's averaged variational principle and compared with numerical simulations. The problem of the threshold for transition to autoresonance in the driven system is studied in detail, focusing on the regime when the weakly nonlinear frequency shift in the problem differs from the typical quadratic dependence on the wave amplitude. The numerical simulations in this regime show a deviation of the autoresonance threshold on the driving amplitude from the usual 3/4 power dependence on the driving frequency chirp rate. The theory of this effect is suggested.

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I. INTRODUCTION

Dark solitons are fundamental objects in physics of nonlinear waves. They have been studied both experimentally and theoretically in different fields of physics, such as nonlinear optics [1], the physics of matter waves in condensates [2], and spin waves in magnets [3]. One of the main problems in applications is how to effectively generate dark solitons with predefined parameters and smallest remaining perturbations. The use of impulse or localized external fields is usually unsuitable to generate pure solitons. In the present work, we propose a method of exciting pure dark solitons by a periodic chirped frequency external field, allowing one to adiabatically control the amplitude of the soliton. The approach is based on the idea of autoresonance taking place in many nonlinear systems driven by perturbations with a slowly varying frequency passing through the linear resonance of the unperturbed system [4]. When the driving amplitude exceeds some threshold value, the phase of the excited nonlinear oscillations of the system locks to that of the drive, so the frequency of the driven oscillation follows the driving frequency continuously at later times. This continuing phase locking is achieved by an automatic self-adjustment of the amplitude of the excited oscillations yielding efficient and robust control of the excited coherent structure by varying the driving frequency. The autoresonance was first used in relativistic particle accelerators [5–7]. Much later it was realized that the autoresonance is a general phenomenon of nonlinear physics and recent applications exist in superconducting Josephson junctions [8], nonlinear optics [9], condensed matter [10], nonlinear waves [11,12], and more. Here, we apply the idea to excitation of dark solitons of the nonlinear Schrodinger (NLS) equation.

The presentation will be as follows. In Sec. II, we present the theory based on Whitham's averaged variational principle for describing the driven-chirped defocusing NLS equation. The results of this theory will allow one to understand the reasons beyond the autoresonance phenomenon in the present application. In Sec. III, we discuss our numerical simulations and compare the results with the theory. Finally, Sec. IV will present our conclusions.

II. AVERAGED VARIATIONAL THEORY

We study solutions of the adiabatically driven, defocusing NLS equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = \varepsilon \exp(i\varphi), \quad \varepsilon \ll 1 \tag{1}$$

where the driving phase $\varphi = kx - \int \omega_d(t)dt$ has constant k and a slowly chirped frequency $\omega_d = \omega_r + \alpha t, \omega_r =$ constant. The driving term of this form can appear, for example, in the problem of two resonant, weakly coupled nonlinear waves [9], where one of the waves plays a role of a prescribed, chirped frequency drive. Various phase locked solutions of the driven NLS equation with *constant* frequency drive are known (see [13] and references therein). Here, we study the formation and subsequent evolution of dark solitons from constant amplitude initial conditions as the result of a *chirped* frequency drive. We solve Eq. (1) subject to the initial condition $\psi(x,t_0) = U_0 =$ constant at some large negative t_0 and periodic boundary condition $\psi(x+\lambda,t) = \psi(x,t)$, $\lambda = 2\pi/k$. A weakly nonlinear limit of this problem was studied in [12] and here we extend the theory to the fully nonlinear case. The aforementioned initial condition can be obtained by starting at earlier times from $\psi \equiv 0$ and driving the system by purely time dependent, chirped frequency oscillation $\varepsilon_0 \exp[-i \int \omega_d(t)dt]$ [12]. We seek an adiabatically varying solution of (1) in the form

$$\psi = U \exp(i\theta), \tag{2}$$

where the amplitude U and phase θ are real and, thus, satisfy

$$U_t + 2U_x\theta_x + U\theta_{xx} = -\varepsilon \sin \Phi, \tag{3}$$

$$U\theta_t - U_{xx} + U\theta_x^2 + 2U^3 = -\varepsilon \cos \Phi, \tag{4}$$

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with $\Phi = \theta - \varphi$. This system can be also obtained from the variational principle $\delta[\int L dt dx] = 0$ with the Lagrangian $L = L_0 + L_1$, where

$$L_0 = \tfrac{1}{2}\left[U_x^2 + U^2\left(\theta_x^2 + \theta_t\right) + U^4\right] \tag{5}$$

and

$$L_1 = \varepsilon U \cos\Phi. \tag{6}$$

At this stage, in view of small ε and the slowly varying driving frequency, we seek two-scale-type solution in the problem $U = U(\Theta, t)$ and $\theta = \xi + V(\Theta, t)$, where $\Theta = kx - \int \Omega dt$ and $\xi = -\int \Omega_0 dt$ are *fast* phase variables, both U and V are 2π periodic in Θ, and the explicit time dependence in U and V indicates *slow* time variation in the driven problem, while Ω and Ω_0 are *slowly* varying internal and external frequencies, respectively. With these definitions, the perturbing Lagrangian becomes

$$L_1 = \frac{\varepsilon}{2}\left(Ue^{iV}e^{i(\xi-\varphi)} + \text{c.c.}\right).$$

Here, we expand in Fourier series $Ue^{iV} = \Sigma_n \alpha_n \exp(in\Theta)$ and leave only one *resonant* contribution in L_1 (single resonance approximation) corresponding to a continuing resonance $\omega_d(t) \approx \Omega_0(t) + \Omega(t)$ in the system, i.e.,

$$L_{1r} = \varepsilon\rho\cos\Psi, \tag{7}$$

where $\Psi = \Theta + \xi - \varphi + \chi = \int(\omega_d - \Omega_0 - \Omega)dt + \chi$ and $\rho = |\alpha_1|$, $\chi = \arg(\alpha_1)$.

Next, we proceed to Whitham's averaging [14]. The idea of using this approach in studying autoresonant wave problems was suggested in [15] and applied to the driven sine-Gordon equation. It was later generalized to two component autoresonant waves [16] in an application to the driven Korteweg–de-Vries problem. Here we use a similar approach in studying autoresonant excitation of spatially modulated solutions of the defocusing NLS equation. We proceed by writing $\theta_x = V_x = kV_\Theta$ and $\theta_t = -\Omega_0 - \Omega V_\Theta = -\Omega_0 - (\Omega/k)V_x$ and substitute these relations in the unperturbed Lagrangian (5)

$$L_0 = \frac{1}{2}\left[U_x^2 + U^2V_x^2\right] - \frac{\Omega}{2k}U^2V_x - \frac{\Omega_0}{2}U^2 + \frac{1}{2}U^4. \tag{8}$$

If one fixes the slow time in the problem, this Lagrangian describes the dynamics (in x) of a two degrees of freedom (U, V) problem. Note that V in this problem is a cyclic variable, and, therefore, the canonical momentum $P_V = \partial L_0/\partial V_x = U^2(V_x - \frac{\Omega}{2k}) \equiv B$ is conserved. Then, in the perturbed problem, B becomes a slow function of time. We define the second canonical momenta $P_U = \partial L_0/\partial U_x = U_x$ and write the Hamiltonian $H_0 = P_U U_x + P_V V_x - L_0$, which after some algebra yields another slow variable in the perturbed problem

$$H_0 = \frac{1}{2}\left(P_U^2 + \frac{B^2}{U^2}\right) + \frac{R}{2}U^2 - \frac{1}{2}U^4 + \frac{\Omega}{2k}B, \tag{9}$$

where $R = \frac{1}{4}(\frac{\Omega}{k})^2 + \Omega_0$. It is convenient to use a different slow variable $A = H_0 - \frac{\Omega}{2k}B$ instead of H_0 and write it as

$$A = \tfrac{1}{2}P_U^2 + V_{\text{eff}}(U), \tag{10}$$

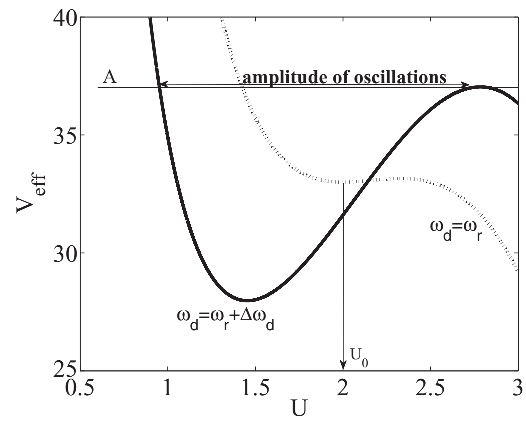

FIG. 1. The effective potential $V_{\text{eff}}(U)$ at two stages of autoresonant excitation: (dotted line) as the driving frequency passes the linear resonance $\omega_d = \omega_r$ and (thick solid line) after the driving frequency shifts by $\Delta\omega_d = 1.62$. The horizontal line in the latter case shows the location of the quasienergy A at this stage.

where

$$V_{\text{eff}} = \frac{B^2}{2U^2} + \frac{R}{2}U^2 - \frac{1}{2}U^4. \tag{11}$$

This allows one to interpret the motion of U as that of a quasiparticle in a slowly varying effective potential, while A plays the role of the slow energy of the particle. In our driven problem, one proceeds from equilibrium (at the minimum of the potential), where $U = U_0$ and excites autoresonant (phase-locked) oscillations of U as both A and B slowly vary in time. Note that, initially, $V'_{\text{eff}}(U_0) = 0$, which yields initial $B^2 = U_0^4(R - 2U_0^2)$ and

$$V''_{\text{eff}}(U_0) = 4R - 12U_0^2 = k^2 \tag{12}$$

(k is the frequency of spatial oscillations of U). Since $\Omega_0 = 2U_0^2$ initially, Eq. (12) yields the initial frequency $\Omega = k(k^2 + 4U_0^2)^{1/2}$. Then, the resonant driving frequency in the problem is

$$\omega_r = \Omega_0 + \Omega = 2U_0^2 + k\left(k^2 + 4U_0^2\right)^{1/2}. \tag{13}$$

We illustrate the effective potential $V_{\text{eff}}(U)$ in Fig. 1 by the dashed line for the case $k = 3$ and $U_0 = 2$ when the quasiparticle is in equilibrium (at $U = U_0$). If the driving frequency passes ω_r, then, beyond the resonance ($\omega_d = \omega_r + \alpha t$), under certain conditions, we expect excitation of autoresonant oscillations of U. The full line in Fig. 1 shows $V_{\text{eff}}(U)$ as it develops in such autoresonant evolution at the time when $\Delta\omega_d = \alpha t = 1.62$ (the details of this calculation will be described in the following paragraph). The quasienergy at this time reaches the value shown by the thin horizontal line in the figure illustrating excitation of large amplitude oscillations of U at this time.

To describe the autoresonant evolution of A and the slow parameters B,Ω, and Ω_0 defining the quasipotential in the driven system, we need to find the slow Lagrangian and use Whitham's averaging [14] to accomplish this goal. To this end, we rewrite L_0 in terms of the slow variables A and B first:

$$L_0 = P_U U_x + BV_x - H_0 = -A - \frac{\Omega}{2k}B + P_U^2 + BV_x. \tag{14}$$

Next, according to Whitham's procedure [14], we fix the slow time in the problem and average the Lagrangian over Θ between zero and 2π. The resulting averaged Lagrangian is

$$\Lambda_0 = \frac{1}{2\pi}\int_0^{2\pi} L_0 d\Theta = -A - \frac{\Omega}{2k}B + kJ(A,B,R), \quad (15)$$

where

$$J(A,B,R) = \frac{1}{2\pi}\int_0^{2\pi} P_U^2 d\Theta = \frac{k}{2\pi}\int_0^{2\pi} P_U U_\Theta d\Theta$$
$$= \frac{k}{2\pi}\int \sqrt{2[A - V_{\text{eff}}(B,R,U)]}dU. \quad (16)$$

We also average the perturbing resonant Lagrangian (7), assuming that the phase mismatch Ψ is slow:

$$\Lambda_{1r} = \varepsilon\rho\cos\Psi.$$

Finally, we use the full averaged Lagrangian $\Lambda = \Lambda_0 + \Lambda_{1r}$ to write the variational evolution equations for the slow dependent variables. The variation with respect to A and B yields

$$kJ_A - 1 + \varepsilon(\rho_A\cos\Psi - \rho\chi_A\sin\Psi) = 0, \quad (17)$$

$$kJ_B - \frac{\Omega}{2k} + \varepsilon(\rho_B\cos\Psi - \rho\chi_B\sin\Psi) = 0. \quad (18)$$

Similarly, the variation with respect to Θ and $\xi = -\int\Omega_0 dt$ gives additional two equations

$$\frac{d}{dt}\left(kJ_\Omega - \frac{B}{2k}\right) = \varepsilon\rho\sin\Psi, \quad (19)$$

$$\frac{d}{dt}\left(kJ_{\Omega_0}\right) = \varepsilon\rho\sin\Psi. \quad (20)$$

The last four equations describe slow evolution of A,B,Ω, and Ω_0 in the problem. Note that Eqs. (19) and (20) yield a conservation law

$$J_{\Omega_0} - J_\Omega + \frac{B}{2k^2} = C,$$

which can be used instead of one of the differential equations above. The weakly nonlinear limit in Whitham's problem was studied in [12] demonstrating stable autoresonant excitation in the problem. Here, we generalized to fully nonlinear evolution. Nonetheless, we will not address the stability problem in this work, but rather the question of the existence of the large amplitude autoresonant state. Consequently, we take the $\varepsilon \to 0$ limit and consider the system of the following *algebraic* equations:

$$kJ_A - 1 = 0, \quad (21)$$

$$kJ_B - \frac{\Omega}{2k} = 0, \quad (22)$$

$$J_{\Omega_0} - J_\Omega + \frac{B}{2k^2} = C. \quad (23)$$

This system, with the addition of the autoresonant condition $\omega_d = \omega_r + \alpha t = \Omega_0 + \Omega$ allows one to find A,B,Ω_0, and Ω as functions of time. In turn, the knowledge of A,B,Ω_0, and Ω at each time yields the spatial form of U and V by simple integration (in x) of the motion of the quasiparticle in the known effective potential. Figure 2 shows the results of such calculations for the amplitude $u = \max|\psi(x)| - \min|\psi(x)|$ of the excited wave over one spatial oscillation versus time (the dashed line) for two sets of parameters. The results are compared with those from the numerical solution of Eq. (1) represented by the solid line (see the next Section for the details of the simulations). One observes an excellent agreement between the two results in the advanced autoresonance stage. The theoretical curves are based on the solution of the variational equations (21)–(23) under the ideal phase-locking assumption and do not include adiabatic modulations around the quasisteady state. The presence of these oscillations in numerics demonstrates modulational stability of the autoresonant solution. The usual adiabaticity condition for the applicability of Whitham's approach in problems involving nonresonant waves is $\frac{1}{P\Omega}\frac{dP}{dt} \ll 1$, where P is a slowly varying parameter in the problem. In contrast, in our driven problem, since the slow modulations are associated with the nonlinear resonance, the adiabaticity condition on the chirped frequency of the drive is stricter, i.e., $\frac{1}{\omega\nu}\frac{d\omega}{dt} \ll 1$, where ν is the frequency of the modulations. If this condition is violated, we expect the escape from the autoresonance and dephasing.

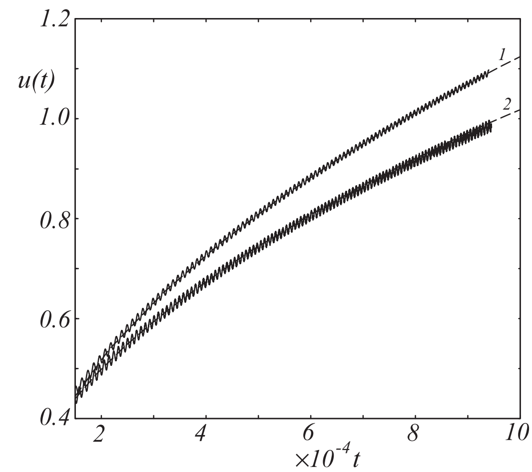

FIG. 2. Comparison of the variational theory (dashed lines) with the numerical solution of the NLS equation (solid lines). Line 1: $U_0 = 2$, $k = 3$, $\varepsilon = 4.85 \times 10^{-5}$, and $\alpha = 6 \times 10^{-6}$; line 2: $U_0 = 1$, $k = 1$, $\varepsilon = 6 \times 10^{-5}$, and $\alpha = 6 \times 10^{-6}$.

III. AUTORESONANT DARK SOLITONS IN NUMERICAL SIMULATIONS

We solve the driven NLS equation (1) numerically using a standard spectral method (see, e.g., [17]). We start with the initial condition $\psi(x,t_0) = U_0$ at some large negative $t = t_0$, such that the driving frequency $\omega_d = \omega_r + \alpha t$ crosses the resonant frequency ω_r (13) at $t = 0$. A typical process of autoresonant excitation of a dark soliton is illustrated in Figs. 3–5. After crossing the linear resonance, the fundamental harmonic of wave number k amplifies, resulting in a significant modulation of the initially homogeneous wave similar to a dark soliton as shown in Fig. 3 for $|\psi(x)|$. The phase of the

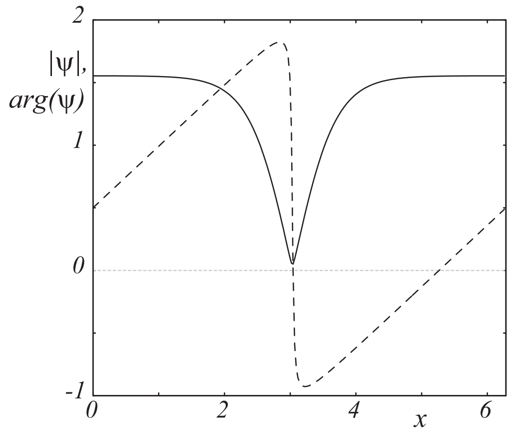

FIG. 3. The amplitude $|\psi(x)|$ (solid line) and the phase $\arg[\psi(x)]$ (dashed line) of the dark soliton at $t = 416$; $U_0 = 1, k = 1, \varepsilon = 0.005$, and $\alpha = 0.003$.

solution $\arg[\psi(x)]$ is also shown in Fig. 3, illustrating a typical stepwise behavior of the phase of a dark soliton. We found that the difference between the amplitude of the soliton in this example and that for the exact dark soliton [1] with the same parameters is less than 1%, but the phase of the numerical solution is approximated well by the exact formula in the core of the soliton only, because of the periodic boundary conditions in our simulations. Importantly, the formation of the soliton depends critically on the driving amplitude ε. Figure 4 shows the depth of the soliton, $\min|\psi(x)|$, versus time. One can see that the depth saturates if the driving parameter ε is below some threshold value ε_{th} and, in the opposite case $\varepsilon > \varepsilon_{th}$, the hole deepens because of autoresonance and $\min|\psi(x)|$ approaches zero. We also found in simulations that if one switches off the drive at some time, the excited soliton preserves its form for a very long time. In Fig. 4 we show an example (line 3), where the drive is switched off at $t = 400$. The soliton was preserved up to $t = 2000$ in these simulations. The main feature of autoresonance is the phase locking between the excited wave and the drive. This effect is illustrated in Fig. 5

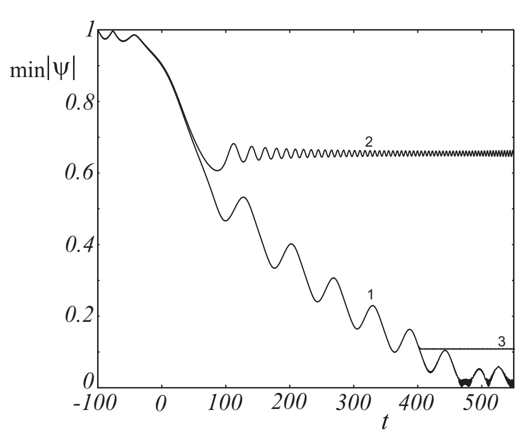

FIG. 4. The depth $\min|\psi|$ of the dark soliton versus time. Line 1: $\varepsilon = 0.005$, line 2: $\varepsilon = 0.0048$, line 3: soliton depth after switching off the drive at $t = 400$. $U_0 = 1, k = 1$, and $\alpha = 0.003$. The threshold value is $\varepsilon_{th} \approx 0.0049$.

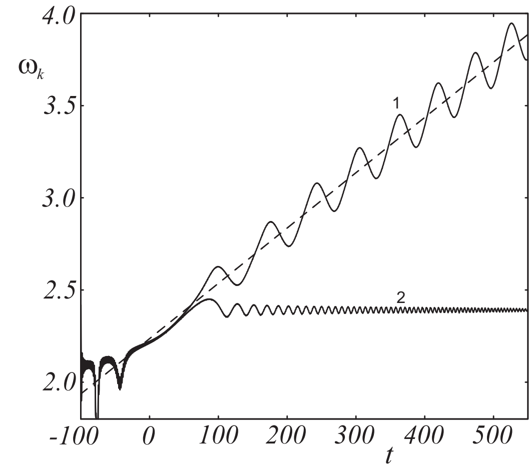

FIG. 5. The soliton frequency ω_k: solid line 1: $\varepsilon = 0.005$, solid line 2: $\varepsilon = 0.0048$. The dashed line shows the driving frequency ω_d. $U_0 = 1$, $k = 1$, and $\alpha = 0.003$.

showing the frequency of the Fourier harmonic associated with the fundamental wave number k:

$$\psi_k(t) = \frac{1}{\lambda}\int_0^{\lambda} \psi(x,t)e^{-ikx}dx.$$

Using the Fourier expansion $\psi = \sum a_n e^{i(n\Theta+\xi)}$ of the solution (2) in terms of Θ, we have $\psi_k(t) \approx a_1(t)\exp[-i(\int(\Omega_0 + \Omega)dt)]$. We use this result to calculate the frequency of $\psi_k(t)$ numerically via

$$\omega_k \approx \Omega_0 + \Omega = -\mathrm{Im}\left(\frac{d\ln\psi_k(t)}{dt}\right).$$

One can see in Fig. 5 (line 1) that for $\varepsilon > \varepsilon_{th}$, the autoresonance $\omega_d(t) \approx \Omega_0 + \Omega$ is preserved, as the system self-adjusts Ω_0 and Ω continuously, while the driven and driving oscillations remain phase locked. In the opposite case, $\varepsilon < \varepsilon_{th}$, the phase locking is destroyed shortly after crossing the linear resonance at $t \approx 0$ (line 2).

In the initial stage of the autoresonance, when the amplitude of the excited wave is still small, the system is described by the following set of variational equations [12]:

$$u_t = -\tilde{\varepsilon}\sin\Phi\ , \tag{24}$$

$$\Phi_t = \omega_d(t) - \Omega_0(u) - \Omega(u) - \frac{\tilde{\varepsilon}}{u}\cos\Phi, \tag{25}$$

where $u = U - U_0$, $\Phi = \theta - \phi$ is the mismatch between of excited and driving waves, $\tilde{\varepsilon} = \varepsilon(1+\rho^{-1})$, and $\rho^2 = 1 + 4U_0^2/k^2$. Furthermore,

$$\Omega_0 + \Omega \approx \omega_r + Q_2u^2 + Q_3u^3 + \cdots\ , \tag{26}$$

where the last two terms describe the nonlinear frequency shift of the excited wave. If $Q_2 \neq 0$, the leading nonlinearity of the frequency is quadratic. This case has been studied in Ref. [12]. Here, we focus on a special case when $Q_2 \approx 0$ and the leading nonlinearity may have a higher power. Bearing in mind this possibility, we consider the approximation of the frequency of

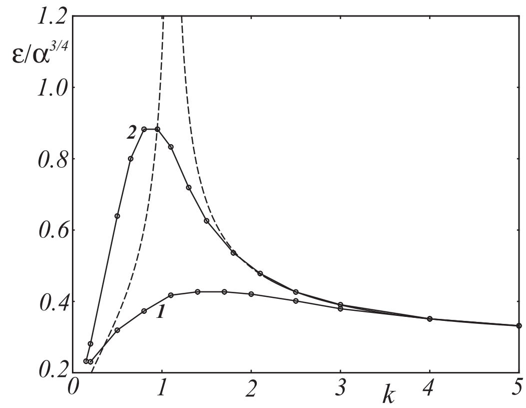

FIG. 6. The threshold ε_{th} versus k. The dashed line–Eq. (32); the solid lines–the numerical solution of the NLS equation: line 1: $\alpha = 10^{-2}$, line 2: $\alpha = 6 \times 10^{-6}$. $U_0 = 2$.

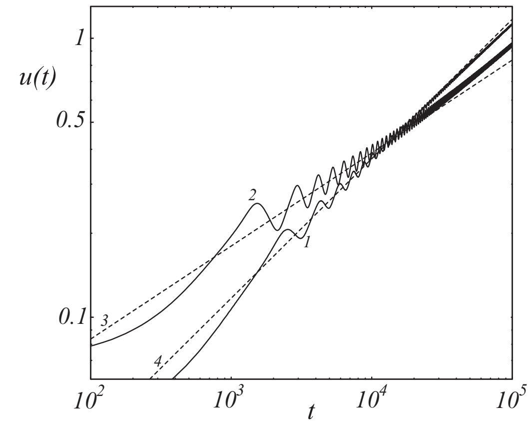

FIG. 7. The amplitude $u(t)$ versus time. Solid line 1: $U_0 = 2$, $k = 3$, $\varepsilon = 4.85 \times 10^{-5}$, and $\alpha = 6 \times 10^{-6}$. The dashed line 4 shows $u = 0.0037 t^{1/2}$. Solid line 2: $U_0 = 2$, $k = 1.1$, $\varepsilon = 1.07 \times 10^{-4}$, and $\alpha = 6 \times 10^{-6}$. The dashed line 3 shows $u = 0.018 t^{1/3}$.

the form

$$\Omega_0 + \Omega = \omega_r + Q u^m , \tag{27}$$

where m can be any integer. Then, by introducing new variables, $\tau = \alpha^{1/2} t$ and $\bar{u} = Q^{1/m} \alpha^{-1/2m} u$, we obtain modified equations

$$\bar{u}_\tau = -\mu \sin \Phi , \tag{28}$$

$$\Phi_\tau = \tau - \bar{u}^m - \frac{\mu}{\bar{u}} \cos \Phi, \tag{29}$$

depending on a single parameter

$$\mu = \tilde{\varepsilon} Q^{1/m} \alpha^{-1/2m - 1/2} \tag{30}$$

proportional to the amplitude of the drive ε. The thresholds for the autoresonance in this system can be found numerically. If $m = 2$, the threshold is $\mu_{th} \approx 0.411$, while for $m = 3$, $\mu_{th} \approx 0.355$. For $m = 2$, the coefficient Q has been found analytically in Ref. [12]:

$$Q = Q_2 = (4 - \rho - \rho^{-1})/4, \tag{31}$$

and using (30) one obtains

$$\varepsilon_{th} = \frac{0.411 \, \alpha^{3/4} \rho}{(1 + \rho)|Q_2|^{1/2}} . \tag{32}$$

Equation (32) yields the threshold for the transition to autoresonance, except in the singular region near $Q_2 \approx 0$, i.e., at $\rho \approx 3.73$. As an example, Fig. 6 shows the dependence of ε_{th} on the wave number k (the dashed line) for $U_0 = 2$. The singularity in this case occurs near $k \approx 1.1$, where $\rho(k) \approx 3.73$. Solid lines in Fig. 6 show the threshold as obtained in the numerical solution of the NLS equation (1). One can see that sufficiently far from the singular region, Eq. (32) is in a good agreement with numerical simulations. However, near the singularity, the numerical results differ significantly from the theoretical prediction of Eq. (32). In the case of small Q_2, high order nonlinear terms in the nonlinear frequency shift dominate, which can be modeled by $m > 2$ in Eq. (27). For finding this exponent, we can use the salient property of the autoresonant phase locking, such that the frequency of the excited wave follows the frequency of the drive, $\omega_d \approx \Omega_0 + \Omega$. Then, since $\omega_d = \omega_r + \alpha t$, Eq. (27) yields the dependence of the amplitude of the driven wave on time:

$$u(t) \approx (\alpha/Q)^{1/m} \, t^{1/m}, \tag{33}$$

which can be tested numerically. Let us consider the nonsingular case first ($k = 3$, $U_0 = 2$). We show the amplitude versus time in this case in the logarithmic scale in Fig. 7 (line 1). One can see that for sufficiently small amplitudes, $u < 0.5$, the amplitude satisfies Eq. (33) with $m = 2$. The value of Q can be also found from the Figure, i.e., $Q \approx 0.438$, which is very close to $Q_2 = 0.433$ given by the theory in Eq. (31). Similarly, in the singular region ($k = 1.1$, $U_0 = 2$), we find the nonlinear frequency shift exponent $m = 3$ (see Fig. 7, line 2) and $Q = Q_3 \approx 1.029$. Then, the threshold value of ε from Eq. (30) is

$$\varepsilon_{th} = \frac{0.355 \, \alpha^{2/3} \rho}{(1 + \rho)|Q_3|^{1/3}} . \tag{34}$$

The dependence of ε_{th} on the chirp rate α, found by direct solution of NLS equation in the singular region, is compared with the prediction of Eq. (34) in Fig. 8. Except very small α, the results are close to those from Eq. (34).

IV. CONCLUSIONS

In conclusion, we have studied the excitation of dark solitons of the defocusing nonlinear Schrodinger equation by passage through resonance with a slowly chirped wavelike perturbation. The excited wave comprises a two-phase NLS excitation and the relevant resonance in the problem is that between the sum $\Omega_0 + \Omega$ of the external and internal frequencies of the NLS wave and the chirped frequency $\omega_d(t)$ of the drive. The system enters the autoresonant excitation stage after the driving frequency passes the linear resonance, provided the driving amplitude exceeds a threshold. Above this threshold, the phase locking is preserved continuously due to the self-adjustment of the slow parameters in the system to stay in the aforementioned resonance with the drive. The theory of these autoresonant excitations is developed using Whitham's averaged variational principle. The idea is based on

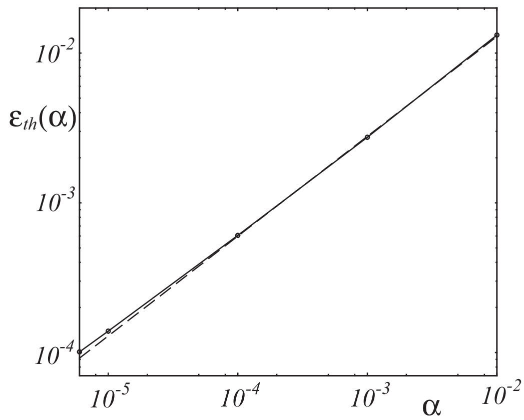

FIG. 8. The threshold $\varepsilon_{th}(\alpha)$ versus α in the singular region, $U_0 = 2$, $k = 1.1$. The dashed line represents Eq. (34); the solid line is from the numerical solution of the NLS equation.

transforming the problem to that of the motion of a quasiparticle in an effective potential with slowly varying parameters. Whitham's averaged Lagrangian yields variational equations describing the evolution of these parameters and, thus, also the evolution of the dark solitons in the driven system. We have solved these equations under the assumption of the ideal phase locking in the system and found an excellent agreement with the numerical solutions of the original driven NLS equation. The problem of the threshold on the driving amplitude for entering the autoresonant regime was also studied both numerically and in theory. The driven, defocusing NLS problem allows for the case when the weakly nonlinear frequency shift of the driven wave differs from the quadratic dependence on the wave amplitude. In this case, the usual scaling of the autoresonance threshold on the driving frequency chirp rate α deviates from the usual $\alpha^{3/4}$ dependence. The theory of this phenomenon was suggested. It seems interesting to further develop Whitham's variational theory in the problem and go beyond the ideal phase-locking assumption. Our numerical simulations show slow oscillations of the autoresonant solution around an ideally phase-locked solution as long as the driving amplitude is above the threshold, indicating stability in the problem. The theory of this stability comprises an important goal for future research. In addition, a parametric-type driving term of form $V(x,t)\psi(x,t)$ appears in some NLS problems (e.g. Bose-Einstein condensates in oscillating trapping potentials). Studying autoresonant solutions in these cases is another interesting goal for the future. Finally, application of autoresonant ideas to excitation and control of more complex, multiphase solutions of both the focusing and defocusing NLS equations and using other boundary conditions also seem to be important future directions.

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