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On expressive power of basic modal intuitionistic logic as a fragment of classical FOL

Grigory K. Olkhovikov [a,b,*]

[a] Dept of Philosophy II, Ruhr-Universität Bochum, Germany
[b] Dept. of Philosophy, Ural Federal University, Russian Federation

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A B S T R A C T

The modal characterization theorem by J. van Benthem characterizes classical modal logic as the bisimulation invariant fragment of first-order logic. In this paper, we prove a similar characterization theorem for intuitionistic modal logic. For this purpose we introduce the notion of modal asimulation as an analogue of bisimulations. The paper treats four different fragments of first-order logic induced by their respective versions of Kripke-style semantics for modal intuitionistic logic. It is shown further that this characterization can be easily carried over to arbitrary first-order definable subclasses of classical first-order models.

1. Introduction

It is well known that Kripke semantics uses a rather limited set of first-order tools (including, first and foremost, restricted quantification over worlds) to interpret the language of basic modal logic. This allows to embed basic modal logic into the first-order logic (FOL) via the so called *standard translation* encoding a given modal propositional formula by a first-order one with a single free variable (this variable represents a world in which this modal formula is supposed to be true). In this way, basic modal logic is shown to correspond to a specific (so called *modal*) fragment of FOL which, in turn, is often called in this context the *correspondence language* for modal logic. Even though it is relatively easy to find out if a given first-order formula is a standard translation of a modal formula, it is not so easy to say when a first-order formula can be *defined* by a translation of a modal formula, which would amount to a description of expressive power of the modal fragment of FOL. A well-known answer to this more difficult question is given by the modal characterization theorem[1] proved by J. van Benthem. The answer is that a first-order formula with a single

* Correspondence to: Ruhr Universität Bochum, Dept of Philosophy II; Universitätstr. 150, D-44780 Bochum, Germany.
E-mail addresses: grigory.olkhovikov@rub.de, grigory.olkhovikov@gmail.com.

[1] See, e.g. [3, Ch.1, Th. 13]. Another standard reference is [4], where the modal characterization theorem is proved as Theorem 2.68 in a very detailed and accessible way.

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free variable is equivalent to a standard translation of a modal propositional formula iff it is invariant w.r.t. a special class of binary relations between models called bisimulations.

The modal characterization theorem has become a template for many other similar results characterizing expressive power of logics other than basic modal one. The developments based on the original result by J. van Benthem were of the following main types. First, one could modify the definition of bisimulation itself. For instance, once one understands the machinery behind the modal characterization theorem, it is fairly easy to design a version of bisimulation for the modal logic with universal modality. The second type of modification involves switching to another correspondence language. For example, in [8] modal μ-calculus is characterized as the bisimulation invariant fragment of monadic second-order logic. Finally, as the third type of modification let us note attempts (with varying degrees of success) to carry over the whole task of characterizing expressivity into a non-Kripkean semantic context and then to find there notions similar to bisimulation. A case in point here would be the so called topo-bisimulations.

Of course, these types are not supposed to be mutually exclusive, for instance, capturing the expressive powers of basic hybrid logic using bisimulations-with-names would be an example of result belonging both to the first and to the second type. An interested reader may find a survey of these as well as further results of this kind together with numerous references to the existing literature on this subject in [3, Ch. 1, Section 6 and Ch. 6, Section 5].

For some years, our own efforts in this direction were centered on obtaining versions of the modal characterization theorem for intuitionistic propositional logic and its extensions. We came up with the term *asimulation* as an umbrella term for modifications of bisimulation required to handle intuitionistic formalisms. It was shown in [10] and [11] that both intuitionistic first-order logic and its propositional fragment, viewed as different fragments of classical first-order logic, admit of a full analogue of modal characterization theorem where invariance with respect to bisimulations is replaced with invariance with respect to asimulations or first-order asimulations, respectively. The present paper extends these results onto the main versions of basic modal intuitionistic logic. This further addition to the set of existing versions of van Benthem-like characterizations answers a question which is both natural and non-trivial. Indeed, intuitionistic modalities are a kind of modalities, so one would expect that a version of modal characterization theorem can be proven for them as well. On the other hand, at least some of intuitionistic modalities involve a rather peculiar pattern of restricted quantification so that the problem of finding a suitable characterization of their expressive power does not seem to be very easy.

The layout of the paper is as follows. Section 2 starts with notational conventions, after which we introduce the main variants of Kripke-style semantics for the basic modal intuitionistic system.

In Section 3 we give a summary of the ideas and results from [10] which provide a common basis for all the results and constructions presented in the main part of the present paper.

This main part begins with Section 4, which contains definitions for all the four variants of extension of basic asimulation notion to be employed in semantic characterizations of their respective variants of basic modal intuitionistic logic. We also formulate here the main results of the paper, although their proofs are postponed till Sections 5 and 6. Then Section 7 is devoted to characterization of modal intuitionistic fragments of FOL modulo first-order definable classes of models. Section 8 gives conclusions and drafts directions for future work.

2. Preliminaries

In this section, we first specify the main notational conventions to be used below and then briefly introduce intuitionistic modal logic with its four attending versions of Kripke semantics.

2.1. Notation

A *formula* is a formula of classical predicate logic without identity whose predicate letters are in vocabulary $\Sigma = \{ R^2, R^2_\Box, R^2_\Diamond, P^1_1, \dots P^1_n, \dots \}$, where the superscripts indicate the arities. We assume $\{ \bot, \top, \to, \vee, \wedge, \forall, \exists \}$ as the set of basic connectives and quantifiers for this variant of classical first-order language, which we call *the correspondence language*. A model is a classical first-order model of the correspondence language. We refer to correspondence formulas with lower-case Greek letters φ, ψ, and χ, and to sets of correspondence formulas with upper-case Greek letters Γ and Δ. If φ is a correspondence formula, then we associate with it the following finite vocabulary $\Sigma_\varphi \subseteq \Sigma$ such that $\Sigma_\varphi = \{ R^2, R^2_\Box, R^2_\Diamond \} \cup \{ P_i \mid P_i \text{ occurs in } \varphi \}$. More generally, we refer with Θ to an arbitrary subset of Σ such that $\{ R^2, R^2_\Box, R^2_\Diamond \} \subseteq \Theta$. If ψ is a formula and every predicate letter occurring in ψ is in Θ, then we call ψ a Θ-formula.

We refer to sequence $x_1, \dots, x_n$ of any objects as $\bar{x}_n$. We identify a sequence consisting of a single element with this element. If all free variables of a formula φ (formulas in Γ) coincide with a variable x, we write $\varphi(x)$ ($\Gamma(x)$).

By *degree* of a classical first-order formula we mean the greatest number of nested quantifiers occurring in it. The degree of a formula φ is denoted by $r(\varphi)$. Its formal definition by induction on the complexity of φ goes as follows:

$$
\begin{aligned}
& r(\bot) = r(\top) = r(\varphi) = 0 && \text{for atomic } \varphi;\\
& r(\varphi \circ \psi) = max(r(\varphi), r(\psi)) && \text{for } \circ \in \{ \wedge, \vee, \to \};\\
& r(Qx\varphi) = r(\varphi) + 1 && \text{for } Q \in \{ \forall, \exists \}.
\end{aligned}
$$

For $k \in \mathbb{N}$, we say that the Θ-formula $\varphi(x)$ such that $r(\varphi) \leq k$ is a (Θ, x, k)-formula.

For a binary relation S and any objects s, t we abbreviate the fact that $s\ S\ t \wedge t\ S\ s$ by $s \overset{\leftrightarrow}{S} t$

We use the following notation for models of classical predicate logic:

$$M = \langle U, \iota \rangle, M_1 = \langle U_1, \iota_1 \rangle, M_2 = \langle U_2, \iota_2 \rangle, \dots, M' = \langle U', \iota' \rangle, M'' = \langle U'', \iota'' \rangle, \dots,$$

where the first element of a model is its domain and the second element is its interpretation of predicate letters. If $k \in \mathbb{N}$ then we write R_k ($R_{\Box k}$, $R_{\Diamond k}$) as an abbreviation for $\iota_k(R)$ ($\iota_k(R_\Box)$, $\iota_k(R_\Diamond)$). If $a \in U$ then we say that (M, a) is a pointed model. Further, we say that $\varphi(x)$ is true at (M, a) and write $M, a \models \varphi(x)$ iff for any variable assignment α in M such that $\alpha(x) = a$ we have $M, \alpha \models \varphi(x)$. It follows from this convention that the truth of a formula $\varphi(x)$ at a pointed model is to some extent independent from the choice of its only free variable. Moreover, for $k \in \mathbb{N}$ we will sometimes write $a \models_k \varphi(x)$ instead of $M_k, a \models \varphi(x)$.

A modal intuitionistic formula is a formula of modal intuitionistic propositional logic, where $\{ \bot, \top, \to, \vee, \wedge, \Box, \Diamond \}$ is the set of basic connectives and modal operators, and $\{ p_n \mid n \in \mathbb{N} \}$ is the set of propositional letters. An intuitionistic formula is a modal intuitionistic formula without any occurrences of either box or diamond. We refer to intuitionistic formulas, both modal and non-modal, with letters I, J, K, possibly with primes or subscripts.

2.2. Definitions of basic modal intuitionistic logic

There exist different versions of basic system of modal intuitionistic logic. In this paper we only consider versions that have a Kripke-style semantics associated with them, and we will view these versions via the lens of their respective Kripke-style semantics.

Quite naturally, a Kripke-style semantics for a given version of intuitionistic modal logic is normally built as an extension of the Kripke semantics for basic intuitionistic propositional logic; that is to say, the models extend the Kripke models for basic propositional logic and the satisfaction clauses for $\bot$, $\top$, $\to$, $\vee$, $\wedge$ are left unchanged (cf. [9, Definition 7.1 and Definition 7.2]).

The new components in the models are one or more additional binary relations between states which are needed to handle the satisfaction clauses for $\Box$ and $\Diamond$. In the most general case, both modal operators are handled by separate relations $R_\Box$ and $R_\Diamond$, although not infrequently one assumes that $R_\Box$ and $R_\Diamond$ do coincide[2] or are otherwise non-trivially related. It is also quite common to assume different conditions connecting $R_\Box$ and $R_\Diamond$ with the accessibility relation R, e. g. to assume that

$$R \circ R_\Box \subseteq R_\Box \circ R.$$

Both the condition that $R_\Box = R_\Diamond$ and the other conditions mentioned in this connection in the existing literature are easily seen to be first-order definable.[3] Since it was shown in [10] and [11] that asimulations are easily scalable according to arbitrary first-order conditions imposed upon the models, we will first concentrate on the minimal case without any restrictions imposed and then accommodate for the possible restrictions in a trivial way, by restricting the domain and the counter-domain of asimulation relations accordingly.

As for the satisfaction clauses employed in the existing literature on Kripke-style semantics for intuitionistic modal operators, the following varieties seem to be the most common and general:[4]

$$M, s \models \Box I \Leftrightarrow \forall t(sR_\Box t \Rightarrow M, t \models I) \tag{$\Box_1$}$$

$$M, s \models \Box I \Leftrightarrow \forall t(sRt \Rightarrow \forall u(tR_\Box u \Rightarrow M, u \models I)) \tag{$\Box_2$}$$

$$M, s \models \Diamond I \Leftrightarrow \exists t(sR_\Diamond t \wedge M, t \models I) \tag{$\Diamond_1$}$$

$$M, s \models \Diamond I \Leftrightarrow \forall t(sRt \Rightarrow \exists u(tR_\Diamond u \wedge M, u \models I)) \tag{$\Diamond_2$}$$

This gives us four possible choices of satisfaction clauses. In the literature, these sets of satisfaction clauses are often viewed as more or less explicit manifestations of one and the same set of semantic intuitions. In this view, the reason why clauses ($\Box_2$) and ($\Diamond_2$) differ from ($\Box_1$) and ($\Diamond_1$), respectively, is that the former clauses lift up to the level of semantical definitions some desirable properties that under ($\Box_1$) and ($\Diamond_1$) are handled by restrictions on the class of models.[5] However, in what follows, we will disregard this circumstance and will simply consider these four systems of clauses as *bona fide* different systems. The reason for this is that, like we said above, we find it convenient, in the context of treating modal intuitionistic formulas via asimulations, to omit whatever restrictions on models that are employed to equate these systems in the existing literature on the subject.

Each of the four semantical choices sketched above induces a different standard translation of modal intuitionistic formulas into classical FOL, thus giving a different fragment of it. Every such standard translation is an obvious extension of the well known notion of the standard translation of propositional intuitionistic formulas, see e.g. [9, Definition 8.7]. More precisely, for $i, j \in \{1, 2\}$ we will denote the (i, j)-standard translation, or the standard translation induced by adopting ($\Box_i$)-clause together with ($\Diamond_j$)-clause above, by ST_{ij}.

Thus the inductive definitions of the (i, j)-standard x-translations run as follows:

$$\mathsf{ST}_{ij}(p_n, x) = P_n(x);$$

$$\mathsf{ST}_{ij}(\bot, x) = \bot;$$

[2] This is the case, e.g., for all the systems mentioned in [13, Ch. 3].

[3] As an example one may consider restrictions on Kripke frames mentioned in [1, Section 4], [13, Ch. 3], or [15, Section 2].

[4] In this form they are given, e.g., in [1, Section 4].

[5] E.g. the property of monotonicity. Cf. the motivation for the clause ($\Box_2$) given in [13, p. 46].

$$\mathsf{ST}_{ij}(\top, x) = \top;$$

$$\mathsf{ST}_{ij}(I \wedge J, x) = \mathsf{ST}_{ij}(I, x) \wedge \mathsf{ST}_{ij}(J, x);$$

$$\mathsf{ST}_{ij}(I \vee J, x) = \mathsf{ST}_{ij}(I, x) \vee \mathsf{ST}_{ij}(J, x);$$

$$\mathsf{ST}_{ij}(I \to J, x) = \forall y(R(x, y) \to (\mathsf{ST}_{ij}(I, y) \to \mathsf{ST}_{ij}(J, y)));$$

$$\mathsf{ST}_{1j}(\Box I, x) = \forall y(R_\Box(x, y) \to \mathsf{ST}_{1j}(I, y));$$

$$\mathsf{ST}_{2j}(\Box I, x) = \forall y(R(x, y) \to \forall z(R_\Box(y, z) \to \mathsf{ST}_{2j}(I, z)));$$

$$\mathsf{ST}_{i1}(\Diamond I, x) = \exists y(R_\Diamond(x, y) \wedge \mathsf{ST}_{i1}(I, y));$$

$$\mathsf{ST}_{i2}(\Diamond I, x) = \forall y(R(x, y) \to \exists z(R_\Diamond(y, z) \wedge \mathsf{ST}_{i2}(I, z))).$$

In the definitions above the variables x, y, and z are assumed to be pairwise different, with y and z fresh for both $\mathsf{ST}_{ij}(I, x)$ and $\mathsf{ST}_{ij}(J, x)$.

3. Basic asimulations and expressive power of intuitionistic propositional logic

Just as Kripke semantics for modal intuitionistic logic is built on top of Kripke semantics for propositional intuitionistic logic, so our characterization of expressive powers for modal intuitionistic logic is framed as an extension of the earlier characterization of expressive powers of propositional intuitionistic logic presented in [10]. Therefore, we shall begin by briefly re-capitaluting the main results and notions of this earlier paper, as well as the intuitions behind them.

It is well known that, when viewed as a fragment of FOL, classical modal propositional logic defines exactly the set of first-order formulas invariant w.r.t. bisimulations. Bisimulations are binary relations between the Kripke models framed so as to capture coincidence of the sets of modal formulas true in the points of Kripke models related by bisimulation. In other words, bisimulations are intended to capture coincidence of modal truth sets for the given points. Of course, one can only infer the existence of a bisimulation from coincidence of modal truth sets when the models in question are 'good enough', that is to say, when the related models are saturated in an appropriate sense.[6]

In a similar fashion, basic asimulations are tailored to capture relations between sets of intuitionistic formulas true in the related states of Kripke models — and that, again, only within an 'ideal' environment provided by saturated models. The difference, however, is that asimulations capture inclusion of truth sets rather than their coincidence. Hence the name 'asimulations', which is intended to stand for something like 'asymmetrical bisimulations'.

Keeping in mind that asimulations are meant to capture inclusion of the truth set of the 'left' state of asimulation pair into the truth set of the 'right' state, one can better understand the intuitive meaning of different versions of asimulation given below. In this section we explain the application of this idea to the basic case of propositional intuitionistic logic. First, consider atomic formulas: since we now care about inclusion rather than coincidence of intuitionistic truth sets, we need to substitute the bi-conditional in the atomic clause of bisimulation with a respective conditional, thus getting clause (base) in Definition 1 below. Presence of $\bot$, $\top$, $\wedge$, and $\vee$ in our language then requires no further provisions since the preservation of the respective formulas will follow from the preservation of their conjuncts or disjuncts. However, the presence of intuitionistic implication calls for an additional clause. It is clear that asimulation will preserve true intuitionistic implications from left to right iff it will preserve false intuitionistic implications from right to left. This means that every counterexample to an intuitionistic implication in the right model must also exist for the respective world in the left model. Now, assume that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are two pointed

[6] Generally one must require here at least *modal saturation*; see, for instance, [4, p. 92, Definition 2.53].

models, and that for an asimulation A we have $s\ A\ s'$. We need to ensure that a counterexample to an implication $I \to J$ at $(\mathcal{M}', s')$ is 'matched' with a similar counterexample at $(\mathcal{M}, s)$. Typically, the existence of such a counterexample means the existence of a successor t' to s' making both I true and J false. If we want to find a similar counterexample for $I \to J$ at $(\mathcal{M}, s)$, we need, again, to find a successor t to s with a similar structure of truth sets. Well, if we require that $t\ A\ t'$, we ensure that the intuitionistic truth set of t is included in the intuitionistic truth set of t', thus getting that J is false at t as well. On the other hand, if we require that $t'\ A\ t$, inverse inclusion of intuitionistic truth sets is obtained, thus ensuring that I is true at t. Therefore, the joint requirement that $t\ \overset{\leftrightarrow}{A}\ t'$ will do the trick, whereas any of its constituent asymmetric A-links would be insufficient. This explains the form of condition (step) in Definition 1.

Summing everything up, we are now able to give the following definition of basic asimulation:

Definition 1. Let (M_1, t), (M_2, u) be two pointed Θ-models. A binary relation A is called a *basic asimulation from* (M_1, t) *to* (M_2, u) iff for any i, j such that $\{i, j\} = \{1, 2\}$, any $a, c \in U_i$, $b, d \in U_j$, any unary predicate letter $P \in \Theta$ the following conditions hold:

$$A \subseteq (U_1 \times U_2) \cup (U_2 \times U_1) \tag{type}$$

$$t\ A\ u \tag{elem}$$

$$(a\ A\ b \wedge a \models_i P(x)) \Rightarrow b \models_j P(x)) \tag{base}$$

$$(a\ A\ b \wedge b\ R_j\ d) \Rightarrow \exists c \in U_i(a\ R_i\ c \wedge c\ \overset{\leftrightarrow}{A}\ d) \tag{step}$$

One immediately sees that the notion of basic asimulation is another member in the series of such notions as partial isomorphism (employed in the formulation of Fraïssé's Theorem, see e.g. [7, Ch. XII, Theorem 2.1]) or bisimulation (Van Benthem's Modal Characterization Theorem). Now, it is well known that these latter notions are typically accompanied by a finitary version, which only preserves properties of truth sets up to a given quantifier rank k. Thus, one speaks of k-isomorphisms and k-bisimulations alongside partial isomorphisms and bisimulations. This is also the case with basic asimulations, and the correspondent finitary notion looks as follows:

Definition 2. Let (M_1, t), (M_2, u) be two pointed Θ-models. A binary relation A is called a *basic* k*-asimulation from* (M_1, t) *to* (M_2, u) iff for any i, j such that $\{i, j\} = \{1, 2\}$, any $(\bar{a}_m, a, c) \in U_i^{m+2}$, $(\bar{b}_m, b, d) \in U_j^{m+2}$, any unary predicate letter $P \in \Theta$, the following conditions hold:

$$A \subseteq \bigcup_{n>0}((U_1^n \times U_2^n) \cup (U_2^n \times U_1^n)) \tag{p-type}$$

$$t\ A\ u \tag{elem}$$

$$((\bar{a}_m, a)\ A\ (\bar{b}_m, b) \wedge a \models_i P(x)) \Rightarrow b \models_j P(x) \tag{p-base}$$

$$\begin{aligned} &((\bar{a}_m, a)\ A\ (\bar{b}_m, b) \wedge b\ R_j\ d \wedge m < k) \Rightarrow \\ &\qquad \Rightarrow \exists c \in U_i(a\ R_i\ c \wedge (\bar{a}_m, a, c)\ \overset{\leftrightarrow}{A}\ (\bar{b}_m, b, d)) \end{aligned} \tag{p-step}$$

In order to explicate the sense in which basic asimulations and basic k-asimulations capture the expressive powers of intuitionistic propositional logic, we need first to define the notion of invariance of a formula w.r.t. a class of binary relations:

Definition 3. Let β be a class of relations such that for any $A \in \beta$ there is a Θ and there are Θ-models M_1 and M_2 such that (p-type) holds. Then a formula $\varphi(x)$ is said to be *invariant with respect to* β, iff for any $A \in \beta$, for any corresponding Θ-models M_1 and M_2, and for any $a \in U_1$ and $b \in U_2$ it is true that:

$$(a\ A\ b \wedge a \models_1 \varphi(x)) \Rightarrow b \models_2 \varphi(x).$$

In view of this definition, when we say that basic asimulations and basic k-asimulations offer a semantic characterization of intuitionistic propositional logic, we mean that the following theorems can be established:

Theorem 1. *Let $i, j \in \{1, 2\}$. A formula $\varphi(x)$ is equivalent to an (i, j)-standard x-translation of an intuitionistic formula iff $\varphi(x)$ is invariant with respect to basic asimulations.*

Theorem 2. *Let $i, j \in \{1, 2\}$. A formula $\varphi(x)$ is invariant with respect to basic k-asimulations iff there exists an intuitionistic formula I such that $r(\mathsf{ST}_{ij}(I, x)) \leq k$ and $\varphi(x)$ is equivalent to $\mathsf{ST}_{ij}(I, x)$.*

Instantiation of i and j above can be arbitrary, since all of the above-defined standard translations coincide on the set of intuitionistic propositional formulas.

Theorem 2 yields as an immediate corollary the following 'parametrized version' of Theorem 1:

Corollary 1. *Let $i, j \in \{1, 2\}$. A formula $\varphi(x)$ is equivalent to an (i, j)-standard x-translation of an intuitionistic formula iff there exists a $k \in \mathbb{N}$ such that $\varphi(x)$ is invariant with respect to basic k-asimulations.*

Theorem 1 and Corollary 1 were stated and proved in [10] as Theorem 4.16 and Theorem 3.12, respectively. Theorem 2 was not proved there; in fact, the proof of Corollary 1 given in [10] established a somewhat weaker proposition. In this paper we do not give a separate proof of Theorem 2 either. However, in the following sections one can find full proofs of analogous propositions w.r.t. modal intuitionistic logic and modal asimulations, namely Theorems 3 and 4 and Corollary 2. Proofs of Theorems 1 and 2 and Corollary 1 can be obtained from these proofs by simply deleting all inductive cases not relevant to the notion of basic asimulation and language of intuitionistic propositional logic.[7]

Before we leave the subject of basic asimulation, we would like to consider a typical example of asimulation relation and offer some comments on it.

Example 1. Consider Σ-models $\mathcal{M}_1$, $\mathcal{M}_2$ with $U_1 = \{w_1, w_2, w_3\}$, $U_2 = \{v_1, v_2\}$, $\iota_1(P) = \{w_3\}$, $\iota_2(P) = \emptyset$. We further set that $\iota_1(R)$, $\iota_2(R)$ are reflexive and transitive closures of relations r_1 and r_2 respectively, assuming that

$$r_1 := \{(w_1, w_2), (w_1, w_3)\}; \qquad r_2 := \{(v_1, v_2)\},$$

and we set all the other binary and unary predicate constants in Σ to $\emptyset$. Then the following binary relation A (see Fig. 1) is, according to the Definition 1, a basic asimulation from (M_1, w_1) to (M_2, v_1):

$$A := \{(w_1, v_1), (v_1, w_2), (w_2, v_1), (w_2, v_2), (v_2, w_2)\}.$$

Regarding the above example one may note, in the first place, that, in view of the foregoing theorems, the example shows that the correspondence formula $\exists y(R(x, y) \wedge P(y))$ is not definable by a standard translation of an intuitionistic propositional formula. Secondly, note the pivotal role of asymmetry of A in this result. Were we to enforce symmetry with respect to the initial link $w_1\ A\ v_1$, we would have to look then for counterparts of w_3, and would consequently fail to construct an asimulation at all. On the other hand, one

[7] However, in [10] these statements were proved for a slightly different language which did not contain $\top$ among its basic connectives. As for Theorem 2, one direction of it, i.e. that if a formula is equivalent to a standard translation of an intuitionistic formula of degree not exceeding k, then this formula is invariant w.r.t. basic k-asimulations, is an easy consequence of [10, Lemma 3.3]. In the other direction, however, one can extract from the proofs given in [10] only a somewhat weaker statement that every formula invariant w.r.t. basic k-asimulations is equivalent to a standard translation of intuitionistic formula of degree not exceeding $k + 2$. Thus the proofs given in this paper, among other things, somewhat strengthen the results of [10].

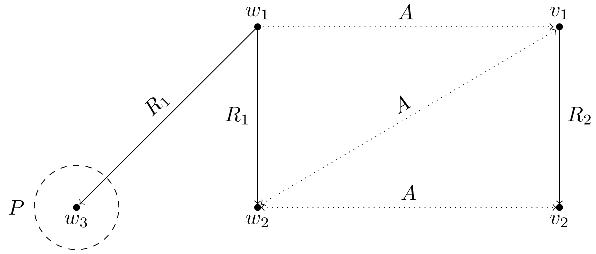

Fig. 1. A is a basic asimulation

should also note that the role of asymmetry in basic asimulations is clearly limited. Indeed, due to the form of clause (step), all the asimulation links necessarily generated from other links have to be symmetric, and, with symmetry enforced, a basic asimulation is just a regular bisimulation. Therefore, every basic asimulation can be 'trimmed' so that, in a sense, *almost* all of the asymmetric links are eliminated. This situation is probably one of the reasons for the existence of results like [14, Theorem A.1] which shows that if one (1) limits one's attention to the class of intended models of intuitionistic propositional logic (i.e. assumes that R is a pre-order, etc.) and (2) only looks at the *monotonic* formulas of the correspondence language, then expressive powers of intuitionistic propositional logic can be described already with bisimulation invariance.

However, the situation looks more complicated when one considers natural extensions of intuitionistic propositional logic, like intuitionistic modal or first-order logic. In the respective versions of asimulation relations for these extensions, the asymmetric links can be reinstated at any later stage. It is thus not clear if any analogues of [14, Theorem A.1] are possible w.r.t. these extensions, and if yes, then what the form of the corresponding restrictions would be like.

4. Characterization of modal intuitionistic formulas: definitions and main results

Our aim in the present paper is to characterize the expressive power of the four fragments of the correspondence language induced by the four above-mentioned versions of (i, j)-standard translation of modal intuitionistic formulas via the suitable extension of basic asimulation. We will begin by giving strict definitions of the four required extensions, and then formulate the two versions of our main result for all the four considered fragments of the correspondence language in one full sweep.

Let us begin with some easy cases. It is clear that the clauses ($\Box_1$) and ($\Diamond_1$) are just definitions of box and diamond in classical modal propositional logic, so the expressive powers induced by these clauses can be captured by a version of the clause used in the definition of bisimulation. One only needs to take into an account the asymmetric setting at hand in order to come up with an appropriate weakening of bisimulation conditions. Thus, since we want classic modal diamonds to be preserved from left to right, we need to ensure that 'verifying successors' of a given state are preserved in the same direction; hence the clause (diam-1) below. On the other hand, if we want classic modal boxes to be preserved from left to right, we need to secure the transition of 'falsifying successors' in the reverse direction, i.e., from right to left. Hence the form of the clause (box-1) below. The clause ($\Box_2$) only differs from ($\Box_1$) in that relation R is substituted here by a composition of relations $R \circ R_\Box$ and, *mutatis mutandis*, can be dealt with in a similar way.

The above considerations contain enough intuitive motivation for an attempt to define extensions of basic asimulations and k-asimulations for two of the four fragments of the correspondence language that we treat in this paper. An extension of basic (k-)asimulation that is intended to capture expressive powers of the set of (i, j)-standard translations of modal intuitionistic formulas we will name (i, j)-modal (k-)asimulation. Thus, the above considerations motivate the definitions of (i, j)-modal (k-)asimulation for all $i \in \{1, 2\}$ and $j = 1$. The respective definitions look as follows:

Definition 4. Let (M_1, t), (M_2, u) be two pointed Θ-models. A binary relation A is called a $(2, 1)$-*modal k-asimulation from* (M_1, t) *to* (M_2, u) iff A is a basic k-asimulation from (M_1, t) to (M_2, u) and for any i, j such that $\{i, j\} = \{1, 2\}$, any $(\bar{a}_m, a, c) \in U_i^{m+2}$, $(\bar{b}_m, b, d) \in U_j^{m+2}$, the following conditions hold:

$$((\bar{a}_m, a)\ A\ (\bar{b}_m, b) \wedge b\ R_j\ d \wedge d\ R_{\Box j}\ f \wedge m + 1 < k) \Rightarrow$$

$$\Rightarrow \exists c, e \in U_i(a\ R_i\ c \wedge c\ R_{\Box i}\ e \wedge (\bar{a}_m, a, c, e)\ A\ (\bar{b}_m, b, d, f)) \quad \text{(p-box-2)}$$

$$((\bar{a}_m, a)\ A\ (\bar{b}_m, b) \wedge a\ R_{\Diamond i}\ c \wedge m < k) \Rightarrow$$

$$\Rightarrow \exists d \in U_j(b\ R_{\Diamond j}\ d \wedge (\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d)) \quad \text{(p-diam-1)}$$

Definition 5. Let (M_1, t), (M_2, u) be two pointed Θ-models. A binary relation A is called a $(2, 1)$-*modal asimulation from* (M_1, t) *to* (M_2, u) iff A is a basic asimulation from (M_1, t) to (M_2, u) and for any i, j such that $\{i, j\} = \{1, 2\}$, any $a, c \in U_i$, $b, d \in U_j$, the following conditions hold:

$$(a\ A\ b \wedge b\ R_j\ d \wedge d\ R_{\Box j}\ f) \Rightarrow \exists c, e \in U_i(a\ R_i\ c \wedge c\ R_{\Box i}\ e \wedge e\ A\ f) \quad \text{(box-2)}$$

$$(a\ A\ b \wedge a\ R_{\Diamond i}\ c) \Rightarrow \exists d \in U_j(b\ R_{\Diamond j}\ d \wedge c\ A\ d) \quad \text{(diam-1)}$$

Definition 6. Let (M_1, t), (M_2, u) be two pointed Θ-models. A binary relation A is called a $(1, 1)$-*modal k-asimulation from* (M_1, t) *to* (M_2, u) iff A is a basic k-asimulation from (M_1, t) to (M_2, u) and for any i, j such that $\{i, j\} = \{1, 2\}$, any $(\bar{a}_m, a, c) \in U_i^{m+2}$, $(\bar{b}_m, b, d) \in U_j^{m+2}$, condition (p-diam-1) is satisfied together with the following condition:

$$((\bar{a}_m, a)\ A\ (\bar{b}_m, b) \wedge b\ R_j\ d \wedge m < k) \Rightarrow$$

$$\Rightarrow \exists c \in U_i(a\ R_{\Box i}\ c \wedge (\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d)) \quad \text{(p-box-1)}$$

Definition 7. Let (M_1, t), (M_2, u) be two pointed Θ-models. A binary relation A is called a $(1, 1)$-*modal asimulation from* (M_1, t) *to* (M_2, u) iff A is a basic asimulation from (M_1, t) to (M_2, u) and for any i, j such that $\{i, j\} = \{1, 2\}$, any $a, c \in U_i$, $b, d \in U_j$, condition (diam-1) is satisfied together with the following condition:

$$(a\ A\ b \wedge b\ R_j\ d) \Rightarrow \exists c \in U_i(a\ R_{\Box i}\ c \wedge c\ A\ d) \quad \text{(box-1)}$$

If, instead of using clause $(\Diamond_1)$, one chooses clause $(\Diamond_2)$, things get somewhat more complicated. The problem is that in this case relations between the truth sets of points in respective Kripke models seem to be insufficient to transfer formulas of the form $\Diamond I$ along asimulation links. Assume, again, that we have two pointed models $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$, and for an asimulation A we have that $s\ A\ s'$. According to $(\Diamond_2)$, a standard translation of $\Diamond I$ is a universal formula of the correspondence language, therefore, we need to ensure that any counterexample to $\Diamond I$ at $(\mathcal{M}', s')$ is matched by an analogous counterexample at $(\mathcal{M}, s)$. Now, such a counterexample must be an R-successor t' to s' in $\mathcal{M}'$ such that none of $R_\Diamond$-successors of t' verifies I, so we need to find an R-successor t of s in $\mathcal{M}$ which has no $R_\Diamond$-successors verifying I. The problem, however, is that this property of successors cannot be defined by any modal intuitionistic formula J in any of the two logics induced by $(\Diamond_2)$, if we do not put any non-trivial restrictions onto $R_\Box$ and $R_\Diamond$. Moreover, it follows from Example 2 below that this indefinability persists even if we require that $(\Sigma \setminus \{R_\Box, R_\Diamond\})$-reducts of the models in question are intended models of intuitionistic propositional logic, that is to say, have monotonic valuations for unary predicates and have R as preorder. Incidentally, Example 3 shows that the same holds for the negation of this property, that is to say, for the property ‘there are some $R_\Diamond$-successors verifying a given formula’.

Therefore, the task at hand cannot be achieved by requiring w.r.t. the above-mentioned successor t of s that it satisfies $t\ A\ t'$, or $t\ \overset{\leftrightarrow}{A}\ t'$, or, it appears, any other natural combination of conditions mentioning A alone. A straightforward solution to this difficulty would be to introduce another binary relation B which would be able to preserve the property 'there are no $R_\diamond$-successors verifying a given formula' in the right-to-left direction, or, by contraposition, the property 'there are some $R_\diamond$-successors verifying a given formula' in the reverse, left-to-right direction. In this way, we will be able to handle the problem in a separate 'compartment' of asimulation. It is a matter of arbitrary choice which of the two properties will be preserved along direct B-links and which according to the reverse ones. We choose to stipulate that direct B-links ensure the preservation of the existence of $R_\diamond$-successors verifying a given formula, whereas the inverse B-links ensure the preservation of absence of $R_\diamond$-successors verifying a given formula.

Under this reading, the new relation B will have no direct connections with the truth sets of the related states, but it will have strong connections with the truth sets of both $R_\diamond$-successors and R-predecessors of these states. Indeed, if $s\ A\ s'$, then every R-successor t' of s' that has no $R_\diamond$-successors verifying I must have a matching R-successor t of s. This means that one must require that $t\ B\ t'$, which motivates the form of clause (diam-2(1)) in Definitions 9 and 11 below. On the other hand, assume that $t\ B\ t'$. If a modal intuitionistic formula I is verified by an $R_\diamond$-successor u of t, there must be a matching $R_\diamond$-successor u' of t' verifying the same formula, else the property of 'no $R_\diamond$-successors verifying a given formula' will not be preserved along the reverse B-link from t' to t. Hence the clause (diam-2(2)) in Definitions 9 and 11. The foregoing observations motivate the following definitions:

Definition 8. Let (M_1, t), (M_2, u) be two pointed Θ-models. An ordered couple of binary relations (A, B) is called a $(2,2)$-*modal k-asimulation from* (M_1, t) *to* (M_2, u) iff A is a basic k-asimulation from (M_1, t) to (M_2, u) and for any i, j such that $\{i, j\} = \{1, 2\}$, any $(\bar{a}_m, a, c) \in U_i^{m+2}$, $(\bar{b}_m, b, d) \in U_j^{m+2}$ condition (p-box-2) is satisfied together with the following conditions:

$$B \subseteq \bigcup_{n>0}((U_1^n \times U_2^n) \cup (U_2^n \times U_1^n)) \tag{p-B-type}$$

$$((\bar{a}_m, a)\ A\ (\bar{b}_m, b) \wedge b\ R_j\ d \wedge m+1 < k) \Rightarrow$$
$$\Rightarrow \exists c \in U_i(a\ R_i\ c \wedge (\bar{a}_m, a, c)\ B\ (\bar{b}_m, b, d)) \tag{p-diam-2(1)}$$

$$((\bar{a}_m, a)\ B\ (\bar{b}_m, b) \wedge a\ R_{\diamond i}\ c \wedge m < k) \Rightarrow$$
$$\Rightarrow \exists d \in U_j(b\ R_{\diamond j}\ d \wedge (\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d)) \tag{p-diam-2(2)}$$

Definition 9. Let (M_1, t), (M_2, u) be two pointed Θ-models. An ordered couple of binary relations (A, B) is called a $(2,2)$-*modal asimulation from* (M_1, t) *to* (M_2, u) iff A is a basic asimulation from (M_1, t) to (M_2, u) and for any i, j such that $\{i, j\} = \{1, 2\}$, any $a, c \in U_i$, $b, d \in U_j$ condition (box-2) is satisfied together with the following conditions:

$$B \subseteq (U_1 \times U_2) \cup (U_2 \times U_1) \tag{B-type}$$

$$(a\ A\ b \wedge b\ R_j\ d) \Rightarrow \exists c \in U_i(a\ R_i\ c \wedge c\ B\ d) \tag{diam-2(1)}$$

$$(a\ B\ b \wedge a\ R_{\diamond i}\ c) \Rightarrow \exists d \in U_j(b\ R_{\diamond j}\ d \wedge c\ A\ d) \tag{diam-2(2)}$$

The only case left is the one where one defines the satisfaction relation by using $(\diamond_2)$ combined with $(\Box_1)$. The respective definitions simply re-shuffle the conditions mentioned in the previous versions:

Definition 10. Let (M_1, t), (M_2, u) be two pointed Θ-models. An ordered couple of binary relation (A, B) is called a $(1,2)$-*modal k-asimulation from* (M_1, t) *to* (M_2, u) iff A is a basic k-asimulation from (M_1, t)

to (M_2, u) and for any $\{i, j\} = \{1, 2\}$, any $(\bar{a}_m, a, c) \in U_i^{m+2}$, $(\bar{b}_m, b, d) \in U_j^{m+2}$, the conditions (p-box-1), (p-diam-2(1)), and (p-diam-2(2)) are satisfied.

Definition 11. Let (M_1, t), (M_2, u) be two pointed Θ-models. An ordered couple of binary relation (A, B) is called a $(1, 2)$-*modal asimulation from* (M_1, t) *to* (M_2, u) iff A is a basic asimulation from (M_1, t) to (M_2, u) and for any $\{i, j\} =\in \{1, 2\}$, any $a \in U_i$, $b, d \in U_j$, the conditions (box-1), (diam-2(1)) and (diam-2(2)) are satisfied.

In what follows, we will identify invariance w.r.t. an $(i, 2)$-modal asimulation (A, B) with invariance with respect to its left projection A.

It turns out that for arbitrary $i, j \in \{1, 2\}$, invariance w.r.t. (i, j)-modal asimulations can be used to characterize modal intuitionistic fragment of FOL described by their respective ST_{ij}. More precisely, one can obtain the following theorems:

Theorem 3. *Let $i, j \in \{1, 2\}$. A formula $\varphi(x)$ is equivalent to an (i, j)-standard x-translation of an intuitionistic formula iff $\varphi(x)$ is invariant with respect to (i, j)-modal asimulations.*

Theorem 4. *Let $i, j \in \{1, 2\}$ A formula $\varphi(x)$ is invariant with respect to (i, j)-modal k-asimulations iff there exists a modal intuitionistic formula I such that $r(\mathsf{ST}_{ij}(I, x)) \leq k$ and $\varphi(x)$ equivalent to $\mathsf{ST}_{ij}(I, x)$.*

In addition to Theorem 4, one can derive the following corollary:

Corollary 2. *Let $i, j \in \{1, 2\}$. A formula $\varphi(x)$ is equivalent to an (i, j)-standard x-translation of a modal intuitionistic formula iff there exists a $k \in \mathbb{N}$ such that $\varphi(x)$ is invariant with respect to (i, j)-modal k-asimulations.*

The proofs of Theorem 4 and Corollary 2 for the case $i = j = 2$ are to be found in Section 5.1. The proof of Theorem 3 for the same case makes up the content of Section 5.2. In Section 6 we show how to modify this proof for the other three cases at hand.

We devote the rest of this section to the analysis of some examples of modal asimulations.

Example 2. Consider Σ-models $\mathcal{M}_1$, $\mathcal{M}_2$ with $U_1 = \{w_1, w_2, w_3, w_4\}$, $U_2 = \{v_1, v_2\}$, $\iota_1(P) = \{w_3\}$, $\iota_2(P) = \{v_2\}$. We further set that $\iota_1(R)$, $\iota_2(R)$ are reflexive and transitive closures of relations $\{(w_1, w_2)\}$ and $\emptyset$, respectively. We assume that

$$\iota_1(R_\Diamond) := \{(w_1, w_4), (w_2, w_3)\},$$

and that

$$\iota_2(R_\Diamond) := \{(v_1, v_2)\}.$$

Finally, we set $R_\Box$ and all the other unary predicate constants in Σ to $\emptyset$ in both models. Then the following couple (A, B) of binary relations is an $(i, 2)$-modal asimulation from (M_1, w_1) to (M_2, v_1) for any $i \in \{1, 2\}$:

$$A := \{(w_1, v_1), (w_2, v_1), (v_1, w_2), (v_2, w_3), (w_3, v_2)\},$$

$$B := \{(w_2, v_1), (v_1, w_2), (w_3, v_2), (v_2, w_3)\}.$$

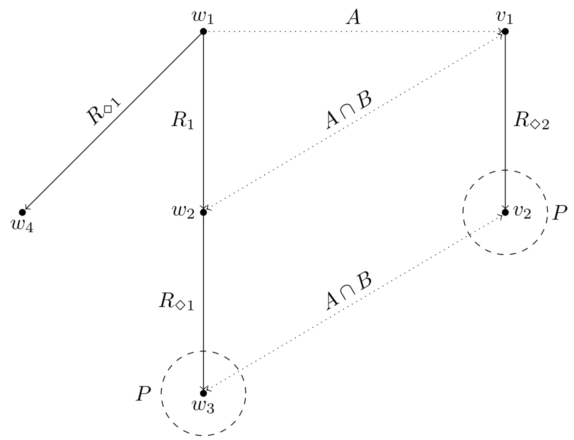

Fig. 2. (A, B) is a $(i, 2)$-modal asimulation for all $i \in \{1, 2\}$.

See Fig. 2. Note that this example shows that formula $\neg\exists y(R_\Diamond(x, y) \wedge P(y))$ is not preserved under $(i, 2)$-modal simulations. Therefore, by Theorem 3 above, this formula is not definable by any $(i, 2)$-standard translation of a modal intuitionistic formula.[8]

Example 3. Consider the following modification of Example 1: we convert the R-link (w_1, w_3) into an $R_\Diamond$-link and add binary relation $B = \{(w_1, v_2), (v_2, w_1), (w_2, v_2), (v_2, w_2)\}$. Then the pair (A, B) turns out to be an $(i, 2)$-modal asimulation from (M_1, w_1) to (M_2, v_1) for any $i \in \{1, 2\}$.

By Theorem 3, this shows that the negation of the correspondence formula from Example 2, i.e. the formula $\exists y(R_\Diamond(x, y) \wedge P(y))$, is also undefinable by an $(i, 2)$-standard translation of a modal intuitionistic formula. Note that this undefinability does not follow already from Example 2, since modal intuitionistic logic is not closed w.r.t. classical negation.

Example 4. Consider Σ-models $\mathcal{M}_1$, $\mathcal{M}_2$ with $U_1 = \{w_0, w_1, w_2, w'\}$, $U_2 = \{v_0, v_1, v_2\}$, $\iota_1(P) = \{w_2, w'\}$, $\iota_2(P) = \{v_2\}$, $\iota_1(P_2) = \{w_2, w_2'\}$, $\iota_2(P_2) = \{v_2, v_2'\}$. We further set that $\iota_1(R)$, $\iota_2(R)$ are reflexive and transitive closures of relations $\{(w_1, w_2)\}$ and $\{(v_1, v_2)\}$, respectively. We assume that $\iota_1(R_\Box) = \{(w_0, w_1), (w_0, w')\}$ and that $\iota_2(R_\Box) = \{(v_0, v_1)\}$. Finally, we set $R_\Diamond$ and all the other unary predicate constants in Σ to $\emptyset$ in both models. Then the following binary relation A (see Fig. 3) is a $(2, 1)$-modal asimulation from (M_1, w_0) to (M_2, v_0):

$$A := \{(w_i, v_i), (v_i, w_i) \mid i \in \{0, 1, 2\}\} \cup \{(v_1, w'), (v_2, w'), (w', v_2)\}.$$

This example displays the phenomenon already mentioned in the previous section: when dealing with the extensions of intuitionistic propositional logic, the asymmetric links between the states of Kripke models can be induced by symmetric links. Note that in this case one cannot construct the required asimulation without the asymmetric link $v_1\ A\ w'$. Indeed, this link is necessarily generated from symmetric A-link $v_0 \overset{\leftrightarrow}{A} w_0$ by condition (box-2) and the fact that $w_0\ R_1\ w_0\ R_{\Box 1}\ w'$. One has to choose here v_1 as the counterpart for w' since v_1 is the only $(R \circ R_\Box)$-successor of v_0. Moreover, turning $v_1\ A\ w'$ into a symmetric link immediately results in a violation of (base) by the evaluation of P in $\mathcal{M}_1$. Finally, note that the reducts of the two models in this examples are intended models of intuitionistic propositional logic.

[8] Note that the relationship $B \subseteq A$ is specific for this example and does **not** always obtain w.r.t. other cases of $(i, 2)$-modal asimulations. We omit a more detailed discussion of this for spatial reasons.

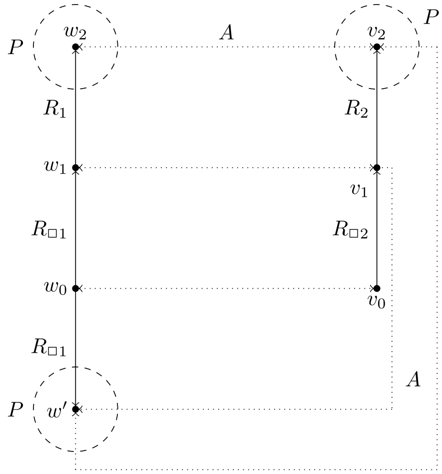

Fig. 3. A is a $(2,1)$-modal asimulation.

5. Characterization of modal intuitionistic formulas: the main case

Theorems 4 and 3 allow for four different instantiations of i and j. The most difficult ones seem to be the two instantiations with $j = 2$, since with them the situation bears the smallest degree of analogy to the asimulation clauses considered earlier in [10] and [11]. Therefore, in the present section we consider in some detail the case $i = j = 2$, whereas in the next section we show how to adapt our proofs to the other cases.

In formulating our lemmas and presenting our proofs we will follow the method of [10] and [11]. This proof procedure itself is ultimately an adaptation of the standard way of proving the van Benthem modal characterization theorem to the peculiar conditions of intuitionistic logic and its extensions. Almost all of the new ideas needed to accommodate this procedure to modal intuitionistic logic are presented in the proofs of the three propositions below. The other parts of the proofs are basically just reiteration of the respective parts of the proofs in [10] and [11] which we could not omit without affecting the intelligibility of the whole narrative.

5.1. Proof of Theorem 4

Proposition 1. *Let $\varphi(x) = \mathsf{ST}_{22}(I, x)$ for some modal intuitionistic formula I, and let $r(\varphi) = k$. Let $\Sigma_\varphi \subseteq \Theta$, let (M_1, t), (M_2, u) be two pointed Θ-models, and let (A, B) be a $(2, 2)$-modal l-asimulation from (M_1, t) to (M_2, u). Then*

$$((\bar{a}_m, a)\ A\ (\bar{b}_m, b) \wedge m + k \leq l \wedge a \models_i \varphi(x)) \Rightarrow b \models_j \varphi(x),$$

for all $i, j \in \{1, 2\}$, $(\bar{a}_m, a) \in U_i^{m+1}$, and $(\bar{b}_m, b) \in U_j^{m+1}$.

Proof. We proceed by induction on the complexity of I. In what follows we will abbreviate the induction hypothesis by IH.

First, we note that since (A, B) is a $(2, 2)$-modal l-asimulation, then A is a basic l-asimulation. Therefore, the case when $\varphi(x)$ is just $P(x)$, for a $P \in \Theta$, is handled by condition (p-base), and the cases related to $\bot$, $\top$, $\wedge$, and $\vee$, are trivial.

There remain the three cases which involve implication and modal operators:

Case 1. Let $I = J \to K$. Then

$$\varphi(x) = \forall y(R(x, y) \to (\mathsf{ST}_{22}(J, y) \to \mathsf{ST}_{22}(K, y))).$$

Assume that:

$$a \models_i \forall y(R(x,y) \to (\mathsf{ST}_{22}(J,y) \to \mathsf{ST}_{22}(K,y))) \tag{1}$$

$$(\bar{a}_m, a)\ A\ (\bar{b}_m, b) \tag{2}$$

$$m + r(\varphi(x)) \leq l \tag{3}$$

Moreover, it follows from the definition of r that:

$$r(\varphi(x)) \geq 1 \tag{4}$$

$$r(\mathsf{ST}_{22}(J,y)) \leq r(\varphi(x)) - 1 \tag{5}$$

$$r(\mathsf{ST}_{22}(K,y)) \leq r(\varphi(x)) - 1 \tag{6}$$

Now, consider an arbitrary $d \in U_j$ such that $b\ R_j\ d$. Since (3) and (4) clearly imply that $m < l$, it follows from (2) and (p-step) that one can choose a $c \in U_i$, such that:

$$a\ R_i\ c \tag{7}$$

$$(\bar{b}_m, b, d)\ A\ (\bar{a}_m, a, c) \tag{8}$$

$$(\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d) \tag{9}$$

So, we reason as follows:

$$d \models_j \mathsf{ST}_{22}(J,y) \quad \text{(premise)} \tag{10}$$

$$m + 1 + r(\mathsf{ST}_{22}(J,y)) \leq l \quad \text{(from (3) and (5))} \tag{11}$$

$$m + 1 + r(\mathsf{ST}_{22}(K,y)) \leq l \quad \text{(from (3) and (6))} \tag{12}$$

$$c \models_i \mathsf{ST}_{22}(J,y) \quad \text{(from (8), (10), (11) by IH)} \tag{13}$$

$$c \models_i \mathsf{ST}_{22}(K,y) \quad \text{(from (1), (7), and (13))} \tag{14}$$

$$d \models_j \mathsf{ST}_{22}(K,y) \quad \text{(from (9), (14), (12) by IH)} \tag{15}$$

Since d was chosen to be an arbitrary R_j-successor of b, this means that

$$b \models_j \forall y(R(x,y) \to (\mathsf{ST}_{22}(J,y) \to \mathsf{ST}_{22}(K,y))),$$

and we are done.

Case 2. Let $I = \Box J$. Then

$$\varphi(x) = \forall y(R(x,y) \to \forall z(R_\Box(y,z) \to \mathsf{ST}_{22}(J,z))).$$

Assume that:

$$a \models_i \forall y(R(x,y) \to \forall z(R_\Box(y,z) \to \mathsf{ST}_{22}(J,z))) \tag{16}$$

$$(\bar{a}_m, a)\ A\ (\bar{b}_m, b) \tag{17}$$

$$m + r(\varphi(x)) \leq l \tag{18}$$

Moreover, it follows from the definition of r that:

$$r(\varphi(x)) \geq 2 \tag{19}$$

$$r(\mathsf{ST}_{22}(J, z)) \leq r(\varphi(x)) - 2 \tag{20}$$

Now, consider arbitrary $d, f \in U_j$ such that $b\ R_j\ d$ and $d\ R_{\Box j}\ f$. Since (18) and (19) clearly imply that $m + 1 < l$, it follows from (17) and (p-box-2) that one can choose $c, e \in U_i$, such that:

$$a\ R_i\ c \tag{21}$$

$$c\ R_{\Box i}\ e \tag{22}$$

$$(\bar{a}_m, a, c, e)\ A\ (\bar{b}_m, b, d, f) \tag{23}$$

So, we reason as follows:

$$e \models_i \mathsf{ST}_{22}(J, z) \qquad \text{(from (16), (21), and (22))} \tag{24}$$

$$m + 2 + r(\mathsf{ST}_{22}(J, z)) \leq l \qquad \text{(from (18) and (20))} \tag{25}$$

$$f \models_j \mathsf{ST}_{22}(J, z) \qquad \text{(from (23), (24), (25) by IH)} \tag{26}$$

Since d was chosen to be an arbitrary R_j-successor of b, and f an arbitrary $R_{\Box j}$-successor of d, this means that

$$b \models_j \forall y(R(x, y) \to \forall z(R_\Box(y, z) \to \mathsf{ST}_{22}(J, z))),$$

and we are done.

Case 3. Let $I = \Diamond J$. Then

$$\varphi(x) = \forall y(R(x, y) \to \exists z(R_\Diamond(y, z) \wedge \mathsf{ST}_{22}(J, z))).$$

Assume that:

$$a \models_i \forall y(R(x, y) \to \exists z(R_\Diamond(y, z) \wedge \mathsf{ST}_{22}(J, z))) \tag{27}$$

$$(\bar{a}_m, a)\ A\ (\bar{b}_m, b) \tag{28}$$

$$m + r(\varphi(x)) \leq l \tag{29}$$

Moreover, it follows from the definition of r that:

$$r(\varphi(x)) \geq 2 \tag{30}$$

$$r(\mathsf{ST}(J, y)) \leq r(\varphi(x)) - 2 \tag{31}$$

Since (29) and (30) clearly imply that $m + 1 < l$, it follows from (28) and (p-diam-2(1)) that one can choose a $c \in U_i$, such that:

$$a\ R_i\ c \tag{32}$$

$$(\bar{a}_m, a, c)\ B\ (\bar{b}_m, b, d) \tag{33}$$

Now from (27) and (32) it follows that we can choose an $e \in U_i$ such that

$$c\ R_{\Diamond i}\ e \tag{34}$$

$$e \models_i \mathsf{ST}_{22}(J, z) \tag{35}$$

Also, by (33), condition (p-diam-2(2)), and the fact that $m + 1 < l$, we have:

$$\exists f \in U_j(d\ R_{\Diamond j}\ f \wedge (\bar{a}_m, a, c, e)\ A\ (\bar{b}_m, b, d, f)) \tag{36}$$

We further get that:

$$m + 2 + r(\mathsf{ST}_{22}(J, z)) \leq l \qquad \text{(from (29) and (31))} \tag{37}$$

$$\exists f \in U_j(d\ R_{\Diamond j}\ f \wedge (f \models_j \mathsf{ST}_{22}(J, z))) \qquad \text{(by IH from (35), (36), and (37))} \tag{38}$$

Since d was chosen to be an arbitrary R_j-successor of b, this means that

$$b \models_j \forall y(R(x, y) \to \exists z(R_\Diamond(y, z) \wedge \mathsf{ST}_{22}(J, z))),$$

and we are done. □

We use Proposition 1 to derive the 'easy' direction of $(2, 2)$-instantiation of Theorem 4:

Corollary 3. *Suppose $\varphi(x)$ is equivalent to $\mathsf{ST}_{22}(I, x)$ for some modal intuitionistic formula I with $r(\mathsf{ST}_{22}(I, x)) \leq k$. Then $\varphi(x)$ is invariant with respect to $(2, 2)$-modal k-asimulations.*

Proof. Let $\varphi(x)$ be logically equivalent to $\mathsf{ST}_{22}(I, x)$ for some modal intuitionistic formula I, and let $r(\mathsf{ST}_{22}(I, x)) = s$. Then it follows from Proposition 1 (setting $i := 1$, $j := 2$, $m := 0$, and $l := s$) that $\mathsf{ST}_{22}(I, x)$ is invariant with respect to $(2, 2)$-modal s-asimulations, and so is $\varphi(x)$.

Therefore, in the assumptions of the Corollary, if $r(\mathsf{ST}_{22}(I, x)) = s$, then $\varphi(x)$ is invariant with respect to $(2, 2)$-modal s-asimulations, and this holds for any natural s. This already yields us the Corollary for the case $r(\mathsf{ST}_{22}(I, x)) = k$.

Assume that $r(\mathsf{ST}_{22}(I, x)) = l < k$. Then, by the above reasoning, $\varphi(x)$ is invariant with respect to $(2, 2)$-modal l-asimulations. It remains to note that if $l < k$, then all $(2, 2)$-modal k-asimulations are, by definition, $(2, 2)$-modal l-asimulations. Therefore, $\varphi(x)$ must be invariant w.r.t. $(2, 2)$-modal k-asimulations as well. □

On our way to the inverse direction of the $(2, 2)$-instantiation of Theorem 4 we first need a new piece of notation. For a formula $\varphi(x)$ in the correspondence language, variable x and a natural l, we denote with $\mathsf{int}(\varphi, x, l)$ the set of all $(2, 2)$-standard x-translations of intuitionistic formulas, which happen to be (Σ_φ, x, l)-formulas. We use this notation to define three types of formulas which are important components in the proofs to follow. Let $\Sigma_\varphi \subseteq \Theta$, let M be a Θ-model and let $a \in U$. Then:

$$\mathsf{tp}_l(\varphi(x), M, a) = \{\psi(x) \in \mathsf{int}(\varphi, x, l) \mid M, a \models \psi(x)\},$$

$$\overline{\mathsf{tp}}_l(\varphi(x), M, a) = \{\psi(x) \in \mathsf{int}(\varphi, x, l) \mid M, a \not\models \psi(x)\},$$

and, further:

$$\mathsf{imp}_l(\varphi(x), M, a) = \bigcap\{\overline{\mathsf{tp}}_l(\varphi(x), M, b) \mid \iota(R_\Diamond)(a, b)\}.$$

We mention the following obvious link between different pieces of notation just defined. Under the above-mentioned assumptions about $\varphi(x)$, l, Θ, M, a, and for arbitrary Θ-model M' and $a' \in U'$ we have:

$$\mathsf{tp}_l(\varphi(x), M, a) \subseteq \mathsf{tp}_l(\varphi(x), M', a') \Leftrightarrow \overline{\mathsf{tp}}_l(\varphi(x), M', a') \subseteq \overline{\mathsf{tp}}_l(\varphi(x), M, a).$$

We then invoke the following well known fact about classical first-order logic:

Lemma 1. *For any finite predicate vocabulary Θ, any variable x and any natural k there are, up to logical equivalence, only finitely many (Θ, x, k)-formulas.*

For a proof of this fact see, e.g. Lemma XII.3.4 in [7, p. 253]. It implies that for every set of formulas which have one of the forms $\mathsf{int}(\varphi, x, l)$, $\mathsf{tp}_l(\varphi(x), M, a)$, $\overline{\mathsf{tp}}_l(\varphi(x), M, a)$, $\mathsf{imp}_l(\varphi(x), M, a)$, there exists a finite subset collecting the logical equivalents for all the formulas in the set. Moreover, it allows us to collect logical equivalents of all $(2, 2)$-standard translations of intuitionistic formulas which are true together with a given formula at some pointed model in a single formula which we will call a complete conjunction:

Definition 12. Let $\varphi(x)$ be a formula. A conjunction $\Psi(x)$ of formulas from $\mathsf{int}(\varphi, x, k)$ is called a *complete (φ, x, k)-conjunction* iff there is a pointed model (M, a) such that $M, a \models \Psi(x) \wedge \varphi(x)$, and for any $\psi(x) \in \mathsf{tp}_k(\varphi(x), M, a)$ we have $\Psi(x) \models \psi(x)$.

The two following lemmas summarize some rather obvious properties of complete conjunctions:

Lemma 2. *For any formula $\varphi(x)$, any natural $k \geq 1$, any Θ such that $\Sigma_\varphi \subseteq \Theta$ and any pointed Θ-model (M, a) such that $M, a \models \varphi(x)$ there is a complete (φ, x, k)-conjunction $\Psi(x)$ such that $M, a \models \Psi(x) \wedge \varphi(x)$.*

Proof. Consider $\mathsf{tp}_k(\varphi(x), M, a)$. This set is non-empty since $\mathsf{ST}_{22}(\top, x)$ will be true at (M, a). Due to Lemma 1, we can choose in this set a non-empty finite subset $\Gamma(x)$ such that any formula from $\mathsf{tp}_k(\varphi(x), M, a)$ is logically equivalent to (and hence follows from) a formula in $\Gamma(x)$. By $\Gamma(x) \subseteq \mathsf{tp}_k(\varphi(x), M, a)$, we also have $M, a \models \bigwedge\Gamma(x)$, therefore, $\bigwedge\Gamma(x)$ is a complete (φ, x, k)-conjunction. $\square$

Lemma 3. *For any formula $\varphi(x)$ and any natural k there are, up to logical equivalence, only finitely many complete (φ, x, k)-conjunctions.*

Proof. It suffices to observe that for any formula $\varphi(x)$ and any natural k, a complete (φ, x, k)-conjunction is a (Σ_φ, x, k)-formula. Our lemma then follows from Lemma 1. $\square$

As a result, we are now able to establish the 'hard' right-to-left direction of Theorem 4:

Proposition 2. *Let $\varphi(x)$ be invariant with respect to $(2, 2)$-modal k-asimulations. Then $\varphi(x)$ is equivalent to a $(2, 2)$-standard x-translation of a modal intuitionistic formula I, such that $r(\mathsf{ST}_{22}(I, x)) \leq k$.*

Proof. We may assume that $\varphi(x)$ is satisfiable, since $\perp$ is obviously invariant with respect to $(2, 2)$-modal k-asimulations and we have, for example, $\perp \leftrightarrow \mathsf{ST}_{22}(\perp, x)$.

We now have to consider two cases.

Case 1. There is no complete (φ, x, k)-conjunction $\Psi(x)$ such that $\Psi(x) \wedge \neg\varphi(x)$ is satisfiable. Then take the set of all complete (φ, x, k)-conjunctions. This set is non-empty, because $\varphi(x)$ is satisfiable, and by Lemma 2, it can be satisfied only together with some complete (φ, x, k)-conjunction. Now, using Lemma 3, choose in it a finite non-empty subset $\{\, \Psi_{i_1}(x) \ldots, \Psi_{i_n}(x) \,\}$ such that any complete (φ, x, k)-conjunction is equivalent to an element of this subset. We can show that $\varphi(x)$ is logically equivalent to $\Psi_{i_1}(x) \vee \ldots \vee \Psi_{i_n}(x)$.

In fact, if $M, a \models \varphi(x)$ then, by Lemma 2, at least one complete (φ, x, k)-conjunction is true at (M, a) and therefore, its equivalent in $\{ \Psi_{i_1}(x) \ldots, \Psi_{i_n}(x) \}$ is also true at (M, a), and so, finally we have

$$M, a \models \Psi_{i_1}(x) \vee \ldots \vee \Psi_{i_n}(x).$$

In the other direction, if $M, a \models \Psi_{i_1}(x) \vee \ldots \vee \Psi_{i_n}(x)$, then for some $1 \leq j \leq n$ we have $M, a \models \Psi_{i_j}(x)$. Then, since $\Psi_{i_j}(x) \wedge \neg\varphi(x)$ is, according to our assumption, unsatisfiable, we must also have $M, a \models \varphi(x)$. So $\varphi(x)$ is logically equivalent to $\Psi_{i_1}(x) \vee \ldots, \vee \Psi_{i_n}(x)$ but the latter formula, being a disjunction of conjunctions of $(2, 2)$-standard x-translations of modal intuitionistic formulas of degree not exceeding k, is itself a $(2, 2)$-standard x-translation of a modal intuitionistic formula of degree not exceeding k, and so we are done.

Case 2. There exists a complete (φ, x, k)-conjunction $\Psi(x)$ such that $\Psi(x) \wedge \neg\varphi(x)$ is satisfiable, so that for some pointed model (M_2, u) we get

$$u \models_2 \Psi(x) \wedge \neg\varphi(x).$$

We infer a contradiction thus showing that this case is impossible.

By Definition 12, we can choose a pointed Σ_φ-model (M_1, t) such that $t \models_1 \Psi(x) \wedge \varphi(x)$, and that any formula $\psi(x) \in \mathsf{tp}_k(\varphi(x), M_1, t)$ follows from $\Psi(x)$. We then construct a $(2, 2)$-modal k-asimulation from (M_1, t) to (M_2, u) and thus obtain a contradiction, since by the choice of (M_1, t) and (M_2, u) we know that φ will not be preserved along this asimulation, contrary to our assumption that φ is $(2, 2)$-modal k-asimulation-invariant.

We define this asimulation as the ordered couple (A, B), where for arbitrary $i, j \in \{ 1, 2 \}$ and $(\bar{a}_m, a) \in U_i^{m+1}$, $(\bar{b}_m, b) \in U_j^{m+1}$ we set:

$$(\bar{a}_m, a)\ A\ (\bar{b}_m, b) \Leftrightarrow (m \leq k \wedge \mathsf{tp}_{k-m}(\varphi(x), M_i, a) \subseteq \mathsf{tp}_{k-m}(\varphi(x), M_j, b)),$$

and, for B:

$$(\bar{a}_m, a)\ B\ (\bar{b}_m, b) \Leftrightarrow (m \leq k \wedge \mathsf{imp}_{k-m-1}(\varphi(x), M_j, b) \subseteq \mathsf{imp}_{k-m-1}(\varphi(x), M_i, a)).$$

From the definition of (A, B) it is evident that conditions (p-type) and (p-B-type) are satisfied. As for condition (elem), note that since every formula in $\mathsf{tp}_k(\varphi(x), M_1, t)$ follows from $\Psi(x)$, and since (M_2, u) verifies $\Psi(x)$, we must have that

$$\mathsf{tp}_k(\varphi(x), M_1, t) \subseteq \mathsf{tp}_k(\varphi(x), M_2, u),$$

which, in turn, means that we will have $t\ A\ u$.

To verify condition (p-base), note that the degree of standard translation of any atomic formula is 0, and the above condition implies that $k - m \geq 0$. Therefore, it is evident that for any $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ and any unary predicate letter $P \in \Sigma_\varphi$ we will have $a \models_i P(x) \Rightarrow b \models_j P(x)$.

To verify condition (p-step), take any $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ such that $m < k$ and any $d \in U_j$ such that $b\ R_j\ d$. In this case we will also have $m + 1 \leq k$.

Then consider the sets $\mathsf{tp}_{k-m-1}(\varphi(x), M_j, d)$ and $\overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d)$. These sets are non-empty, since by our assumption we have $k - m - 1 \geq 0$. Therefore, as we have $r(\mathsf{ST}(\perp, x)) = r(\mathsf{ST}(\top, x)) = 0$, we will also have $\mathsf{ST}(\perp, x) \in \overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d)$ and $\mathsf{ST}(\top, x) \in \mathsf{tp}_{k-m-1}(\varphi(x), M_j, d)$. Then, according to our Lemma 1, there are finite non-empty sets of logical equivalents for both sets. Choosing these finite sets, we in fact choose some finite subsets

$$\{\,\mathsf{ST}(I_1,x)\ldots\mathsf{ST}(I_n,x)\,\}\subseteq \mathsf{tp}_{k-m-1}(\varphi(x),M_j,d),$$

and

$$\{\,\mathsf{ST}(J_1,x)\ldots\mathsf{ST}(J_q,x)\,\}\subseteq \overline{\mathsf{tp}}_{k-m-1}(\varphi(x),M_j,d)$$

such that

$$(\forall\psi(x)\in \mathsf{tp}_{k-m-1}(\varphi(x),M_j,d))(\mathsf{ST}(I_1,x)\wedge\ldots\wedge\mathsf{ST}(I_n,x)\models\psi(x));$$
$$(\forall\chi(x)\in \overline{\mathsf{tp}}_{k-m-1}(\varphi(x),M_j,d))(\chi(x)\models\mathsf{ST}(J_1,x)\vee\ldots\vee\mathsf{ST}(J_q,x)).$$

But then we obtain that

$$b\not\models_j \mathsf{ST}((I_1\wedge\ldots\wedge I_n)\to(J_1\vee\ldots\vee J_q),x).$$

In fact, d falsifies this implication for (M_j,b). But every formula in both sets

$$\{\,\mathsf{ST}(I_1,x)\ldots\mathsf{ST}(I_n,x)\,\},\quad \{\,\mathsf{ST}(J_1,x)\ldots\mathsf{ST}(J_q,x)\,\}$$

is, by their choice, a $(\Sigma_\varphi,x,k-m-1)$-formula, and so the standard translation of implication under consideration must be a $(\Sigma_\varphi,x,k-m)$-formula. Note, further, that by $(\bar a_m,a)\ A\ (\bar b_m,b)$ we have

$$\mathsf{tp}_{k-m}(\varphi(x),M_i,a)\subseteq \mathsf{tp}_{k-m}(\varphi(x),M_j,b)$$

and therefore this implication must be false at (M_i,a) as well. But then take any $c\in U_i$ such that $a\ R_i\ c$ and c verifies the conjunction in the antecedent of the formula but falsifies its consequent. We must conclude then, by the choice of $\{\,\mathsf{ST}(I_1,x)\ldots\mathsf{ST}(I_n,x)\,\}$, that $c\models_i \mathsf{tp}_{k-m-1}(\varphi(x),M_j,d)$ and so, by the definition of A, and given that $m+1\le k$, that

$$(\bar b_m,b,d)\ A\ (\bar a_m,a,c).$$

Since, in addition, c falsifies every formula from $\{\,\mathsf{ST}(J_1,x)\ldots\mathsf{ST}(J_q,x)\,\}$, then, by the choice of this set, we must conclude that:

$$\overline{\mathsf{tp}}_{k-m-1}(\varphi(x),M_j,d)\subseteq \overline{\mathsf{tp}}_{k-m-1}(\varphi(x),M_i,c)$$

whence, by contraposition we get that:

$$\mathsf{tp}_{k-m-1}(\varphi(x),M_i,c)\subseteq \mathsf{tp}_{k-m-1}(\varphi(x),M_j,d).$$

But then, again by the definition of A, and the fact that $m+1\le k$, we must also have $(\bar a_m,a,c)\ A\ (\bar b_m,b,d)$, and so condition (p-step) holds.

To verify condition (p-box-2), take any $(\bar a_m,a)\ A\ (\bar b_m,b)$ such that $m+1<k$ and any $d,f\in U_j$ such that $b\ R_j\ d$ and $d\ R_{\Box j}\ f$. In this case we will also have $m+2\le k$. Then consider $\overline{\mathsf{tp}}_{k-m-2}(\varphi(x),M_j,f)$. This set is non-empty, since by our assumption we have $k-m-2\ge 0$. Therefore, as we have $r(\mathsf{ST}_{22}(\perp,x))=0$, we will also have $\mathsf{ST}_{22}(\perp,x)\in\overline{\mathsf{tp}}_{k-m-2}(\varphi(x),M_j,f)$. Then, according to our Lemma 1, there exists a finite non-empty set of logical equivalents for $\overline{\mathsf{tp}}_{k-m-2}(\varphi(x),M_j,f)$. Choosing this finite set, we in fact choose some finite $\{\,\mathsf{ST}_{22}(J_1,x)\ldots\mathsf{ST}_{22}(J_q,x)\,\}\subseteq\overline{\mathsf{tp}}_{k-m}(\varphi(x),M_j,f)$ such that

$$(\forall\psi(x)\in\overline{\mathsf{tp}}_{k-m-2}(\varphi(x),M_j,f))(\psi(x)\models\mathsf{ST}_{22}(J_1,x)\vee\ldots\vee\mathsf{ST}_{22}(J_q,x)).$$

But then we obtain that

$$b \not\models_j \mathsf{ST}_{22}(\Box(J_1 \vee \ldots \vee J_q), x).$$

In fact, d, f jointly falsify this boxed disjunction for (M_j, b). But, given that

$$\{\, \mathsf{ST}_{22}(J_1, x) \ldots \mathsf{ST}_{22}(J_q, x) \,\} \subseteq \overline{\mathsf{tp}}_{k-m-2}(\varphi(x), M_j, f),$$

the standard translation of the boxed disjunction under consideration must be in $\overline{\mathsf{tp}}_{k-m}(\varphi, M_j, b)$. Note, further, that by $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ we have

$$\mathsf{tp}_{k-m}(\varphi(x), M_i, a) \subseteq \mathsf{tp}_{k-m}(\varphi(x), M_j, b),$$

thus:

$$\overline{\mathsf{tp}}_{k-m}(\varphi(x), M_j, b) \subseteq \overline{\mathsf{tp}}_{k-m}(\varphi(x), M_i, a),$$

and therefore this boxed disjunction must be false at (M_i, a) as well. But then take any $c, e \in U_i$ such that $a\ R_i\ c$, $c\ R_{\Box i}\ e$ and c, e falsify the boxed disjunction under consideration. By choice of $\{\, \mathsf{ST}(J_1, x) \ldots \mathsf{ST}(J_q, x) \,\}$ it follows that

$$\overline{\mathsf{tp}}_{k-m-2}(\varphi(x), M_j, f) \subseteq \overline{\mathsf{tp}}_{k-m-2}(\varphi(x), M_i, e),$$

and thus

$$\mathsf{tp}_{k-m-2}(\varphi(x), M_i, e) \subseteq \mathsf{tp}_{k-m-2}(\varphi(x), M_j, f).$$

But then, again by the definition of A, and given the fact that $m + 2 \leq k$, we must also have $(\bar{a}_m, a, c, e)\ A\ (\bar{b}_m, b, d, f)$, and so condition (p-box-2) holds.

To verify condition (p-diam-2(1)), take any $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ such that $m + 1 < k$ and any $d \in U_j$ such that $b\ R_j\ d$. In this case we will also have $m + 2 \leq k$. Then consider $\mathsf{imp}_{k-m-2}(\varphi(x), M_j, d)$. This set is non-empty, since by our assumption we have $k - m - 2 \geq 0$. Therefore, as we have $r(\mathsf{ST}_{22}(\bot, x)) = 0$, we will also have $\mathsf{ST}_{22}(\bot, x) \in \mathsf{imp}_{k-m-2}(\varphi(x), M_j, d)$. Then, according to our Lemma 1, there exists a finite non-empty set of logical equivalents for $\mathsf{imp}_{k-m-2}(\varphi(x), M_j, d)$. Choosing this finite set, we in fact choose some finite

$$\{\, \mathsf{ST}_{22}(J_1, x) \ldots \mathsf{ST}_{22}(J_q, x) \,\} \subseteq \mathsf{imp}_{k-m-2}(\varphi(x), M_j, d)$$

such that

$$(\forall \psi(x) \in \mathsf{imp}_{k-m-2}(\varphi(x), M_j, d))(\psi(x) \models \mathsf{ST}_{22}(J_1, x) \vee \ldots \vee \mathsf{ST}_{22}(J_q, x)).$$

But then we obtain that

$$b \not\models_j \mathsf{ST}_{22}(\Diamond(J_1 \vee \ldots \vee J_q), x).$$

In fact, d falsifies this disjunction for (M_j, b). But, given that

$$\{\, \mathsf{ST}_{22}(J_1, x) \ldots \mathsf{ST}_{22}(J_q, x) \,\} \subseteq \mathsf{imp}_{k-m-2}(\varphi(x), M_j, d),$$

the standard translation of the modalized disjunction under consideration must be in $\overline{\mathsf{tp}}_{k-m}(\varphi(x), M_j, b)$. Note, further, that by $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ we have

$$\mathsf{tp}_{k-m}(\varphi(x), M_i, a) \subseteq \mathsf{tp}_{k-m}(\varphi(x), M_j, b),$$

thus:

$$\overline{\mathsf{tp}}_{k-m}(\varphi(x), M_j, b) \subseteq \overline{\mathsf{tp}}_{k-m}(\varphi(x), M_i, a),$$

and therefore the modalized disjunction must be false at (M_i, a) as well. But then take any $c \in U_i$ such that $a\ R_i\ c$, and for every e, such that $c\ R_{\diamond i}\ e$, we have

$$e \not\models_j \mathsf{ST}_{22}(J_1, x) \vee \ldots \vee \mathsf{ST}_{22}(J_q, x).$$

By choice of $\{\,\mathsf{ST}_{22}(J_1, x) \ldots \mathsf{ST}_{22}(J_q, x)\,\}$ it follows that

$$\mathsf{imp}_{k-m-2}(\varphi(x), M_j, d) \subseteq \mathsf{imp}_{k-m-2}(\varphi(x), M_i, c).$$

But then, again by the definition of B, and given the fact that $m + 2 \leq k$, we must also have $(\bar{a}_m, a, c)\ B\ (\bar{b}_m, b, d)$, and so condition (p-diam-2(1)) holds.

Finally, to verify condition (p-diam-2(2)), take any $(\bar{a}_m, a)\ B\ (\bar{b}_m, b)$ such that $m < k$ and any $c \in U_i$ such that $a\ R_{\diamond i}\ c$. In this case we will also have $m + 1 \leq k$. Then consider $\mathsf{tp}_{k-m-1}(\varphi(x), M_i, c)$. This set is non-empty, since by our assumption we have $k - m - 1 \geq 0$. Therefore, as we have $r(\mathsf{ST}_{22}(\top, x)) = 0$, we will also have $\mathsf{ST}_{22}(\top, x) \in \mathsf{tp}_{k-m-1}(\varphi(x), M_i, c)$. Then, according to our Lemma 1, there exists a finite non-empty set of logical equivalents for $\mathsf{tp}_{k-m-1}(\varphi(x), M_i, c)$. Choosing this finite set, we in fact choose some finite $\{\,\mathsf{ST}_{22}(I_1, x) \ldots \mathsf{ST}_{22}(I_p, x)\,\} \subseteq \mathsf{tp}_{k-m-1}(\varphi(x), M_i, c)$ such that

$$(\forall \psi(x) \in \mathsf{tp}_{k-m-1}(\varphi(x), M_i, c))(\mathsf{ST}_{22}(I_1, x) \wedge \ldots \wedge \mathsf{ST}_{22}(I_p, x) \models \psi(x)).$$

But then we obtain that

$$\mathsf{ST}_{22}((I_1 \wedge \ldots \wedge I_p), x) \notin \mathsf{imp}_{k-m-1}(\varphi(x), M_i, a).$$

Note, further, that by $(\bar{a}_m, a)\ B\ (\bar{b}_m, b)$ we have

$$\mathsf{imp}_{k-m-1}(\varphi(x), M_j, b) \subseteq \mathsf{imp}_{k-m-1}(\varphi(x), M_i, a),$$

thus:

$$\mathsf{ST}_{22}((I_1 \wedge \ldots \wedge I_p), x) \notin \mathsf{imp}_{k-m-1}(\varphi(x), M_j, b).$$

But then take any $d \in U_j$ such that $b\ R_{\diamond j}\ d$ and we have

$$d \models_j \mathsf{ST}_{22}(I_1, x) \wedge \ldots \wedge \mathsf{ST}_{22}(I_p, x).$$

By choice of $\{\,\mathsf{ST}_{22}(I_1, x) \ldots \mathsf{ST}_{22}(I_p, x)\,\}$ it follows that

$$\mathsf{tp}_{k-m-1}(\varphi(x), M_i, c) \subseteq \mathsf{tp}_{k-m-1}(\varphi(x), M_j, d).$$

But then, again by the definition of A, and given the fact that $m + 1 \leq k$, we must also have $(\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d)$, and so condition (p-diam-2(2)) holds.

Therefore (A, B) is a $(2,2)$-modal k-asimulation from (M_1, t) to (M_2, u), and we have got our contradiction in place. □

Theorem 4 now immediately follows from Corollary 3 and Proposition 2. It remains to supply a proof for Corollary 2.

This proof is as follows:

If $\varphi(x)$ is equivalent to a $(2,2)$-standard x-translation of a modal intuitionistic formula I, then let $r(\mathsf{ST}_{22}(I, x)) = k$. By Theorem 4 we get then that $\varphi(x)$ is invariant w.r.t. basic k-asimulations.

In the other direction, if $\varphi(x)$ is invariant w.r.t. basic k-asimulations, then, by Theorem 4, there must be a modal intuitionistic formula I such that $\varphi(x)$ is logically equivalent to $r(\mathsf{ST}_{22}(I, x))$.

5.2. Proof of Theorem 3

We now turn to the proof of $(2,2)$-instantiation of Theorem 3. The 'only if' direction we again have by Corollary 3:

Corollary 4. *If $\varphi(x)$ is equivalent to a $(2,2)$-standard x-translation of an intuitionistic formula, then $\varphi(x)$ is invariant with respect to $(2,2)$-modal asimulations.*

Proof. Let $\varphi(x)$ be logically equivalent to $\mathsf{ST}_{22}(I, x)$ for some intuitionistic formula I. For an arbitrary $\Theta \supseteq \Sigma_\varphi$, Θ-models M_1 and M_2, and arbitrary $t \in U_1$, $u \in U_2$ let (A, B) be a $(2,2)$-modal asimulation from (M_1, t) to (M_2, u), so that we have $t\ A\ u$. Assume that

$$t \models_1 \varphi(x).$$

Then consider the ordered couple (A', B') such that:

$$\begin{aligned} A' = \{\, &\langle(\bar{a}_m, a), (\bar{b}_m, b;)\rangle \mid \\ &\mid \exists i, j(\{\, i, j\,\} = \{\,1, 2\,\} \wedge (\bar{a}_m, a) \in U_i^{m+1} \wedge (\bar{b}_m, b) \in U_j^{m+1} \wedge a\ A\ b)\,\}; \\ B' = \{\, &\langle(\bar{a}_m, a), (\bar{b}_m, b;)\rangle \mid \\ &\mid \exists i, j(\{\, i, j\,\} = \{\,1, 2\,\} \wedge (\bar{a}_m, a) \in U_i^{m+1} \wedge (\bar{b}_m, b) \in U_j^{m+1} \wedge a\ B\ b)\,\}. \end{aligned}$$

It is straightforward to verify that (A', B') is a $(2,2)$-modal k-asimulation from (M_1, t) to (M_2, u) for every $k \in \mathbb{N}$. Moreover, we still have $t\ A'\ u$. By Corollary 3, there is a natural k, such that $\varphi(x)$ is invariant with respect to $(2,2)$-modal k-asimulation, therefore we have

$$u \models_2 \varphi(x).$$

Since the $(2,2)$-modal asimulation (A, B) was chosen arbitrarily, this means that $\varphi(x)$ is invariant with respect to $(2,2)$-modal asimulations. □

Note that the proof of the above corollary shows that every $(2,2)$-modal asimulation induces a couple of binary relations which turns out to be a $(2,2)$-modal k-asimulation for every natural k. In other words, the following lemma holds:

Lemma 4. *If (A, B) is a $(2,2)$-modal asimulation from (M_1, t) to (M_2, u), then the couple (A', B'), defined as in the proof of Corollary 4, is a $(2,2)$-modal k-asimulation from (M_1, t) to (M_2, u) for every $k \in \mathbb{N}$.*

Lemma 4 admits of a converse along the lines of [10, Lemma 4.4(2)]. More precisely, if (A, B) is a $(2, 2)$-modal k-asimulation for every natural k, then one can define a $(2, 2)$-modal asimulation (A'', B'') stipulating that $a\ A''\ b$ iff there exist $\bar{a}_m$, $\bar{b}_m$, such that $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$, and similarly for B''.

To proceed, we need to introduce some further notions and results from classical model theory. For a model M and $\bar{a}_n \in U$ let $[M, \bar{a}_n]$ be the extension of M with $\bar{a}_n$ as new individual constants interpreted as themselves. It is easy to see that there is a simple relation between the truth of a formula at a sequence of elements of a Θ-model and the truth of its substitution instance in an extension of the above-mentioned kind; namely, for any Θ-model M, any Θ-formula $\varphi(\bar{y}_n, \bar{w}_m)$ and any $\bar{a}_n, \bar{b}_m \in U$ it is true that:

$$[M, \bar{a}_n], \bar{b}_m \models \varphi(\bar{a}_n, \bar{w}_m) \Leftrightarrow M, \bar{a}_n, \bar{b}_m \models \varphi(\bar{y}_n, \bar{w}_m).$$

We will call a theory of M (and write $Th(M)$) the set of all first-order sentences true at M. We will call an n-type of M a set of formulas $\Gamma(\bar{w}_n)$ consistent with $Th(M)$.

Definition 13. Let M be a Θ-model. M is *ω-saturated* iff for all $k \in \mathbb{N}$ and for all $\bar{a}_n \in U$, every k-type $\Gamma(\bar{w}_k)$ of $[M, \bar{a}_n]$ is satisfiable in $[M, \bar{a}_n]$.

Definition of ω-saturation normally requires satisfiability of 1-types only. However, our modification is equivalent to the more familiar version: see e.g. [6, Lemma 4.31, p. 73].

It is known that every model can be elementarily extended to an ω-saturated model; in other words, the following lemma holds:

Lemma 5. *Let M be a Θ-model. Then there is an ω-saturated extension M' of M such that for all $\bar{a}_n \in U$ and every Θ-formula $\varphi(\bar{w}_n)$:*

$$M, \bar{a}_n \models \varphi(\bar{w}_n) \Leftrightarrow M', \bar{a}_n \models \varphi(\bar{w}_n).$$

The latter lemma is a trivial corollary of e.g. [5, Lemma 5.1.14, p. 216].

A very useful property of ω-saturated models is that one can define among them $(2, 2)$-modal asimulations more or less according to the strategy assumed in the proof of Proposition 2. In order to do this, however, we need to re-define the types used in the above proof. We collect the required changes in the following definition:

Definition 14. Let M be a Θ-model, $t \in U$ and let x be a variable in the correspondence language. Then we define $\mathsf{int}_x(\Theta)$ to be the set of all Θ-formulas that are $(2, 2)$-standard x-translations of modal intuitionistic formulas. We further set:

$$\mathsf{tp}_x(M, t) = \{\psi(x) \in \mathsf{int}_x(\Theta) \mid M, t \models \psi(x)\};$$

$$\overline{\mathsf{tp}}_x(M, t) = \{\psi(x) \in \mathsf{int}_x(\Theta) \mid M, t \not\models \psi(x)\};$$

$$\mathsf{imp}_x(M, t) = \bigcap \{\overline{\mathsf{tp}}_x(M, u) \mid \iota(R_\diamond)(t, u)\}.$$

The analogue of 'contrapositive' scheme mentioned above holds, namely, for arbitrary models M, M', and elements $a \in U$, and $a' \in U'$ we have:

$$\mathsf{tp}_x(M, a) \subseteq \mathsf{tp}_x(M', a') \Leftrightarrow \overline{\mathsf{tp}}_x(M', a') \subseteq \overline{\mathsf{tp}}_x(M, a).$$

The following proposition gives the precise version of the above statement:

Proposition 3. *Let M_1, M_2 be ω-saturated Θ-models, let $t \in U_1$, and $u \in U_2$ be such that $\mathsf{tp}_x(M_1, t) \subseteq \mathsf{tp}_x(M_2, u)$ for some variable x in the correspondence language. Then the ordered couple (A, B) such that:*

$$A = \{ \langle a, b\rangle \mid \exists i, j(\{ i, j \} = \{ 1, 2 \} \wedge \mathsf{tp}_x(M_i, a) \subseteq \mathsf{tp}_x(M_j, b)) \}$$

and:

$$B = \{ \langle a, b\rangle \mid \exists i, j(\{ i, j \} = \{ 1, 2 \} \wedge \mathsf{imp}_x(M_i, a) \supseteq \mathsf{imp}_x(M_j, b)) \}$$

is a $(2, 2)$-modal asimulation from (M_1, t) to (M_2, u).

Proof. It is obvious that $t\ A\ u$, and that conditions (type) and (B-type) are satisfied as well. Since for any unary predicate letter P and variable x formula $P(x)$ is a standard x-translation of an atomic intuitionistic formula, condition (base) is trivially satisfied for A.

To verify condition (step), choose any i, j such that $\{i, j\} = \{ 1, 2 \}$, and $a \in U_i$, $b, d \in U_j$ such that $a\ A\ b$, that is to say, $\mathsf{tp}_x(M_i, a) \subseteq \mathsf{tp}_x(M_j, b)$ and $b\ R_j\ d$. Now consider $\mathsf{tp}_x(M_j, d)$ and $\overline{\mathsf{tp}}_x(M_j, d)$ and choose an arbitrary finite subset for each of these types, so that we have:

$$\{ \mathsf{ST}_{22}(I_1, x) \ldots \mathsf{ST}_{22}(I_p, x) \} \subseteq \mathsf{tp}_x(M_j, d),$$
$$\{ \mathsf{ST}_{22}(J_1, x) \ldots \mathsf{ST}_{22}(J_q, x) \} \subseteq \overline{\mathsf{tp}}_x(M_j, d).$$

We immediately get that:

$$b \not\models_j \mathsf{ST}_{22}((I_1 \wedge \ldots \wedge I_p) \to (J_1 \vee \ldots \vee J_q), x).$$

Since by contraposition of $a\ A\ b$ we have that $\overline{\mathsf{tp}}_x(M_j, b) \subseteq \overline{\mathsf{tp}}_x(M_i, a)$, we obtain that:

$$a \not\models_i \mathsf{ST}_{22}((I_1 \wedge \ldots \wedge I_p) \to (J_1 \vee \ldots \vee J_q), x).$$

This means that every finite subset of the type

$$\{ R(a, x) \} \cup \mathsf{tp}_x(M_j, d) \cup \{\neg\psi(x) \mid \psi(x) \in \overline{\mathsf{tp}}_x(M_j, d)\}$$

is satisfiable at $[M_i, a]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_i, a])$ and, by ω-saturation of both M_1 and M_2, it must be satisfied in $[M_i, a]$ by some $c \in U_i$. So for any such c we will have $a\ R_i\ c$ and, moreover,

$$\mathsf{tp}_x(M_j, d) = \mathsf{tp}_x(M_i, c),$$

whence we get that $c \overset{\leftrightarrow}{A} d$ and that condition (step) is verified.

To verify condition (box-2), choose any i, j such that $\{ i, j \} = \{ 1, 2 \}$, any $a \in U_i$, $b \in U_j$ such that $a\ A\ b$, that is to say, $\mathsf{tp}_x(M_i, a) \subseteq \mathsf{tp}_x(M_j, b)$ and choose any $d, f \in U_j$ for which we have $b\ R_j\ d$ and $d\ R_{\Box j}\ f$.

Consider $\overline{\mathsf{tp}}_x(M_j, f)$. If $\{ \mathsf{ST}_{22}(J_1, x) \ldots \mathsf{ST}_{22}(J_q, x) \}$ is a finite subset of this type, then we have

$$b \not\models_j \mathsf{ST}_{22}(\Box(J_1 \vee \ldots \vee J_q), x).$$

Since by contraposition of $a\ A\ b$ we have that $\overline{\mathsf{tp}}_x(M_j, b) \subseteq \overline{\mathsf{tp}}_x(M_i, a)$, we obtain that

$$a \not\models_i \mathsf{ST}_{22}(\Box(J_1 \vee \ldots \vee J_q), x).$$

This means that every finite subset of the type

$$\{\, R(a,y), R_{\Box}(y,x) \,\} \cup \{\neg\psi(x) \mid \psi(x) \in \overline{\mathsf{tp}}_x(M_j, f)\}$$

is satisfiable at $[M_i, a]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_i, a])$ and, by ω-saturation of both M_1 and M_2, it must be satisfied in $[M_i, a]$ by some $c, e \in U_i$. So for any such c and e we will have $a\ R_i\ c$, $c\ R_{\Box i}\ e$ and, moreover,

$$(\forall \psi \in \overline{\mathsf{tp}}_x(M_j, f))(e \not\models_i \psi(x)).$$

Thus we have that $\overline{\mathsf{tp}}_x(M_j, f) \subseteq \overline{\mathsf{tp}}_x(M_i, e)$, and further, by contraposition, that $\mathsf{tp}_x(M_i, e) \subseteq \mathsf{tp}_x(M_j, f)$. Thus we get that $e\ A\ f$ and condition (box-2) is verified.

To verify condition (diam-2(1)), choose any i, j such that $\{\, i, j \,\} = \{\, 1, 2 \,\}$, any $a \in U_i$, $b \in U_j$ such that $a\ A\ b$, that is to say, $\mathsf{tp}_x(M_i, a) \subseteq \mathsf{tp}_x(M_j, b)$ and choose any $d \in U_j$ for which we have $b\ R_j\ d$.

Consider $\mathsf{imp}_x(M_j, d)$. If $\{\, \mathsf{ST}_{22}(J_1, x) \dots \mathsf{ST}_{22}(J_q, x) \,\}$ is a finite subset of this type, then we have

$$b \not\models_j \mathsf{ST}_{22}(\Diamond(J_1 \vee \ldots \vee J_q), x).$$

Since by contraposition of $a\ A\ b$ we have that $\overline{\mathsf{tp}}_x(M_j, b) \subseteq \overline{\mathsf{tp}}_x(M_i, a)$, we obtain that

$$a \not\models_i \mathsf{ST}_{22}(\Diamond(J_1 \vee \ldots \vee J_q), x).$$

This means that every finite subset of the type

$$\{\, R(a,x) \,\} \cup \{\forall y(R_{\Diamond}(x,y) \to \neg\psi(y)) \mid \psi(x) \in \mathsf{imp}_x(M_j, d)\}$$

is satisfiable at $[M_i, a]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_i, a])$ and, by ω-saturation of both M_1 and M_2, it must be satisfied in $[M_i, a]$ by some $c \in U_i$. So for any such c we will have $a\ R_i\ c$ and, moreover,

$$(\forall \psi(x) \in \mathsf{imp}_x(M_j, d))(\forall e \in U_i)(R_{\Diamond i}(c, e) \Rightarrow e \not\models_i \psi(x)).$$

Thus we have that $\mathsf{imp}_x(M_j, d) \subseteq \mathsf{imp}_x(M_i, c)$, and therefore, that $c\ B\ d$. Thus condition (diam-2(1)) is verified.

Finally, to verify condition (diam-2(2)), choose any i, j such that $\{\, i, j \,\} = \{\, 1, 2 \,\}$, any $a \in U_i$, $b \in U_j$ such that $a\ B\ b$, that is to say, $\mathsf{imp}_x(M_j, b) \subseteq \mathsf{imp}_x(M_i, a)$ and choose any $c \in U_i$ for which we have $a\ R_{\Diamond i}\ c$.

Consider $\mathsf{tp}_x(M_i, c)$. If $\{\, \mathsf{ST}_{22}(I_1, x) \dots \mathsf{ST}_{22}(I_p, x) \,\}$ is a finite subset of this type, then we have

$$c \models_i \mathsf{ST}_{22}((I_1 \wedge \ldots \wedge I_p), x).$$

Therefore, the set $\{\, \mathsf{ST}_{22}(I_1, x) \dots \mathsf{ST}_{22}(I_p, x) \,\}$ is disjoint from $\mathsf{imp}_x(M_i, a)$, and thus it is also disjoint from $\mathsf{imp}_x(M_j, b)$. Therefore, the formula $\mathsf{ST}_{22}(I_1 \wedge \ldots \wedge I_p, x)$ is also verified by some $R_{\Diamond j}$-successor of b. More formally, this means that every finite subset of the type

$$\{\, R_{\Diamond}(b,x) \,\} \cup \{\psi(x) \mid \psi(x) \in \mathsf{tp}_x(M_i, c)\}$$

is satisfiable at $[M_j, b]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_j, b])$ and, by ω-saturation of both M_1 and M_2, it must be satisfied in $[M_j, b]$ by some $d \in U_j$. So for any such d we will have $b\ R_{\Diamond j}\ d$ and, moreover,

$$(\forall \psi(x) \in \mathsf{tp}_x(M_i, c))(d \models_j \psi(x)).$$

Thus we have that $\mathsf{tp}_x(M_i, c) \subseteq \mathsf{tp}_x(M_j, c)$, and therefore, that $c\ A\ d$. The condition (diam-2(2)) is verified. □

We are prepared now to prove the hard part of $(2, 2)$-instantiation of Theorem 3:

Lemma 6. *Let $\varphi(x)$ be invariant with respect to $(2, 2)$-modal asimulations. Then $\varphi(x)$ is equivalent to $(2, 2)$-standard translation of a modal intuitionistic formula.*

Proof. We may assume that $\varphi(x)$ is satisfiable, for $\perp$ is clearly invariant with respect to $(2, 2)$-modal asimulations and $\perp \leftrightarrow \mathsf{ST}_{22}(\perp, x)$ is a valid formula. Throughout this proof, we will write $\mathrm{ic}(\varphi(x))$ for the following set:

$$\{\psi(x) \in \mathsf{int}_x(\Sigma_\varphi) \mid \varphi(x) \models \psi(x)\}$$

Our strategy will be to show that $\mathrm{ic}(\varphi(x)) \models \varphi(x)$. Once this is done, we will apply compactness of first-order logic and conclude that $\varphi(x)$ is equivalent to a finite conjunction of standard $(2, 2)$-modal x-translations of intuitionistic formulas and hence to a standard x-translation of the corresponding intuitionistic conjunction.

To show this, take any Σ_φ-model M_1 and $a \in U_1$ such that $a \models_1 \mathrm{ic}(\varphi(x))$. Then, of course, we also have $\mathrm{ic}(\varphi(x)) \subseteq \mathsf{tp}_x(M_1, a)$. Such a model exists, because $\varphi(x)$ is satisfiable and $\mathrm{ic}(\varphi(x))$ will be satisfied in any model satisfying $\varphi(x)$. Then we can also choose a Σ_φ-model M_2 and $b \in U_2$ such that $b \models_2 \varphi(x)$ and $\mathsf{tp}_x(M_2, b) \subseteq \mathsf{tp}_x(M_1, a)$.

For suppose otherwise. Then for any Σ_φ-model M such that $U \subseteq \mathbb{N}$ and any $c \in U$ such that $M, c \models \varphi(x)$ we can choose a modal intuitionistic formula $I_{(M,c)}$ such that $\mathsf{ST}_{22}(I_{(M,c)}, x)$ is in $\mathsf{tp}_x(M, c)$ but not in $\mathsf{tp}_x(M_1, a)$. Then consider the set

$$S = \{\, \varphi(x) \,\} \cup \{\, \neg\mathsf{ST}_{22}(I_{(M,c)}, x) \mid M, c \models \varphi(x) \,\}.$$

Let $\{\, \varphi(x), \neg\mathsf{ST}_{22}(I_1, x) \dots, \neg\mathsf{ST}_{22}(I_q, x) \,\}$ be a finite subset of this set. If this set is unsatisfiable, then we must have $\varphi(x) \models \mathsf{ST}_{22}(I_1, x) \vee \dots \vee \mathsf{ST}_{22}(I_q, x)$, but then we will also have $(\mathsf{ST}_{22}(I_1, x) \vee \dots \vee \mathsf{ST}_{22}(I_q, x)) \in \mathrm{ic}(\varphi(x)) \subseteq \mathsf{tp}_x(M_1, a)$, and hence $(\mathsf{ST}_{22}(I_1, x) \vee \dots \vee \mathsf{ST}_{22}(I_q, x))$ will be true at (M_1, a). But then at least one of $\mathsf{ST}_{22}(I_1, x) \dots \mathsf{ST}_{22}(I_q, x)$ must also be true at (M_1, a), which contradicts the choice of these formulas. Therefore, every finite subset of S is satisfiable, and, by compactness, S itself is satisfiable as well. But then, by the Löwenheim–Skolem property, we can take a Σ_φ-model M' such that $U' \subseteq \mathbb{N}$ and $g \in U'$ such that S is true at (M', g) and this will be a model for which we will have both $M', g \models \mathsf{ST}_{22}(I_{(M',g)}, x)$ by choice of $I_{(M',g)}$ and $M', g \not\models \mathsf{ST}_{22}(I_{(M',g)}, x)$ by satisfaction of S, a contradiction.

Therefore, we will assume in the following that some Σ_φ-model M_2 and some $b \in U_2$ are such that $a \models_1 \mathrm{ic}(\varphi(x))$, $b \models_2 \varphi(x)$, and $\mathsf{tp}_x(M_2, b) \subseteq \mathsf{tp}_x(M_1, a)$. According to Lemma 5, there exist ω-saturated elementary extensions M', M'' of M_1 and M_2, respectively. We have:

$$M_1, a \models \varphi(x) \Leftrightarrow M', a \models \varphi(x) \tag{39}$$

$$M'', b \models \varphi(x) \tag{40}$$

Also, since M_1, M_2 are elementarily equivalent to M', M'', respectively, we have

$$\mathsf{tp}_x(M'', b) = \mathsf{tp}_x(M_2, b) \subseteq \mathsf{tp}_x(M_1, a) = \mathsf{tp}_x(M', a).$$

By ω-saturation of M', M'' and Proposition 3, the ordered couple (A, B) such that:

$$A = \{ \langle c, d\rangle \mid \exists \mu, \mu'(\{ \mu, \mu' \} = \{ M', M'' \} \wedge \mathsf{tp}_x(\mu, c) \subseteq \mathsf{tp}_x(\mu', d)) \}$$
$$B = \{ \langle c, d\rangle \mid \exists \mu, \mu'(\{ \mu, \mu' \} = \{ M', M'' \} \wedge \mathsf{imp}_x(\mu, c) \supseteq \mathsf{imp}_x(\mu', d)) \}$$

is a $(2, 2)$-modal asimulation from (M'', b) to (M', a). But then by (40) and invariance of $\varphi(x)$ we get $M', a \models \varphi(x)$, and further, by (39) we conclude that $M_1, a \models \varphi(x)$. Therefore, $\varphi(x)$ in fact follows from $\mathrm{ic}(\varphi(x))$. $\square$

Theorem 3 now follows from Corollary 4 and Lemma 6.

6. Other cases

We now briefly show how to obtain the proofs for the other three instantiations of Theorems 4 and 3. The general scheme of the proofs in these cases is very similar to the proofs given in the previous section. The main difference is that in the other cases we need to assume different sets of conditions in the definitions of modal k-asimulations and modal asimulations *simpliciter*. This affects the three propositions of the previous section, namely, Propositions 1, 2, and 3, in that some parts of their proofs become irrelevant and some new parts need to be supplied instead. Accordingly, when treating the other three instantiations of our main results below, we mainly concentrate on how to revise the proofs of these propositions.

6.1. Case $i = 1$, $j = 2$

In order to obtain the proofs of Theorems 4 and 3 one needs to revise the proofs given in Section 5 in the following way:

Ad Proposition 1:

Revise the inductive step in case where $I = \square J$ as follows:

In this case we have

$$\varphi(x) = \forall y(R_\square(x, y) \to \mathsf{ST}_{12}(J, y)).$$

Assume that:

$$a \models_i \forall y(R_\square(x, y) \to \mathsf{ST}_{12}(J, y)) \tag{41}$$
$$(\bar{a}_m, a)\ A\ (\bar{b}_m, b) \tag{42}$$
$$m + r(\varphi(x)) \le l \tag{43}$$

Moreover, it follows from the definition of r that:

$$r(\varphi(x)) \ge 1 \tag{44}$$
$$r(\mathsf{ST}_{12}(J, y)) \le r(\varphi(x)) - 1 \tag{45}$$

Now, consider arbitrary $d \in U_j$ such that $b\ R_{\square j}\ d$. Since (43) and (44) clearly imply that $m < l$, it follows from (42) and (p-box-1) that one can choose a $c \in U_i$, such that:

$$a\ R_{\square i}\ c \tag{46}$$
$$(\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d) \tag{47}$$

So, we reason as follows:

$$c \models_i \mathsf{ST}_{12}(J, y) \qquad \text{(from (41) and (46))} \tag{48}$$

$$m + 1 + r(\mathsf{ST}_{12}(J, y)) \leq l \qquad \text{(from (43) and (45))} \tag{49}$$

$$d \models_j \mathsf{ST}_{12}(J, y) \qquad \text{(from (47), (48), (49) by IH)} \tag{50}$$

Since d was chosen to be an arbitrary $R_{\Box j}$-successor of b, this means that

$$b \models_j \forall y(R_\Box(x, y) \to \mathsf{ST}_{12}(J, y)),$$

and we are done.

Ad Proposition 2:

Replace the verification of condition (p-box-2) with the following verification of condition (p-box-1):

Take any $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ such that $m < k$ and any $d \in U_j$ such that $b\ R_{\Box j}\ d$. In this case we will also have $m + 1 \leq k$. Then consider $\overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d)$. This set is non-empty, since by our assumption we have $k - m - 1 \geq 0$. Therefore, as we have $r(\mathsf{ST}_{12}(\bot, x)) = 0$, we will also have $\mathsf{ST}_{12}(\bot, x) \in \overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d)$. Then, according to Lemma 1, there exists a finite non-empty set of logical equivalents for $\overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d)$. Choosing this finite set, we in fact choose some finite $\{\,\mathsf{ST}_{12}(J_1, x) \dots \mathsf{ST}_{12}(J_q, x)\,\} \subseteq \overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d)$ such that

$$(\forall \psi(x) \in \overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d))(\psi(x) \models \mathsf{ST}_{12}(J_1, x) \vee \dots \vee \mathsf{ST}_{12}(J_q, x)).$$

But then we obtain that

$$b \not\models_j \mathsf{ST}_{12}(\Box(J_1 \vee \dots \vee J_q), x).$$

In fact, d falsifies this boxed disjunction for (M_j, b). But, given that

$$\{\,\mathsf{ST}_{12}(J_1, x) \dots \mathsf{ST}_{12}(J_q, x)\,\} \subseteq \overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d),$$

the standard translation of boxed disjunction under consideration must be in $\overline{\mathsf{tp}}_{k-m}(\varphi(x), M_j, b)$. Note, further, that by $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ we have

$$\mathsf{tp}_{k-m}(\varphi(x), M_i, a) \subseteq \mathsf{tp}_{k-m}(\varphi(x), M_j, b),$$

thus:

$$\overline{\mathsf{tp}}_{k-m}(\varphi(x), M_j, b) \subseteq \overline{\mathsf{tp}}_{k-m}(\varphi(x), M_i, a),$$

and therefore this boxed disjunction must be false at (M_i, a) as well. But then take any $c \in U_i$ such that $a\ R_{\Box i}\ c$ and c falsifies the disjunction under consideration. By choice of $\{\,\mathsf{ST}_{12}(J_1, x) \dots \mathsf{ST}_{12}(J_q, x)\,\}$ it follows that

$$\overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_j, d) \subseteq \overline{\mathsf{tp}}_{k-m-1}(\varphi(x), M_i, c),$$

and thus

$$\mathsf{tp}_{k-m-1}(\varphi(x), M_i, c) \subseteq \mathsf{tp}_{k-m-1}(\varphi(x), M_j, d).$$

But then, again by the definition of A, and given the fact that $m + 1 \leq k$, we must also have $(\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d)$, and so condition (p-box-1) holds.

Ad Proposition 3:

Replace the verification of condition (box-2) with the following verification of condition (box-1):

Choose any i, j such that $\{ i, j \} = \{ 1, 2 \}$, any $a \in U_i$, $b \in U_j$ such that $a\ A\ b$, that is to say, $\mathsf{tp}_x(M_i, a) \subseteq \mathsf{tp}_x(M_j, b)$ and choose any $d \in U_j$ for which we have $b\ R_{\Box j}\ d$.

Consider $\overline{\mathsf{tp}}_x(M_j, d)$. If $\{ \mathsf{ST}_{12}(J_1, x) \dots \mathsf{ST}_{12}(J_q, x) \}$ is a finite subset of this type, then we have

$$b \not\models_j \mathsf{ST}_{12}(\Box(J_1 \vee \ldots \vee J_q), x).$$

Since by contraposition of $a\ A\ b$ we have that $\overline{\mathsf{tp}}_x(M_j, b) \subseteq \overline{\mathsf{tp}}_x(M_i, a)$, we obtain that

$$a \not\models_i \mathsf{ST}_{12}(\Box(J_1 \vee \ldots \vee J_q), x).$$

This means that every finite subset of the type

$$\{ R_\Box(a, x) \} \cup \{\neg\psi(x) \mid \psi(x) \in \overline{\mathsf{tp}}_x(M_j, d)\}$$

is satisfiable at $[M_i, a]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_i, a])$ and, by ω-saturation of both M_1 and M_2, it must be satisfied in $[M_i, a]$ by some $c \in U_i$. So for any such c we will have $a\ R_{\Box i}\ c$ and, moreover,

$$(\forall\psi \in \overline{\mathsf{tp}}_x(M_j, d))(c \not\models_i \psi(x)).$$

Thus we have that $\overline{\mathsf{tp}}_x(M_j, d) \subseteq \overline{\mathsf{tp}}_x(M_i, c)$, and further, by contraposition, that $\mathsf{tp}_x(M_i, c) \subseteq \mathsf{tp}_x(M_j, d)$. Hence we get that $c\ A\ d$ and condition (box-2) is verified.

6.2. Case $i = 2$, $j = 1$

The changes in three Propositions for this case will look as follows:

Ad Proposition 1:

Revise the inductive step for the case $I = \Diamond J$ as follows:

In this case we have

$$\varphi(x) = \exists y(R_\Diamond(x, y) \wedge \mathsf{ST}_{21}(J, y)).$$

Assume that:

$$a \models_i \exists y(R_\Diamond(x, y) \wedge \mathsf{ST}_{21}(J, y)) \tag{51}$$

$$(\bar{a}_m, a)\ A\ (\bar{b}_m, b) \tag{52}$$

$$m + r(\varphi(x)) \leq l \tag{53}$$

Moreover, it follows from the definition of r that:

$$r(\varphi(x)) \geq 1 \tag{54}$$

$$r(\mathsf{ST}_{21}(J, y)) \leq r(\varphi(x)) - 1 \tag{55}$$

Now, by (51) choose a $c \in U_i$ such that

$$a\ R_{\diamond i}\ c \tag{56}$$

$$c \models_i \mathsf{ST}_{21}(J, y) \tag{57}$$

Since (53) and (54) clearly imply that $m < l$, it follows from (52) and (p-diam-1) that one can choose a $d \in U_j$, such that:

$$b\ R_{\diamond j}\ d \tag{58}$$

$$(\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d) \tag{59}$$

So, we get that:

$$m + 1 + r(\mathsf{ST}_{21}(J, y)) \leq l \qquad \text{(from (53) and (55))} \tag{60}$$

$$d \models_j \mathsf{ST}_{21}(J, y) \qquad \text{(from (57), (59), (60) by IH)} \tag{61}$$

Finally, from (58) and (61) we infer that:

$$b \models_j \exists y(R_\diamond(x, y) \wedge \mathsf{ST}_{21}(J, y)),$$

and we are done.

Ad Proposition 2:

Replace the verification of conditions (p-diam-2(1)) and (p-diam-2(2)) with the following verification of condition (p-diam-1):

Take any $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ such that $m < k$ and any $c \in U_i$ such that $a\ R_{\diamond i}\ c$. In this case we will also have $m + 1 \leq k$. Then consider $\mathsf{tp}_{k-m-1}(\varphi(x), M_i, c)$. This set is non-empty, since by our assumption we have $k - m - 1 \geq 0$. Therefore, as we have $r(\mathsf{ST}_{21}(\top, x)) = 0$, we will also have $\mathsf{ST}_{21}(\top, x) \in \mathsf{tp}_{k-m-1}(\varphi(x), M_i, c)$. Then, according to Lemma 1, there exists a finite non-empty set of logical equivalents for $\mathsf{tp}_{k-m-1}(\varphi(x), M_i, c)$. Choosing this finite set, we in fact choose some finite $\{\, \mathsf{ST}_{21}(I_1, x) \dots \mathsf{ST}_{21}(I_p, x) \,\} \subseteq \mathsf{tp}_{k-m-1}(\varphi(x), M_i, c)$ such that

$$(\forall \psi(x) \in \mathsf{tp}_{k-m-1}(\varphi(x), M_i, c))(\mathsf{ST}_{21}(I_1, x) \wedge \dots \wedge \mathsf{ST}_{21}(I_p, x) \models \psi(x)).$$

But then we obtain that

$$a \models_i \mathsf{ST}_{21}(\diamond(I_1 \wedge \dots \wedge I_p), x).$$

In fact, c verifies this modalized conjunction for (M_i, a). But, given that

$$\{\, \mathsf{ST}_{21}(I_1, x) \dots \mathsf{ST}_{21}(I_p, x) \,\} \subseteq \mathsf{tp}_{k-m-1}(\varphi(x), M_i, c),$$

the standard translation of modalized conjunction under consideration must be in $\mathsf{tp}_{k-m}(\varphi(x), M_i, a)$. Note, further, that by $(\bar{a}_m, a)\ A\ (\bar{b}_m, b)$ we have

$$\mathsf{tp}_{k-m}(\varphi(x), M_i, a) \subseteq \mathsf{tp}_{k-m}(\varphi(x), M_j, b),$$

and therefore this modalized conjunction must be true at (M_j, b) as well. But then take any $d \in U_j$ such that $b\ R_{\diamond j}\ d$ and d verifies the conjunction under consideration. By choice of $\{\, \mathsf{ST}_{21}(I_1, x) \dots \mathsf{ST}_{21}(I_p, x) \,\}$ it follows that

$$\mathsf{tp}_{k-m-1}(\varphi(x), M_i, c) \subseteq \mathsf{tp}_{k-m-1}(\varphi(x), M_j, d).$$

But then, again by the definition of A, and given the fact that $m+1 \leq k$, we must also have $(\bar{a}_m, a, c)\ A\ (\bar{b}_m, b, d)$, and so condition (p-diam-1) holds.

Ad Proposition 3:

Replace the verification of conditions (diam-2(1)) and (diam-2(2)) with the following verification of condition (diam-1):

Choose any i, j such that $\{ i, j \} = \{ 1, 2 \}$, any $a \in U_i$, $b \in U_j$ such that $a\ A\ b$, that is to say, $\mathsf{tp}_x(M_i, a) \subseteq \mathsf{tp}_x(M_j, b)$ and choose any $c \in U_i$ for which we have $a\ R_{\Diamond i}\ c$.

Consider $\mathsf{tp}_x(M_i, c)$. If $\{ \mathsf{ST}_{21}(I_1, x) \dots \mathsf{ST}_{21}(I_p, x) \}$ is a finite subset of this type, then we have

$$a \models_i \mathsf{ST}_{21}(\Diamond(I_1 \wedge \ldots \wedge I_p), x).$$

By $\mathsf{tp}_x(M_i, a) \subseteq \mathsf{tp}_x(M_j, b)$, we obtain that

$$b \models_j \mathsf{ST}_{12}(\Diamond(I_1 \wedge \ldots \wedge I_p), x).$$

This means that every finite subset of the type

$$\{ R_\Diamond(b, x) \} \cup \mathsf{tp}_x(M_i, c)$$

is satisfiable at $[M_j, b]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([M_j, b])$ and, by ω-saturation of both M_1 and M_2, it must be satisfied in $[M_j, b]$ by some $d \in U_j$. So for any such d we will have $b\ R_{\Diamond j}\ d$ and, moreover,

$$d \models_j \mathsf{tp}_x(M_i, c).$$

Thus we have that $\mathsf{tp}_x(M_i, c) \subseteq \mathsf{tp}_x(M_j, d)$. Thus we get that $c\ A\ d$ and condition (diam-1) is verified.

Another important revision of the proofs given in Section 5 for the case $i = 2$, $j = 1$ is the omission of every reference to relation B, since asimulations are now being defined as single relations rather than ordered couples of relations.

Finally, in order to accommodate the proofs in Section 5 to the case $i = j = 1$ one just needs to combine the revisions given in the present section in a straightforward way.

7. Characterization modulo first-order definable classes of models

In this section we show how to account for the impact of numerous sets of restrictions normally imposed on the Kripke models on the expressive powers of intuitionistic modal logic.

Indeed, Theorem 3 establishes a criterion for the equivalence of a first-order formula to an (i, j)-standard translation of modal intuitionistic formula over arbitrary first-order models for $i, j \in \{1, 2\}$. However, as we mentioned in Section 2, in modal intuitionistic logic only models satisfying some rather complicated set of restrictions are normally allowed. But, as we have also mentioned in that section, in most cases to be found in the existing literature, these restrictions can be formalized in first-order logic and thus jointly define a subclass $\varkappa$ of the class of first-order models. One is naturally interested in the criterion of equivalence of a first-order formula to an (i, j)-standard translation of modal intuitionistic formula over some intended subclass of models defined by a given set of such restrictions. Our answer is that this criterion is just invariance with respect to (i, j)-modal asimulations between the intended models.

More precisely, we define:

Definition 15. Let $\varkappa$ be a class of models. Then:

1. $\varkappa(\Theta) = \{ M \in \varkappa \mid M \text{ is a } \Theta\text{-model} \}$;
2. $\varkappa(\Theta)$ is first-order axiomatizable iff there is a set Ax of Θ-sentences, such that a Θ-model M is in $\varkappa$ iff $M \models Ax$;
3. A set Γ of Θ-formulas is $\varkappa$-satisfiable iff Γ is satisfied in some model of $\varkappa$;
4. A Θ-formula φ $\varkappa$-follows from Γ ($\Gamma \models^{\varkappa} \varphi$) iff $\Gamma \cup \{ \neg\varphi \}$ is $\varkappa$-unsatisfiable;
5. Θ-formulas φ and ψ are $\varkappa$-equivalent iff $\varphi \models^{\varkappa} \psi$ and $\psi \models^{\varkappa} \varphi$.

It is clear that for any class $\varkappa$, such that Ax first-order axiomatizes $\varkappa(\Theta)$, any set Γ of Θ-formulas and any Θ-formula φ, Γ is $\varkappa$-satisfiable iff $\Gamma \cup Ax$ is satisfiable, and $\Gamma \models^{\varkappa} \varphi$ iff $\Gamma \cup Ax \models \varphi$.

We say, further, that a formula $\varphi(x)$ is $\varkappa$-invariant with respect to (i,j)-modal asimulations (where $i,j \in \{1,2\}$) iff it is invariant with respect to the class of (i,j)-modal asimulations connecting models in $\varkappa$.

Now for the criterion of $\varkappa$-equivalence to an (i,j)-standard translation of modal intuitionistic formula:

Theorem 5. *Let $\varkappa$ be a class of first-order models such that $\varkappa(\Theta)$ is first-order axiomatizable for all finite Θ, and let $\varphi(x)$ be $\varkappa$-invariant with respect to (i,j)-modal asimulations for some $i,j \in \{1,2\}$. Then $\varphi(x)$ is $\varkappa$-equivalent to an (i,j)-standard translation of a modal intuitionistic formula.*

Proof. Let Ax be the set of first-order sentences that axiomatizes $\varkappa(\Sigma_\varphi)$. We may assume that $\varphi(x)$ is $\varkappa(\Sigma_\varphi)$-satisfiable, otherwise $\varphi(x)$ is $\varkappa$-equivalent to $\mathsf{ST}_{ij}(\bot, x)$ and we are done. In what follows we will write $\mathrm{ic}_\varkappa(\varphi(x))$ for the set

$$\{\psi(x) \in \mathsf{int}_x(\Sigma_\varphi) \mid \varphi(x) \models^{\varkappa} \psi(x)\}.$$

Our strategy will be to show that $\mathrm{ic}_\varkappa(\varphi(x)) \models^{\varkappa} \varphi(x)$. Once this is done we will conclude that

$$Ax \cup \mathrm{ic}_\varkappa(\varphi(x)) \models \varphi(x).$$

Then we apply compactness of first-order logic and obtain that $\varphi(x)$ is equivalent to a finite conjunction $\bigwedge \Psi(x)$ of formulas from this set. But it follows then that $\varphi(x)$ is $\varkappa$-equivalent to the conjunction of the set $\mathrm{ic}_\varkappa(\varphi(x)) \cap \Psi(x)$. In fact, by our choice of $\mathrm{ic}_\varkappa(\varphi(x))$ we have

$$\varphi(x) \models^{\varkappa} \bigwedge(\mathrm{ic}_\varkappa(\varphi(x)) \cap \Psi(x)).$$

And by $\Psi(x) \subseteq Ax \cup \mathrm{ic}_\varkappa(\varphi(x))$ we have

$$Ax \cup (\mathrm{ic}_\varkappa(\varphi(x)) \cap \Psi(x)) \models \varphi(x)$$

and hence

$$\mathrm{ic}_\varkappa(\varphi(x)) \cap \Psi(x) \models^{\varkappa} \varphi(x).$$

To show that $\mathrm{ic}_\varkappa(\varphi(x)) \models^{\varkappa} \varphi(x)$, take any Σ_φ-model M_1 and $a \in U_1$ such that $M_1 \in \varkappa$ and $a \models_1 \mathrm{ic}_\varkappa(\varphi(x))$. Then, of course, we will also have $\mathrm{ic}_\varkappa(\varphi(x)) \subseteq \mathsf{tp}_x(M_1, a)$. Such a model exists, because $\varphi(x)$ is $\varkappa(\Sigma_\varphi)$-satisfiable and $\mathrm{ic}_\varkappa(\varphi(x))$ will be $\varkappa$-satisfied in any Σ_φ-model satisfying $\varphi(x)$. Then we can also choose a Σ_φ-model M_2 and $b \in U_2$ such that $M_2 \in \varkappa$, $b \models_2 \varphi(x)$, and $\mathsf{tp}_x(M_2, b) \subseteq \mathsf{tp}_x(M_1, a)$.

For suppose otherwise. Then for any Σ_φ-model $M \in \varkappa$ such that $U \subseteq \mathbb{N}$ and any $c \in U$ such that $M, c \models \varphi(x)$ we can choose a modal intuitionistic formula $I_{(M,c)}$ such that $\mathsf{ST}_{ij}(I_{(M,c)}, x)$ is in $\mathsf{tp}_x(M, c)$

but not in $\mathsf{tp}_x(M_1, a)$. Then consider the set

$$S = \{ \varphi(x) \} \cup \{ \neg \mathsf{ST}_{ij}(I_{(M,c)}, x) \mid M \in \varkappa, M, c \models \varphi(x) \}.$$

Let $\{ \varphi(x), \neg\mathsf{ST}_{ij}(I_1, x) \ldots, \neg\mathsf{ST}_{ij}(I_q, x) \}$ be a finite subset of this set. If this set is $\varkappa$-unsatisfiable, then we must have

$$\varphi(x) \models^{\varkappa} \mathsf{ST}_{ij}(I_1, x) \vee \ldots \vee \mathsf{ST}_{ij}(I_q, x),$$

but then we will also have

$$(\mathsf{ST}_{ij}(I_1, x) \vee \ldots \vee \mathsf{ST}_{ij}(I_q, x)) \in \mathrm{ic}_{\varkappa}(\varphi(x)) \subseteq \mathsf{tp}_x(M_1, a),$$

and hence $(\mathsf{ST}_{ij}(I_1, x) \vee \ldots \vee \mathsf{ST}_{ij}(I_q, x))$ will be true at (M_1, a). But then at least one of $\mathsf{ST}_{ij}(I_1, x) \ldots,$ $\mathsf{ST}_{ij}(I_q, x)$ must also be true at (M_1, a), which contradicts the choice of these formulas. Therefore, every finite subset of S is $\varkappa$-satisfiable. But then every finite subset of the set $S \cup Ax$ is satisfiable as well. By compactness of first-order logic $S \cup Ax$ is satisfiable, and, by Löwenheim–Skolem property of first-order logic, there is a Σ_φ-model M' and $g \in U'$ such that $U' \subseteq \mathbb{N}$ and (M', g) satisfies $S \cup Ax$. But then we must have $M' \in \varkappa$, since M' is a Σ_φ-model satisfying the set of axioms for $\varkappa(\Sigma_\varphi)$.

For this model and for this element in it we will have both $M', g \models \mathsf{ST}_{ij}(I_{(M',g)}, x)$ by choice of $I_{(M',g)}$ and $M', g \not\models \mathsf{ST}_{ij}(I_{(M',g)}, x)$ by the satisfaction of S, a contradiction.

Therefore, for any given Σ_φ-model M_1 such that $M_1 \in \varkappa$ and for any $a \in U_1$ satisfying $\mathrm{ic}_{\varkappa}(\varphi(x))$ we can choose a Σ_φ-model M_2 and $b \in U_2$ such that $M_2 \in \varkappa$, $b \models_2 \varphi(x)$, and $\mathsf{tp}_x(M_2, b) \subseteq \mathsf{tp}_x(M_1, a)$. Then, reasoning exactly as in the proof of Theorem 3, we conclude that $a \models_1 \varphi(x)$. Therefore, $\varphi(x)$ in fact $\varkappa$-follows from $\mathrm{ic}_{\varkappa}(\varphi(x))$. □

Theorem 6. *Let $\varkappa$ be a class of first-order models such that for all finite Θ class $\varkappa(\Theta)$ is first-order axiomatizable. Then a formula $\varphi(x)$ is $\varkappa$-invariant with respect to (i, j)-modal asimulations iff it is $\varkappa$-equivalent to an (i, j)-standard translation of a modal intuitionistic formula.*

Proof. From left to right our theorem follows from Theorem 5. In the other direction, assume that $\varphi(x)$ is $\varkappa$-equivalent to $\mathsf{ST}_{ij}(I, x)$ and assume that for some Θ such that $\Sigma_\varphi \subseteq \Theta$, some Θ-models M_1, M_2, and some $a \in U_1$ and $b \in U_2$ such that $M_1, M_2 \in \varkappa$, A is a (i, j)-modal asimulation from (M_1, a) to (M_2, b), and $a \models_1 \varphi(x)$. Then, since $\mathsf{ST}_{ij}(I, x)$ is $\varkappa$-equivalent to $\varphi(x)$ and M_1 is in $\varkappa$, we also have $a \models_1 \mathsf{ST}_{ij}(I, x)$. From Theorem 3 it follows that $b \models_2 \mathsf{ST}_{ij}(I, x)$, but since $\mathsf{ST}_{ij}(I, x)$ is $\varkappa$-equivalent to $\varphi(x)$ and M_2 is in $\varkappa$, we also have $b \models_2 \varphi(x)$. Therefore, $\varphi(x)$ is $\varkappa$-invariant with respect to (i, j)-modal asimulations. □

Thus Theorems 5 and 6 imply that (i, j)-modal asimulations as criteria for equivalence to a standard translation of a modal intuitionistic formula are easily scalable down to any first-order definable class of models.

8. Conclusion

In this paper we have defined and vindicated four different versions of modal asimulations, capturing the four different fragments of classical first-order logic induced by the corresponding systems of satisfaction clauses for modal intuitionistic logic. In doing so, we were concentrating on different variants of Kripke semantics for intuitionistic modalities present in the existing literature.

However, it is easy to see that our approach is but an instance of a quite general algorithm that can be easily generalized to deal with a much wider class of extensions of intuitionistic propositional logic. We give a description of one instance of such general algorithm (subsuming also the four cases at hand and some other results in the relevant literature, like, for example, [2]) in [12]. It is not clear at the moment how far generalizations of this type can reach; exploring this area is perhaps the most promising line of further research in the direction set by the present paper together with [10] and [11].

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