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The complement of a σ-compact subset of a space with a π-tree also has a π-tree

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We prove that the complement of a σ-compact subset of a topological space that has a π-tree also has a π-tree. To do this, we construct the *foliage hybrid operation*, which deals with *foliage trees* (that is, set-theoretic trees with a 'leaf' at each node). Then using this operation we modify a π-tree of a space and get a π-tree for its subspace.

1. Introduction

We study topological spaces that have a π-tree; this notion is equivalent to the notion of a Lusin π-base, which was introduced in [1] (see details in Definition 10 and Remark 11). The Sorgenfrey line and the Baire space $\mathcal{N}$ (that is, ${}^{\omega}\omega$ with the product topology) are examples of spaces with a π-tree [1]. Every space that has a π-tree shares many good properties with the Baire space. One reason for this is expressed in Lemma 13, another two are the following: if a space X has a π-tree, then X can be mapped onto $\mathcal{N}$ by a

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continuous one-to-one map [1] and also X can be mapped onto $\mathcal{N}$ by a continuous open map [1] (hence X can be mapped by a continuous open map onto an arbitrary Polish space).

In this paper we prove Theorem 44, which states that if a space X has a π-tree and $Y \subseteq X$ is the complement of a σ-compact subset of X, then Y also has a π-tree. This result reflects the following property of the Baire space: if $Y \subseteq \mathcal{N}$ is the complement of a σ-compact subset of $\mathcal{N}$, then Y is homeomorphic to $\mathcal{N}$ (this property of $\mathcal{N}$ can be easily derived from the Alexandrov–Urysohn characterization of the Baire space and from the characterization of its Polish subspaces — see Theorems 3.11 and 7.7 in [2]).

Theorem 44 is a corollary to Theorem 41, which in combination with Lemma 13 allows to find many more subspaces Y of a space X with a π-tree such that Y also has a π-tree; for example, a dense Y such that $|X \setminus Y| = 2^{\aleph_0}$ (Theorem 44 does not allow to find such Y in the Sorgenfrey line because every σ-compact subset of the Sorgenfrey line is at most countable). In contrast to Theorems 41 and 44, a dense open subspace Y of a space X that has a π-tree can be without π-tree even if X is separable metrizable (this result is in preparation for publication). Both Theorems 41 and 44 are corollaries to Theorem 37, which is the main technical result of this paper.

2. Notation and terminology

We use standard set-theoretic notation from [3,4], according to which ω = the set of natural numbers = the set of finite ordinals = the first limit ordinal = the first infinite cardinal = $\aleph_0$, and each ordinal is equal to the set of smaller ordinals, so that $n = \{0, \dots, n-1\}$ for all $n \in \omega$. We use terminology from [5] when we work with (topological) spaces. Also we use several less common notations:

Notation 1. The symbol := means "equals by definition"; the symbol :⟷ is used to show that an expression on the left side is the abbreviation for expression on the right side;

✎ $x \subset y \quad :\longleftrightarrow \quad x \subseteq y$ and $x \neq y$;
✎ $\forall v \neq w \in A\ \varphi(v,w) \quad :\longleftrightarrow \quad \forall v, w \in A\, [\, v \neq w \rightarrow \varphi(v,w)\,]$;
✎ $\exists\, v \neq w \in A\ \varphi(v,w) \quad :\longleftrightarrow \quad \exists\, v, w \in A\, [\, v \neq w$ and $\varphi(v,w)\,]$;
✎ $A \equiv \bigsqcup_{\lambda \in \Lambda} B_\lambda \quad :\longleftrightarrow \quad A = \bigcup_{\lambda \in \Lambda} B_\lambda$ and $\forall \lambda \neq \lambda' \in \Lambda\, [\, B_\lambda \cap B_{\lambda'} = \varnothing\,]$;
✎ $A \equiv B_0 \sqcup \ldots \sqcup B_n \quad :\longleftrightarrow \quad A \equiv \bigsqcup_{i \in \{0,\dots,n\}} B_i$.

When we work with sequences, we use the following notation:

Notation 2. Suppose that α, β are ordinals, $n \in \omega$, and s, t are transfinite sequences (that is, s and t are functions whose domains are ordinals). Then:

✎ $\mathsf{length}\, s$:= the domain of s;
✎ $\langle r_0, \dots, r_{n-1} \rangle$:= the sequence r such that $\mathsf{length}\, r = n$ and $r(i) = r_i$ for all $i < n$;
in particular, $\langle\rangle$:= the empty sequence (= the empty set);
✎ ${}^{x}A$:= the set of functions from x to A;
in particular, ${}^{0}A = \{\langle\rangle\}$;
✎ ${}^{<\alpha}A := \bigcup_{\beta<\alpha} {}^{\beta}A$;
in particular, ${}^{<\omega}A$ is the set of finite sequences in A;
✎ if $s = \langle s_0, \dots, s_{n-1} \rangle$, then
$s\hat{\ }\langle a \rangle := \langle s_0, \dots, s_{n-1}, a \rangle$;
✎ $s \restriction x$:= the restriction of s to x;
in particular, $s \restriction 0 = \langle\rangle$ for any s;
✎ note that
$s \subseteq t$ iff $\mathsf{length}\, s \leqslant \mathsf{length}\, t$ and $s = t \restriction \mathsf{length}\, s$.

Also we work with partial orders and we use the following notation:

Notation 3. Suppose that $\mathcal{P} = (Q, <)$ is a strict partially ordered set; that is, $<$ is irreflexive and transitive on Q. Let $x, y \in Q$ and $A, B \subseteq Q$. Then:

✎ $\mathsf{nodes}\,\mathcal{P} = \mathsf{nodes}(Q, <) := Q$
(we use the word node because we intend to work with trees);
✎ $x <_{\mathcal{P}} y \quad :\longleftrightarrow \quad x < y$;
✎ $x \leqslant_{\mathcal{P}} y \quad :\longleftrightarrow \quad x <_{\mathcal{P}} y$ or $x = y$;
✎ $x \parallel_{\mathcal{P}} y \quad :\longleftrightarrow \quad x \nleqslant_{\mathcal{P}} y$ and $x \ngtr_{\mathcal{P}} y$;
✎ $x{\uparrow^{\circ}}_{\mathcal{P}} := \{v \in \mathsf{nodes}\,\mathcal{P} : v <_{\mathcal{P}} x\}, \quad x{\downarrow_{\circ}}_{\mathcal{P}} := \{v \in \mathsf{nodes}\,\mathcal{P} : v >_{\mathcal{P}} x\}$;
✎ $x{\uparrow^{\bullet}}_{\mathcal{P}} := \{v \in \mathsf{nodes}\,\mathcal{P} : v \leqslant_{\mathcal{P}} x\}, \quad x{\downarrow_{\bullet}}_{\mathcal{P}} := \{v \in \mathsf{nodes}\,\mathcal{P} : v \geqslant_{\mathcal{P}} x\}$;
✎ $A{\uparrow}_{\mathcal{P}} := \bigcup\{v{\uparrow^{\bullet}}_{\mathcal{P}} : v \in A\}, \quad A{\downarrow}_{\mathcal{P}} := \bigcup\{v{\downarrow_{\bullet}}_{\mathcal{P}} : v \in A\}$;
✎ $(x, y)_{\mathcal{P}} := x{\downarrow_{\circ}}_{\mathcal{P}} \cap y{\uparrow^{\circ}}_{\mathcal{P}}, \quad [x, y]_{\mathcal{P}} := x{\downarrow_{\bullet}}_{\mathcal{P}} \cap y{\uparrow^{\bullet}}_{\mathcal{P}}$;
✎ $[x, y)_{\mathcal{P}} := x{\downarrow_{\bullet}}_{\mathcal{P}} \cap y{\uparrow^{\circ}}_{\mathcal{P}}, \quad (x, y]_{\mathcal{P}} := x{\downarrow_{\circ}}_{\mathcal{P}} \cap y{\uparrow^{\bullet}}_{\mathcal{P}}$;
✎ $x \sqsubset_{\mathcal{P}} y \quad :\longleftrightarrow \quad x <_{\mathcal{P}} y$ and $(x, y)_{\mathcal{P}} = \varnothing$;
✎ $\mathsf{sons}_{\mathcal{P}}(x) := \{\, s \in \mathsf{nodes}\,\mathcal{P} \,:\, x \sqsubset_{\mathcal{P}} s \,\}$;
✎ A is $\mathcal{P}$-**cofinal** in $B \quad :\longleftrightarrow \quad A \subseteq B$ and $B \subseteq A{\uparrow}_{\mathcal{P}}$;
✎ A is an **antichain** in $\mathcal{P} \quad :\longleftrightarrow \quad \forall v \neq w \in A\,[\, v \parallel_{\mathcal{P}} w \,]$;
✎ A is a **chain** in $\mathcal{P} \quad :\longleftrightarrow \quad \forall v, w \in A\,[\, v \leqslant_{\mathcal{P}} w$ or $v >_{\mathcal{P}} w \,]$;
✎ $\mathcal{P}$ has **bounded chains** $\quad :\longleftrightarrow \quad$ for each nonempty chain C in $\mathcal{P}$ there is $z \in \mathsf{nodes}\,\mathcal{P}$ such that $C \subseteq z{\uparrow^{\bullet}}_{\mathcal{P}}$;
✎ $\max\mathcal{P} := \{\, m \in \mathsf{nodes}\,\mathcal{P} \,:\, m{\downarrow_{\circ}}_{\mathcal{P}} = \varnothing \,\}, \quad \min\mathcal{P} := \{\, m \in \mathsf{nodes}\,\mathcal{P} \,:\, m{\uparrow^{\circ}}_{\mathcal{P}} = \varnothing \,\}$;
✎ for $\mathcal{P}$ with the least node,
$0_{\mathcal{P}} :=$ the least node of $\mathcal{P}$.

When a partially ordered set is a (set-theoretic) tree [4,3], we use the following terminology:

Notation 4. Suppose that $\mathcal{T}$ is a tree; that is, $\mathcal{T}$ is a strict partially ordered set such that for each $x \in \mathsf{nodes}\,\mathcal{T}$, the set $x{\uparrow^{\circ}}_{\mathcal{T}}$ is well-ordered by $<_{\mathcal{T}}$. Let $x \in \mathsf{nodes}\,\mathcal{T}$, let α be an ordinal, and let κ be a cardinal. Then:

✎ $\mathsf{height}_{\mathcal{T}}(x) :=$ the ordinal isomorphic to $(x{\uparrow^{\circ}}_{\mathcal{T}}, <_{\mathcal{T}})$;
✎ $\mathsf{level}_{\mathcal{T}}(\alpha) := \big\{v \in \mathsf{nodes}\,\mathcal{T} : \mathsf{height}_{\mathcal{T}}(v) = \alpha\big\}$;
✎ $\mathsf{height}\,\mathcal{T} :=$ the minimal ordinal β such that $\mathsf{level}_{\mathcal{T}}(\beta) = \varnothing$;
✎ B is a **branch** in $\mathcal{T} \quad :\longleftrightarrow \quad B$ is a $\subseteq$-maximal chain in $\mathcal{T}$;
✎ $\mathsf{branches}\,\mathcal{T} := \{B \subseteq \mathsf{nodes}\,\mathcal{T} : B$ is a branch in $\mathcal{T}\}$;
✎ if $A \subseteq \mathsf{nodes}\,\mathcal{T}$ is an antichain in $\mathcal{T}$ and $x \in A{\downarrow}_{\mathcal{T}}$, then
$\mathsf{root}_{\mathcal{T}}(x, A) :=$ the r in A such that $x \in r{\downarrow_{\bullet}}_{\mathcal{T}}$;
✎ $\mathcal{T}$ is κ-**branching** $\quad :\longleftrightarrow \quad \forall v \in \mathsf{nodes}\,\mathcal{T} \smallsetminus \max\mathcal{T}\,\big[\, |\mathsf{sons}_{\mathcal{T}}(v)| = \kappa \,\big]$;
✎ $\mathcal{T}$ is an α,κ-**tree** $\quad :\longleftrightarrow \quad \mathcal{T}$ is isomorphic to the tree $({}^{<\alpha}\kappa, \subset)$.

The following example illustrates the usage of the above terminology:

Example 5. Let $\mathcal{T} = ({}^{<\omega}A, \subset)$, where A is nonempty. Then $\mathcal{T}$ is an $|A|$-branching tree with the least node, $\mathsf{nodes}\,\mathcal{T} = {}^{<\omega}A$, $0_{\mathcal{T}} = \langle\rangle$, and $\max\mathcal{T} = \varnothing$. Suppose that $a, b, c, d \in A$ are different. Then we have:

$$\langle c, a, b, a\rangle{\uparrow^{\circ}}_{\mathcal{T}} = \big\{\langle\rangle, \langle c\rangle, \langle c, a\rangle, \langle c, a, b\rangle\big\}, \quad \mathsf{height}_{\mathcal{T}}\big(\langle c, a, b, a\rangle\big) = 4, \quad \mathsf{height}_{\mathcal{T}}\big(\langle\rangle\big) = 0,$$
$$\mathsf{level}_{\mathcal{T}}(2) = \big\{\langle x, y\rangle : x, y \in A\big\}, \quad \mathsf{level}_{\mathcal{T}}(0) = \big\{\langle\rangle\big\}, \quad \mathsf{level}_{\mathcal{T}}(\omega) = \varnothing, \quad \mathsf{height}\,\mathcal{T} = \omega,$$
$$\mathsf{sons}_{\mathcal{T}}\big(\langle c, a\rangle\big) = \big\{\langle c, a, x\rangle : x \in A\big\}, \quad \mathsf{root}_{\mathcal{T}}\Big(\langle c, b, a, d\rangle, \big\{\langle a\rangle, \langle c, b\rangle, \langle d\rangle\big\}\Big) = \langle c, b\rangle.$$

Also we list here several simple facts about trees, which we use in this paper:

Lemma 6. *Suppose that* $\mathcal{T}$ *is a tree. Then*:

(a) $\mathsf{max}\,\mathcal{T} = \{v \in \mathsf{nodes}\,\mathcal{T} : \mathsf{sons}_{\mathcal{T}}(v) = \varnothing\}$.
(b) *If* $x, y, z \in \mathsf{nodes}\,\mathcal{T}$, $x \geqslant_{\mathcal{T}} y$, *and* $y \parallel_{\mathcal{T}} z$, *then* $x \parallel_{\mathcal{T}} z$.
(c) *If* C *is a chain in* $\mathcal{T}$, *then there is* $B \in \mathsf{branches}\,\mathcal{T}$ *such that* $C \subseteq B$.
(d) *If* $B \in \mathsf{branches}\,\mathcal{T}$ *and* $x \in (\mathsf{nodes}\,\mathcal{T}) \smallsetminus B$, *then there is* $b \in B$ *such that* $x \parallel_{\mathcal{T}} b$.
(e) *If* $B \in \mathsf{branches}\,\mathcal{T}$ *and* $b \in B \smallsetminus \mathsf{max}\,\mathcal{T}$, *then* $B \cap \mathsf{sons}_{\mathcal{T}}(b) \neq \varnothing$.
(f) *If* $B \in \mathsf{branches}\,\mathcal{T}$ *and* $b \in B$, *then* $b{\downarrow}_{\mathcal{T}} \subseteq B$.
(g) *If* $m \in \mathsf{max}\,\mathcal{T}$, *then* $m{\downarrow}_{\mathcal{T}}$ *is a branch in* $\mathcal{T}$.
(h) *If* $\mathcal{T}$ *has bounded chains, then* $\mathsf{branches}\,\mathcal{T} = \{m{\downarrow}_{\mathcal{T}} : m \in \mathsf{max}\,\mathcal{T}\}$.
(i) *The following are equivalent*:
  ➢ $\mathcal{T}$ *is an* $\omega, \aleph_0$*-tree.*
  ➢ $\mathcal{T}$ *has the least node,* $\mathcal{T}$ *is* $\aleph_0$*-branching,* $\mathsf{max}\,\mathcal{T} = \varnothing$, *and* $\mathsf{height}\,\mathcal{T} \leqslant \omega$. □

3. Foliage trees

Informally, a foliage tree is a tree with a leaf at each node, where by a leaf we mean an arbitrary set. Here is the formal definition:

Definition 7. A **foliage tree** is a pair $\mathbf{F} = (\mathcal{T}, l)$ such that $\mathcal{T}$ is a (set-theoretic) tree and l is a function with $\mathsf{domain}\, l = \mathsf{nodes}\,\mathcal{T}$. For each $x \in \mathsf{nodes}\,\mathcal{T}$, the $l(x)$ is called the **leaf** of $\mathbf{F}$ at node x and is denoted by $\mathbf{F}_x$. The tree $\mathcal{T}$ is called the **skeleton** of $\mathbf{F}$ and is denoted by $\mathsf{skeleton}\,\mathbf{F}$.

Convention 8. Let $\mathbf{F}$ be a foliage tree and let $\mathcal{O}$ be an operation or a notion that is defined on trees. Then we use $\mathcal{O}(\mathbf{F})$ as the abbreviation for $\mathcal{O}(\mathsf{skeleton}\,\mathbf{F})$. For example,

✎ $\mathsf{nodes}\,\mathbf{F} := \mathsf{nodes}(\mathsf{skeleton}\,\mathbf{F})$,
✎ $0_{\mathbf{F}} := 0_{\mathsf{skeleton}\,\mathbf{F}}$,
✎ $\mathbf{F}$ has bounded chains $\;:\longleftrightarrow\;$ $\mathsf{skeleton}\,\mathbf{F}$ has bounded chains,
✎ $x \sqsubset_{\mathbf{F}} y \;:\longleftrightarrow\; x \sqsubset_{\mathsf{skeleton}\,\mathbf{F}} y$.

Notation 9. Let $\mathbf{F}$ be a foliage tree, let $\varnothing \neq A \subseteq \mathsf{nodes}\,\mathbf{F}$, and let $z \in \mathsf{nodes}\,\mathbf{F}$. Then:

✎ $\mathsf{fruit}_{\mathbf{F}}(A) := \bigcap\{\mathbf{F}_x : x \in A\}$;
✎ $\mathsf{yield}\,\mathbf{F} := \bigcup\{\mathsf{fruit}_{\mathbf{F}}(B) : B \in \mathsf{branches}\,\mathbf{F}\}$;
✎ $\mathsf{flesh}\,\mathbf{F} := \bigcup\{\mathbf{F}_x : x \in \mathsf{nodes}\,\mathbf{F}\}$;
✎ $\mathsf{flesh}_{\mathbf{F}}(A) := \bigcup\{\mathbf{F}_x : x \in A\}$;
✎ $\mathsf{shoot}_{\mathbf{F}}(z) := \{\, \mathsf{flesh}_{\mathbf{F}}(C) : C$ is a cofinite subset of $\mathsf{sons}_{\mathbf{F}}(z)\,\}$;
✎ $\mathsf{scope}_{\mathbf{F}}(p) := \{y \in \mathsf{nodes}\,\mathbf{F} : \mathbf{F}_y \ni p\}$;
✎ for a space X and a point p in X,
  $\mathsf{nbhds}(p, X) :=$ the family of (not necessarily open) neighbourhoods of p in X;
✎ for arbitrary sets γ and δ,
  $\gamma \gg \delta \;:\longleftrightarrow\; \gamma$ *π-refines* $\delta \;:\longleftrightarrow\; \forall D \in \delta \smallsetminus \{\varnothing\}\ \exists G \in \gamma \smallsetminus \{\varnothing\}\ [\, G \subseteq D\,]$.

Definition 10. Let $\mathbf{F}$ be a foliage tree, X a space, α an ordinal, and κ a cardinal. Then:

✎ $\mathbf{F}$ has **nonempty leaves** $\;:\longleftrightarrow\;$ $\forall x \in \mathsf{nodes}\,\mathbf{F}\ [\,\mathbf{F}_x \neq \varnothing\,]$;
✎ $\mathbf{F}$ is **nonincreasing** $\;:\longleftrightarrow\;$ $\forall x, y \in \mathsf{nodes}\,\mathbf{F}\ [\, y \geqslant_{\mathbf{F}} x \;\rightarrow\; \mathbf{F}_y \subseteq \mathbf{F}_x\,]$;

✎ $\mathbf{F}$ is **splittable** $:\longleftrightarrow$ $\forall x, y \in \mathsf{nodes}\,\mathbf{F}\ [\, x \parallel_{\mathbf{F}} y \ \rightarrow\ \mathbf{F}_x \cap \mathbf{F}_y = \varnothing \,]$;
✎ $\mathbf{F}$ is **complete** $:\longleftrightarrow$ $\mathsf{nodes}\,\mathbf{F} \neq \varnothing$ and $\forall B \in \mathsf{branches}\,\mathbf{F}\ \big[\, \mathsf{fruit}_{\mathbf{F}}(B) \neq \varnothing \,\big]$;
✎ $\mathbf{F}$ has **strict branches** $:\longleftrightarrow$ $\mathsf{nodes}\,\mathbf{F} \neq \varnothing$ and $\forall B \in \mathsf{branches}\,\mathbf{F}\ \big[\, \mathsf{fruit}_{\mathbf{F}}(B)$ is a singleton$\,\big]$;
✎ $\mathbf{F}$ is **locally strict** $:\longleftrightarrow$ $\forall x \in \mathsf{nodes}\,\mathbf{F} \smallsetminus \max \mathbf{F}\ [\, \mathbf{F}_x \equiv \bigsqcup_{s \in \mathsf{sons}_{\mathbf{F}}(x)} \mathbf{F}_s \,]$;
✎ $\mathbf{F}$ is **open** in X $:\longleftrightarrow$ $\forall z \in \mathsf{nodes}\,\mathbf{F}\ [\, \mathbf{F}_z$ is an open subset of $X\,]$;
✎ $\mathbf{F}$ is a **foliage** α,κ**-tree** $:\longleftrightarrow$ $\mathsf{skeleton}\,\mathbf{F}$ is an α,κ-tree (see Notation 4);
✎ $\mathbf{F}$ is a **Baire foliage tree** on X $:\longleftrightarrow$ $\mathbf{F}$ is an open in X locally strict foliage $\omega,\aleph_0$-tree with strict branches and such that $\mathbf{F}_{0_{\mathbf{F}}} = X$;
✎ $\mathbf{F}$ **grows into** X $:\longleftrightarrow$ $\forall p \in X\ \forall U \in \mathsf{nbhds}(p, X)\ \exists z \in \mathsf{scope}_{\mathbf{F}}(p)\ \big[\, \mathsf{shoot}_{\mathbf{F}}(z) \gg \{U\} \,\big]$;
✎ $\mathbf{F}$ is a π**-tree** on X $:\longleftrightarrow$ $\mathbf{F}$ is a Baire foliage tree on X and $\mathbf{F}$ grows into X.

Note that leaves of a π-tree on X are closed-and-open in X and that the set of these leaves forms a countable π-base and pseudo-base for X.

The notion of a π-tree is equivalent to the notion of a Lusin π-base, which was introduced in [1]; the only difference is that a Lusin π-base is a family indexed by nodes of the tree $({}^{<\omega}\omega, \subset)$, while a π-tree is a foliage tree whose skeleton is isomorphic to $({}^{<\omega}\omega, \subset)$. From a topological point of view, there is no difference between these two notions because of the following remark:

Remark 11. For any space X, the following are equivalent:

➢ X has a π-tree.
➢ X has a Lusin π-base. □

Recall that the Baire space $\mathcal{N}$ is the set ${}^{\omega}\omega$ endowed with the Tychonov product topology, where ω carries the discrete topology. The Baire space has a basis $\big\{\{p \in {}^{\omega}\omega : x \subseteq p\} : x \in {}^{<\omega}\omega\big\}$, which is called [2] the standard basis for ${}^{\omega}\omega$. This standard basis can be viewed as a foliage tree:

Notation 12. We denote by $\mathbf{S}$ the foliage tree such that

➢ $\mathsf{skeleton}\,\mathbf{S} := ({}^{<\omega}\omega, \subset)$ and
➢ $\mathbf{S}_x := \{p \in {}^{\omega}\omega : x \subseteq p\}$ for all $x \in {}^{<\omega}\omega$.

We call this foliage tree the **standard foliage tree** of ${}^{\omega}\omega$.

Lemma 13.

(a) $\mathbf{S}$ *is a π-tree on the Baire space* $\mathcal{N} = ({}^{\omega}\omega, \tau_{\mathcal{N}})$.
(b) $\mathbf{S}$ *is a Baire foliage tree on a space* $({}^{\omega}\omega, \tau)$ iff $\tau \supseteq \tau_{\mathcal{N}}$.
(c) *A space X has a Baire foliage tree* iff
X is homeomorphic to some space $({}^{\omega}\omega, \tau)$ such that $\tau \supseteq \tau_{\mathcal{N}}$.
(d) *A space X has a π-tree* iff
X is homeomorphic to some space $({}^{\omega}\omega, \tau)$ such that $\mathbf{S}$ is a π-tree on $({}^{\omega}\omega, \tau)$.

Proof. Part (a) and the $\rightarrow$ direction of (b) follow from the fact that $\{\mathbf{S}_x : x \in {}^{<\omega}\omega\}$ is a basis for the Baire space. The $\leftarrow$ direction of (b) follows from (a). The $\rightarrow$ direction of (c) is a reformulation of Lemma 3.3 from [1] and the $\leftarrow$ direction of (c) follows from (b). The $\rightarrow$ direction of (d) is a reformulation of Lemma 3.9 from [1], the opposite direction of (d) is trivial. □

Lemma 14. *Suppose that* $\mathbf{F}$ *is a foliage tree. Then*:

(a) *If* $\mathbf{F}$ *is nonincreasing,* $\varnothing \neq A \subseteq B \subseteq \mathsf{nodes}\,\mathbf{F}$*, and* A *is* $\mathbf{F}$*-cofinal in* B*, then* $\mathsf{fruit}_{\mathbf{F}}(A) = \mathsf{fruit}_{\mathbf{F}}(B)$*.*
(b) *If* $\mathbf{F}$ *has the least node and* $\mathsf{height}\,\mathbf{F} \leqslant \omega$*, then the following are equivalent*:
 ➢ $\mathbf{F}$ *is locally strict*;
 ➢ $\mathbf{F}$ *is splittable and* $\mathsf{flesh}\,\mathbf{F} = \mathsf{yield}\,\mathbf{F}$*.* □

4. Hybrid operation

In this paper we build a π-tree for a subspace Y of a space X that already has a π-tree by using the *foliage hybrid operation* (see Definition 27 in Section 5). The foliage hybrid operation deals with foliage trees and we construct it by using another operation — the *hybrid operation* — which deals with trees. These two operations are quite complicated, you can look at pictures that illustrate all definitions in [6].

In this section we build the hybrid operation (see Definition 19), prove that the result of the hybrid operation is always a tree (see Proposition 22), and establish properties of this operation (see Proposition 23).

The hybrid operation modifies a given tree $\mathcal{T}$ in two steps: first we cut out several pieces from $\mathcal{T}$, after that we engraft special trees onto the places of cut out pieces. The special trees that are engrafted onto $\mathcal{T}$ are called *grafts*, the cut out pieces are called *explants*, and the parts of grafts that replace explants are called *implants*:

Definition 15. Let $\mathcal{T}$ be a tree. Then a **graft** for $\mathcal{T}$ is a tree $\mathcal{G}$ such that:

(a) $|\mathsf{nodes}\,\mathcal{G}| > 1$;
(b) $\mathcal{G}$ has the least node;
(c) $0_{\mathcal{G}} \in \mathsf{nodes}\,\mathcal{T}$ and $\max \mathcal{G} \subseteq \mathsf{nodes}\,\mathcal{T}$;
(d) $\max \mathcal{G} \subseteq (0_{\mathcal{G}})\!\downarrow_{\!\circ}\!{}_{\mathcal{T}}$;
(e) $\max \mathcal{G}$ is an antichain in $\mathcal{T}$;
(f) $\mathsf{implant}\,\mathcal{G} \cap \mathsf{nodes}\,\mathcal{T} = \varnothing$,

where the set

$$\mathsf{implant}\,\mathcal{G} := \mathsf{nodes}\,\mathcal{G} \setminus \big(\{0_{\mathcal{G}}\} \cup \max \mathcal{G}\big)$$

is called the **implant** of $\mathcal{G}$. The set

$$\mathsf{explant}(\mathcal{T}, \mathcal{G}) := (0_{\mathcal{G}})\!\downarrow_{\!\circ}\!{}_{\mathcal{T}} \setminus (\max \mathcal{G})\!\downarrow_{\mathcal{T}}$$

is called the **explant** of $\mathcal{T}$ and $\mathcal{G}$.

Note that $\max \mathcal{G}$ may be empty and then $(\max \mathcal{G})\!\downarrow_{\mathcal{T}} = \varnothing\!\downarrow_{\mathcal{T}} = \varnothing$. The following example is given to clarify Definition 15.

Example 16. Suppose that $\mathcal{T} = ({}^{<\omega}\!A, \subset)$ is a tree from Example 5 and $a, b, c, d \in A$ are different. Then $\big\{\langle a, d\rangle, \langle a, b, c\rangle\big\} \subseteq \langle a\rangle\!\downarrow_{\!\circ}\!{}_{\mathcal{T}}$ and $\big\{\langle a, d\rangle, \langle a, b, c\rangle\big\}$ is an antichain in $\mathcal{T}$. Let IMP be a set disjoint from $\mathsf{nodes}\,\mathcal{T}$ and let $\mathcal{G}$ be a tree such that

➢ $\mathsf{nodes}\,\mathcal{G} = \big\{\langle a\rangle, \langle a, d\rangle, \langle a, b, c\rangle\big\} \cup \mathsf{IMP}$,
➢ $0_{\mathcal{G}} = \langle a\rangle$, and
➢ $\max \mathcal{G} = \big\{\langle a, d\rangle, \langle a, b, c\rangle\big\}$.

Then $\mathcal{G}$ is a graft for $\mathcal{T}$, $\mathsf{implant}\,\mathcal{G} = \mathsf{IMP}$, and

$$\mathsf{explant}(\mathcal{T},\mathcal{G}) = \left\{ s \in {}^{<\omega}A : \langle a\rangle \subset s \right\} \smallsetminus \left\{ s \in {}^{<\omega}A : \langle a,d\rangle \subseteq s \ \text{ or } \ \langle a,b,c\rangle \subseteq s \right\}.$$

We want to engraft onto $\mathcal{T}$ many grafts at once, so we need to find conditions which guarantee that different grafts do not conflict with each other (for example, nodes of one graft should not lie in the explant of another graft).

Definition 17. Let $\mathcal{T}$ be a tree. Then γ is a **consistent** family of grafts for $\mathcal{T}$ iff

(a) $\forall \mathcal{G} \in \gamma \ [\, \mathcal{G}$ is a graft for $\mathcal{T}\,]$;
(b) $\forall \mathcal{D} \neq \mathcal{E} \in \gamma \ [\, \mathsf{implant}\,\mathcal{D} \cap \mathsf{implant}\,\mathcal{E} = \varnothing \,]$;
(c) $\forall \mathcal{D} \neq \mathcal{E} \in \gamma$
 ➢ $0_{\mathcal{D}} \parallel_{\mathcal{T}} 0_{\mathcal{E}}$ or
 ➢ $0_{\mathcal{D}} \in (\max \mathcal{E}){\downarrow}_{\mathcal{T}}$ or
 ➢ $0_{\mathcal{E}} \in (\max \mathcal{D}){\downarrow}_{\mathcal{T}}$.

The set

$$\mathsf{support}(\mathcal{T},\gamma) \coloneqq \mathsf{nodes}\,\mathcal{T} \smallsetminus \bigcup_{\mathcal{G}\in\gamma} \mathsf{explant}(\mathcal{T},\mathcal{G})$$

is called the **support** of $\mathcal{T}$ for γ.

Lemma 18. *Suppose that γ is a consistent family of grafts for a tree $\mathcal{T}$ and $\mathcal{G} \in \gamma$. Then:*

(a) $\mathsf{nodes}\,\mathcal{G} \equiv \{0_{\mathcal{G}}\} \sqcup \max \mathcal{G} \sqcup \mathsf{implant}\,\mathcal{G}$;
(b) $\{0_{\mathcal{G}}\} \cup \max \mathcal{G} \cup \min \mathcal{T} \subseteq \mathsf{support}(\mathcal{T},\gamma)$;
(c) $\mathsf{implant}\,\mathcal{G} \cap \mathsf{support}(\mathcal{T},\gamma) = \varnothing$;
(d) $\forall s \in \mathsf{support}(\mathcal{T},\gamma) \ \left[\, s >_{\mathcal{T}} 0_{\mathcal{G}} \ \longleftrightarrow \ s \in (\max \mathcal{G}){\downarrow}_{\mathcal{T}} \,\right]$;
(e) $\forall s \in \mathsf{support}(\mathcal{T},\gamma) \ \forall e \in \mathsf{explant}(\mathcal{T},\mathcal{G}) \ \left[\, s \leqslant_{\mathcal{T}} 0_{\mathcal{G}} \ \longleftrightarrow \ s <_{\mathcal{T}} e \,\right]$;
(f) $\forall \mathcal{D} \neq \mathcal{E} \in \gamma \ [\, 0_{\mathcal{D}} \neq 0_{\mathcal{E}}$ and $\max \mathcal{D} \cap \max \mathcal{E} = \varnothing \,]$. $\square$

Now we can give a definition of the hybrid operation:

Definition 19. Let γ be a consistent family of grafts for a tree $\mathcal{T}$. Then the **hybrid** of $\mathcal{T}$ and γ — in symbols, $\mathsf{hybrid}(\mathcal{T},\gamma)$ — is a pair $(H,<)$ such that:

(a) $H \coloneqq \mathsf{support}(\mathcal{T},\gamma) \cup \bigcup_{\mathcal{G}\in\gamma} \mathsf{implant}\,\mathcal{G}$
 (note that all these sets are pairwise disjoint by (b) of Definition 17 and (c) of Lemma 18);
(b) $<$ is a relation on H defined by:
 $x < y \quad :\longleftrightarrow$
 (b1) $x, y \in \mathsf{support}(\mathcal{T},\gamma)$ and $x <_{\mathcal{T}} y$
  or
 (b2) $\exists \mathcal{G} \in \gamma$ such that
  ➢ $x, y \in \mathsf{implant}\,\mathcal{G}$ and
  ➢ $x <_{\mathcal{G}} y$
  or
 (b3) $\exists \mathcal{G} \in \gamma$ such that

➢ $x \in \mathsf{support}(\mathcal{T}, \gamma)$ and
➢ $y \in \mathsf{implant}\,\mathcal{G}$ and
➢ $x \leqslant_{\mathcal{T}} 0_{\mathcal{G}}$

or

(b4) $\exists \mathcal{G} \in \gamma$ such that
➢ $x \in \mathsf{implant}\,\mathcal{G}$ and
➢ $y \in \mathsf{support}(\mathcal{T}, \gamma)$ and
➢ $y \in (\max \mathcal{G}){\downarrow}_{\mathcal{T}}$ and
➢ $x <_{\mathcal{G}} \mathsf{root}_{\mathcal{T}}\big(y, \max \mathcal{G}\big)$

or

(b5) $\exists \mathcal{D} \neq \mathcal{E} \in \gamma$ such that
➢ $x \in \mathsf{implant}\,\mathcal{D}$ and
➢ $y \in \mathsf{implant}\,\mathcal{E}$ and
➢ $0_{\mathcal{E}} \in (\max \mathcal{D}){\downarrow}_{\mathcal{T}}$ and
➢ $x <_{\mathcal{D}} \mathsf{root}_{\mathcal{T}}(0_{\mathcal{E}}, \max \mathcal{D})$.

We could give a shorter (but less suitable for our aims) definition for the hybrid operation in the following equivalent way:

Remark 20. Clause (b) of Definition 19 is equivalent to the assertion that $<$ is the transitive closure of relation

$$\Big(<_{\mathcal{T}} \cup \bigcup_{\mathcal{G} \in \gamma} <_{\mathcal{G}} \Big) \cap (H \times H).$$

Proof. Let $\lhd := \big(<_{\mathcal{T}} \cup \bigcup_{\mathcal{G} \in \gamma} <_{\mathcal{G}} \big) \cap (H \times H)$. We have $\lhd \subseteq <$ by (a)–(b) of Lemma 21 and $<$ is transitive by Proposition 22 (we do not use Remark 20 in the proofs of Lemma 21 and Proposition 22).

It remains to show that if $\lhd \subseteq \lessdot$ and $\lessdot$ is a transitive relation on H, then $< \subseteq \lessdot$. Suppose $(x, y) \in <$; this means that one of conditions (b1)–(b5) of Definition 19 holds. For example, if (b3) holds, then $x \in \mathsf{support}(\mathcal{T}, \gamma)$, $y \in \mathsf{implant}\,\mathcal{G}$, $x \leqslant_{\mathcal{T}} 0_{\mathcal{G}}$, and $0_{\mathcal{G}} <_{\mathcal{G}} y$, so $x \trianglelefteq 0_{\mathcal{G}} \lhd y$. Then $x \leq 0_{\mathcal{G}} \lessdot y$, whence $(x, y) \in \lessdot$ by transitivity. The other cases are similar. □

Lemma 21. *Suppose that γ is a consistent family of grafts for a tree $\mathcal{T}$, $\mathcal{H} = \mathsf{hybrid}(\mathcal{T}, \gamma)$, and $\mathcal{G} \in \gamma$. Then:*

(a) $\mathsf{nodes}\,\mathcal{G} \subseteq \mathsf{nodes}\,\mathcal{H}$ and
$\forall x, y \in \mathsf{nodes}\,\mathcal{G}\ [\, x <_{\mathcal{H}} y \ \leftrightarrow\ x <_{\mathcal{G}} y \,]$;
(b) $\mathsf{support}(\mathcal{T}, \gamma) = \mathsf{nodes}\,\mathcal{H} \cap \mathsf{nodes}\,\mathcal{T}$ and
$\forall x, y \in \mathsf{support}(\mathcal{T}, \gamma)\ [\, x <_{\mathcal{H}} y \ \leftrightarrow\ x <_{\mathcal{T}} y \,]$;
(c) $\forall h \in \mathsf{nodes}\,\mathcal{H}\ \forall i \in \mathsf{implant}\,\mathcal{G}\ [\, h \geqslant_{\mathcal{H}} i \ \rightarrow\ h >_{\mathcal{H}} 0_{\mathcal{G}} \,]$;
(d) $\forall h \in \mathsf{nodes}\,\mathcal{H}\ \forall i \in \mathsf{implant}\,\mathcal{G}\ [\, h \leqslant_{\mathcal{H}} 0_{\mathcal{G}} \ \rightarrow\ h <_{\mathcal{H}} i \,]$;
(e) $\forall h \in \mathsf{nodes}\,\mathcal{H} \setminus \mathsf{implant}\,\mathcal{G}\ \forall i \in \mathsf{implant}\,\mathcal{G}\ [\, h \leqslant_{\mathcal{H}} 0_{\mathcal{G}} \ \leftrightarrow\ h <_{\mathcal{H}} i \,]$;
(f) $\forall h \in \mathsf{nodes}\,\mathcal{H} \setminus \mathsf{implant}\,\mathcal{G}\ \big[\, h >_{\mathcal{H}} 0_{\mathcal{G}} \ \leftrightarrow\ h \in (\max \mathcal{G}){\downarrow}_{\mathcal{H}} \,\big]$;
(g) $\forall g \in \mathsf{nodes}\,\mathcal{G}\ \big[\, g{\Uparrow}_{\mathcal{H}} \equiv g{\Uparrow}_{\mathcal{G}} \sqcup (0_{\mathcal{G}}){\Uparrow}_{\mathcal{H}} \,\big]$.

Proof. (a)–(e) are straightforward; (f) follows from (b) of Lemma 21, (d) and (f) of Lemma 18, and (e) of Definition 15; (g) can be proved by using (a)–(f) of Lemma 21, (d) and (f) of Lemma 18, and (e) of Definition 15. □

First we show that a result of the hybrid operation is always a tree:

Proposition 22. *Suppose that γ is a consistent family of grafts for a tree $\mathcal{T}$. Then $\mathsf{hybrid}(\mathcal{T},\gamma)$ is a tree.*

Proof. Let $\mathcal{H} := \mathsf{hybrid}(\mathcal{T},\gamma)$. The irreflexivity of $<_{\mathcal{H}}$ is trivial, let us prove that $x <_{\mathcal{H}} y <_{\mathcal{H}} z$ implies $x <_{\mathcal{H}} z$. We consider several cases:

(i) $z \in \mathsf{support}(\mathcal{T},\gamma)$.
- (i.1) $y \in \mathsf{support}(\mathcal{T},\gamma)$.
  The case $x \in \mathsf{support}(\mathcal{T},\gamma)$ is trivial. If there is $\mathcal{D} \in \gamma$ such that $x \in \mathsf{implant}\mathcal{D}$, then $x <_{\mathcal{D}} \mathsf{root}_{\mathcal{T}}(y, \max\mathcal{D})$. Since $y <_{\mathcal{T}} z$, we have $\mathsf{root}_{\mathcal{T}}(z, \max\mathcal{D}) = \mathsf{root}_{\mathcal{T}}(y, \max\mathcal{D})$, so $x <_{\mathcal{H}} z$.
- (i.2) $\exists\mathcal{E}\in\gamma\ [\, y \in \mathsf{implant}\mathcal{E}\,]$.
  The case $x \in \mathsf{implant}\mathcal{E}$ is trivial. If $x \notin \mathsf{implant}\mathcal{E}$, then $x \leqslant_{\mathcal{H}} 0_{\mathcal{E}}$ by (e) of Lemma 21 and $0_{\mathcal{E}} <_{\mathcal{H}} z$ by (c) of Lemma 21. Therefore $x \leqslant_{\mathcal{H}} 0_{\mathcal{E}} <_{\mathcal{H}} z$ and we may use (i.1), since $0_{\mathcal{E}} \in \mathsf{support}(\mathcal{T},\gamma)$.

(ii) $\exists\mathcal{G}\in\gamma\ [\, z \in \mathsf{implant}\mathcal{G}\,]$.
- (ii.1) $y \in \mathsf{implant}\mathcal{G}$.
  The case $x \in \mathsf{implant}\mathcal{G}$ is trivial. If $x \notin \mathsf{implant}\mathcal{G}$, then using (e) of Lemma 21 twice, we get $x <_{\mathcal{H}} z$.
- (ii.2) $y \notin \mathsf{implant}\mathcal{G}$.
  By (e) of Lemma 21, $x <_{\mathcal{H}} y \leqslant_{\mathcal{H}} 0_{\mathcal{G}}$. Since $0_{\mathcal{G}} \in \mathsf{support}(\mathcal{T},\gamma)$, we have $x <_{\mathcal{H}} 0_{\mathcal{G}}$ by (i), so $x <_{\mathcal{H}} z$ by (d) of Lemma 21.

Now we prove that for each $z \in \mathsf{nodes}\mathcal{H}$, the set $z\mathring{\upharpoonleft}_{\mathcal{H}}$ is a chain in $\mathcal{H}$. We must show that $x, y \in z\mathring{\upharpoonleft}_{\mathcal{H}}$ implies $x \leqslant_{\mathcal{H}} y$ or $x >_{\mathcal{H}} y$. Again, we consider several cases:

(i) $z \in \mathsf{support}(\mathcal{T},\gamma)$.
- (i.1) $y \in \mathsf{support}(\mathcal{T},\gamma)$.
  - (i.1.1) $x \in \mathsf{support}(\mathcal{T},\gamma)$.
    This case is trivial.
  - (i.1.2) $\exists\mathcal{E}\in\gamma\ [\, x \in \mathsf{implant}\mathcal{D}\,]$.
    By (c) of Lemma 21, $0_{\mathcal{D}} \in z\mathring{\upharpoonleft}_{\mathcal{H}}$, so $0_{\mathcal{D}} \in \mathsf{support}(\mathcal{T},\gamma)$ implies $y \leqslant_{\mathcal{H}} 0_{\mathcal{D}}$ or $y >_{\mathcal{H}} 0_{\mathcal{D}}$ by (i.1.1). If $y \leqslant_{\mathcal{H}} 0_{\mathcal{D}}$, then $y <_{\mathcal{H}} x$ by (d) of Lemma 21. If $y >_{\mathcal{H}} 0_{\mathcal{D}}$, then $y \in (\max\mathcal{D})\lfloor_{\mathcal{T}}$ by (d) of Lemma 18. Then $\mathsf{root}_{\mathcal{T}}(y, \max\mathcal{D}) = \mathsf{root}_{\mathcal{T}}(z, \max\mathcal{D})$, so $x <_{\mathcal{D}} \mathsf{root}_{\mathcal{T}}(y, \max\mathcal{D})$, whence $x <_{\mathcal{H}} y$.
- (i.2) $\exists\mathcal{E}\in\gamma\ [\, y \in \mathsf{implant}\mathcal{E}\,]$.
  - (i.2.1) $x \in \mathsf{support}(\mathcal{T},\gamma)$.
    This case is the same as (i.1.2).
  - (i.2.2) $\exists\mathcal{E}\in\gamma\ [\, x \in \mathsf{implant}\mathcal{D}\,]$.
    The case $\mathcal{D} = \mathcal{E}$ is trivial. If $\mathcal{D} \neq \mathcal{E}$, then by (c) of Lemma 21, $0_{\mathcal{D}}, 0_{\mathcal{E}} \in z\mathring{\upharpoonleft}_{\mathcal{H}}$, so $0_{\mathcal{D}} \leqslant_{\mathcal{H}} 0_{\mathcal{E}}$ or $0_{\mathcal{D}} >_{\mathcal{H}} 0_{\mathcal{E}}$ by (i.1.1). By (f) of Lemma 18, $0_{\mathcal{D}} \neq 0_{\mathcal{E}}$, so by (d) of Lemma 18 we may assume without loss of generality that $0_{\mathcal{E}} \in (\max\mathcal{D})\lfloor_{\mathcal{T}}$. Since $0_{\mathcal{E}} <_{\mathcal{T}} z$, we have $\mathsf{root}_{\mathcal{T}}(0_{\mathcal{E}}, \max\mathcal{D}) = \mathsf{root}_{\mathcal{T}}(z, \max\mathcal{D})$, so $x <_{\mathcal{H}} 0_{\mathcal{E}}$, whence $x <_{\mathcal{H}} y$ by transitivity.

(ii) $\exists\mathcal{G}\in\gamma\ [\, z \in \mathsf{implant}\mathcal{G}\,]$.
- (ii.1) $y \in \mathsf{implant}\mathcal{G}$ or $x \in \mathsf{implant}\mathcal{G}$.
  This case is similar to case (ii.1) from the proof of transitivity.
- (ii.2) $x, y \notin \mathsf{implant}\mathcal{G}$.
  By (e) of Lemma 21, $x, y \in (0_{\mathcal{G}})\upharpoonleft_{\mathcal{H}}$. Then either $\{x, y\} \cap \{0_{\mathcal{G}}\} \neq \varnothing$ or $x, y \in (0_{\mathcal{G}})\mathring{\upharpoonleft}_{\mathcal{H}}$ and the proof from (i) for $z := 0_{\mathcal{G}}$ works.

It remains to prove that for each $z \in \mathsf{nodes}\mathcal{H}$ and each nonempty $A \subseteq z\mathring{\upharpoonleft}_{\mathcal{H}}$, there is a $<_{\mathcal{H}}$-minimal node in A. We consider several cases:

(i) $z \in \mathsf{support}(\mathcal{T}, \gamma)$.
Consider a nonempty set

$$B := \big(A \cap \mathsf{support}(\mathcal{T}, \gamma)\big) \cup \{\, 0_{\mathcal{G}} : \mathcal{G} \in \gamma \text{ and } A \cap \mathsf{implant}\mathcal{G} \neq \varnothing \,\}.$$

We have $B \subseteq \mathsf{support}(\mathcal{T}, \gamma)$, so it follows by (c) of Lemma 21 that $B \subseteq z{\upharpoonleft}_{\mathcal{T}}$. Then there is a $<_{\mathcal{T}}$-minimal node m in B. Note that $m \in \mathsf{support}(\mathcal{T}, \gamma)$.

(i.1) $m \in A$.
Let us show that m is a $<_{\mathcal{H}}$-minimal node of A. Suppose $x \in A$ and $x \leqslant_{\mathcal{H}} m$.

(i.1.1) $x \in \mathsf{support}(\mathcal{T}, \gamma)$.
In this case $x \in B$ and $x \leqslant_{\mathcal{T}} m$, so $x = m$.

(i.1.2) $\exists \mathcal{E} \in \gamma\, [\, x \in \mathsf{implant}\mathcal{D} \,]$.
By (c) of Lemma 21, $0_{\mathcal{D}} <_{\mathcal{T}} m$. But $0_{\mathcal{D}} \in B$, since $x \in A \cap \mathsf{implant}\mathcal{D}$. This contradicts the $<_{\mathcal{T}}$-minimality of m in B.

(i.2) $m \notin A$.
In this case $m = 0_{\mathcal{G}}$ for some $\mathcal{G} \in \gamma$ such that $A \cap \mathsf{implant}\mathcal{G} \neq \varnothing$. Since A is a chain in $\mathcal{H}$, it follows that $A \cap \mathsf{implant}\mathcal{G}$ is a chain in $\mathcal{G}$. Then it is not hard to prove that there is a $<_{\mathcal{G}}$-minimal node l in $A \cap \mathsf{implant}\mathcal{G}$. Let us show that l is a $<_{\mathcal{H}}$-minimal node of A. Suppose $x \in A$ and $x \leqslant_{\mathcal{H}} l$.

(i.2.1) $x \in \mathsf{support}(\mathcal{T}, \gamma)$.
Then $x <_{\mathcal{H}} l$ (since $l \in \mathsf{implant}\mathcal{G} \not\ni x$), so by (e) of Lemma 21, $x \leqslant_{\mathcal{T}} 0_{\mathcal{G}} = m$. Since $x \in A \not\ni m$, we have $x <_{\mathcal{T}} m$ and $x \in B$. This contradicts the $<_{\mathcal{T}}$-minimality of m in B.

(i.2.2) $\exists \mathcal{E} \in \gamma\, [\, x \in \mathsf{implant}\mathcal{D} \,]$.
We have $0_{\mathcal{D}}, 0_{\mathcal{G}} \in B$ and $0_{\mathcal{D}} \leqslant_{\mathcal{H}} 0_{\mathcal{G}}$ by (c) and (e) of Lemma 21. Then $0_{\mathcal{D}} = 0_{\mathcal{G}}$ by the $<_{\mathcal{T}}$-minimality of $0_{\mathcal{G}} = m$ in B, so $\mathcal{D} = \mathcal{G}$ by (f) of Lemma 18. This implies $x \in A \cap \mathsf{implant}\mathcal{G}$ and $x \leqslant_{\mathcal{G}} l$, so $x = l$ by the $<_{\mathcal{G}}$-minimality of l in $A \cap \mathsf{implant}\mathcal{G}$.

(ii) $\exists \mathcal{G} \in \gamma\, [\, z \in \mathsf{implant}\mathcal{G} \,]$.
By (g) of Lemma 21,

$$A \subseteq z{\upharpoonleft}_{\mathcal{G}} \cup (0_{\mathcal{G}}){\upharpoonleft}_{\mathcal{H}}.$$

If $A \cap (0_{\mathcal{G}}){\upharpoonleft}_{\mathcal{H}} \neq \varnothing$, then a $<_{\mathcal{H}}$-minimal node of $A \cap (0_{\mathcal{G}}){\upharpoonleft}_{\mathcal{H}}$, which exists by (i), is a $<_{\mathcal{H}}$-minimal node of A. Otherwise, $A \subseteq z{\upharpoonleft}_{\mathcal{G}}$, and then a $<_{\mathcal{G}}$-minimal node of A is a $<_{\mathcal{H}}$-minimal node of A. □

Now we establish several properties of the hybrid operation:

Proposition 23. *Suppose that γ is a consistent family of grafts for a tree $\mathcal{T}$ and $\mathcal{H} = \mathsf{hybrid}(\mathcal{T}, \gamma)$. Then:*

(a) *For each $x \in \mathsf{nodes}\mathcal{H}$,*

$$\mathsf{sons}_{\mathcal{H}}(x) = \begin{cases} \mathsf{sons}_{\mathcal{G}}(x), & \text{if } x \in \{0_{\mathcal{G}}\} \cup \mathsf{implant}\mathcal{G} \text{ for some } \mathcal{G} \in \gamma; \\ \mathsf{sons}_{\mathcal{T}}(x), & \text{otherwise } \big(\text{i.e., when } x \in \mathsf{support}(\mathcal{T}, \gamma) \smallsetminus \{0_{\mathcal{G}} : \mathcal{G} \in \gamma\}\big). \end{cases}$$

(b) *If $x, y \in \mathsf{nodes}\mathcal{H}$ and $x \parallel_{\mathcal{H}} y$, then there are $x' \in x{\upharpoonleft}_{\mathcal{H}}$ and $y' \in y{\upharpoonleft}_{\mathcal{H}}$ such that*
(b1) $\big[\, x', y' \in \mathsf{support}(\mathcal{T}, \gamma)$ and $x' \parallel_{\mathcal{T}} y' \,\big]$ or
(b2) $\exists \mathcal{G} \in \gamma \big[\, x', y' \in \mathsf{nodes}\mathcal{G}$ and $x' \parallel_{\mathcal{G}} y' \,\big]$.

(c) *If $\mathcal{T}$ has the least node, then $\mathcal{H}$ has the least node, $0_{\mathcal{H}} = 0_{\mathcal{T}}$, and $0_{\mathcal{H}} \in \mathsf{support}(\mathcal{T}, \gamma)$.*

(d) *If $\max \mathcal{T} = \varnothing$, then $\max \mathcal{H} = \varnothing$.*

(e) *If $\mathcal{T}$ is κ-branching and $\forall \mathcal{G} \in \gamma\, [\, \mathcal{G}$ is κ-branching$\,]$, then $\mathcal{H}$ is κ-branching.*

(f) *If $\mathsf{height}\mathcal{T} \leqslant \omega$ and $\forall \mathcal{G} \in \gamma\, [\, \mathsf{height}\mathcal{G} \leqslant \omega \,]$, then $\mathsf{height}\mathcal{H} \leqslant \omega$.*

Proof. (a) Suppose $x \in \mathsf{nodes}\mathcal{H}$. We consider two cases:

Case 1. $\exists \mathcal{G} \in \gamma \left[x \in \{0_\mathcal{G}\} \cup \mathsf{implant}\mathcal{G} \right]$.

First we prove $\mathsf{sons}_\mathcal{H}(x) \supseteq \mathsf{sons}_\mathcal{G}(x)$. If not, then there is $s \in \mathsf{sons}_\mathcal{G}(x) \setminus \mathsf{sons}_\mathcal{H}(x)$. Then $x <_\mathcal{H} s$ by (a) of Lemma 21, so $s \notin \mathsf{sons}_\mathcal{H}(x)$ implies there is $v \in (x,s)_\mathcal{H}$. We have $v \notin (0_\mathcal{G}){\upharpoonright}_\mathcal{H}$, so $v \in s{\upharpoonright}_\mathcal{G} \subseteq \mathsf{nodes}\mathcal{G}$ by (g) of Lemma 21, whence $v \in (x,s)_\mathcal{G}$. This contradicts $s \in \mathsf{sons}_\mathcal{G}(x)$.

Now we prove $\mathsf{sons}_\mathcal{H}(x) \subseteq \mathsf{sons}_\mathcal{G}(x)$. If not, then there is $s \in \mathsf{sons}_\mathcal{H}(x) \setminus \mathsf{sons}_\mathcal{G}(x)$. We consider several subcases:

(i) $x \in \mathsf{implant}\mathcal{G}$.
  (i.1) $s \in \mathsf{nodes}\mathcal{G}$.
    Then $x <_\mathcal{G} s$, so $(x,s)_\mathcal{G} \neq \varnothing$, whence $(x,s)_\mathcal{H} \neq \varnothing$ by (a) of Lemma 21. This contradicts $s \in \mathsf{sons}_\mathcal{H}(x)$.
  (i.2) $s \notin \mathsf{nodes}\mathcal{G}$.
    Then $x <_\mathcal{H} s$ implies $x <_\mathcal{H} r <_\mathcal{H} s$ for some $r \in \max\mathcal{G}$. This contradicts $s \in \mathsf{sons}_\mathcal{H}(x)$.

(ii) $x = 0_\mathcal{G}$.
  (ii.1) $s \in \mathsf{implant}\mathcal{G}$.
    This case is similar to (i.1).
  (ii.2) $s \notin \mathsf{implant}\mathcal{G}$.
    Then $s \in \mathsf{sons}_\mathcal{H}(0_\mathcal{G})$ with (f) of Lemma 21 implies $s \in \max\mathcal{G}$, so $(0_\mathcal{G}, s)_\mathcal{H} = \varnothing$ implies $(0_\mathcal{G}, s)_\mathcal{G} = \varnothing$. This contradicts $s \notin \mathsf{sons}_\mathcal{G}(0_\mathcal{G})$.

Case 2. $x \in \mathsf{support}(\mathcal{T},\gamma) \setminus \{0_\mathcal{G} : \mathcal{G} \in \gamma\}$.

First we prove $\mathsf{sons}_\mathcal{H}(x) \subseteq \mathsf{sons}_\mathcal{T}(x)$. If not, then there is $s \in \mathsf{sons}_\mathcal{H}(x) \setminus \mathsf{sons}_\mathcal{T}(x)$. We consider two subcases:

(i) $s \notin \mathsf{support}(\mathcal{T},\gamma)$.
  Then there is $\mathcal{E} \in \gamma$ such that $s \in \mathsf{implant}\mathcal{E}$. Then $x \leqslant_\mathcal{H} 0_\mathcal{E} <_\mathcal{H} s$ by (e) of Lemma 21, so $x <_\mathcal{H} 0_\mathcal{E} <_\mathcal{H} s$ by Case 2. This contradicts $s \in \mathsf{sons}_\mathcal{H}(x)$.

(ii) $s \in \mathsf{support}(\mathcal{T},\gamma)$.
  Then $x <_\mathcal{T} s$, so $s \notin \mathsf{sons}_\mathcal{T}(x)$ implies there is $v \in (x,s)_\mathcal{T}$. Since $(x,s)_\mathcal{H} = \varnothing$, we have $v \notin \mathsf{support}(\mathcal{T},\gamma)$, so there is $\mathcal{E} \in \gamma$ such that $v \in \mathsf{explant}(\mathcal{T},\mathcal{E})$. Then $x \leqslant_\mathcal{T} 0_\mathcal{E} <_\mathcal{T} v <_\mathcal{T} s$ by (e) of Lemma 18, so $x \leqslant_\mathcal{H} 0_\mathcal{E} <_\mathcal{H} s$, whence $x <_\mathcal{H} 0_\mathcal{E} <_\mathcal{H} s$ by Case 2. This contradicts $s \in \mathsf{sons}_\mathcal{H}(x)$.

Now we prove $\mathsf{sons}_\mathcal{H}(x) \supseteq \mathsf{sons}_\mathcal{T}(x)$. If not, then there is $s \in \mathsf{sons}_\mathcal{T}(x) \setminus \mathsf{sons}_\mathcal{H}(x)$. Again, there are two subcases:

(i) $s \notin \mathsf{support}(\mathcal{T},\gamma)$.
  Then there is $\mathcal{E} \in \gamma$ such that $s \in \mathsf{explant}(\mathcal{T},\mathcal{E})$. Then $x \leqslant_\mathcal{T} 0_\mathcal{E} <_\mathcal{T} s$ by (e) of Lemma 18, so $x <_\mathcal{T} 0_\mathcal{E} <_\mathcal{T} s$ by Case 2. This contradicts $s \in \mathsf{sons}_\mathcal{T}(x)$.

(ii) $s \in \mathsf{support}(\mathcal{T},\gamma)$.
  Then $x <_\mathcal{H} s$, so $s \notin \mathsf{sons}_\mathcal{H}(x)$ implies there is $v \in (x,s)_\mathcal{H}$. Since $(x,s)_\mathcal{T} = \varnothing$, we have $v \notin \mathsf{support}(\mathcal{T},\gamma)$, so there is $\mathcal{E} \in \gamma$ such that $v \in \mathsf{implant}\mathcal{E}$. Then $x \leqslant_\mathcal{H} 0_\mathcal{E} <_\mathcal{H} v <_\mathcal{H} s$ by (e) of Lemma 21, so $x \leqslant_\mathcal{T} 0_\mathcal{E} <_\mathcal{T} s$, whence $x <_\mathcal{T} 0_\mathcal{E} <_\mathcal{T} s$ by Case 2. This contradicts $s \in \mathsf{sons}_\mathcal{T}(x)$.

(b) Suppose $x, y \in \mathsf{nodes}\mathcal{H}$ and $x \parallel_\mathcal{H} y$. We consider several cases:

(i) $x, y \in \mathsf{support}(\mathcal{T},\gamma)$.
  Then by (b) of Lemma 21, $x' := x$ and $y' := y$ satisfy (b1) of Proposition 23.

(ii) $\left|\{x,y\} \cap \mathsf{support}(\mathcal{T},\gamma)\right| = 1$.
We may assume without loss of generality that $x \in \mathsf{support}(\mathcal{T},\gamma)$ and $y \notin \mathsf{support}(\mathcal{T},\gamma)$. Then there is $\mathcal{G} \in \gamma$ such that $y \in \mathsf{implant}\mathcal{G}$.
(ii.1) $x \parallel_{\mathcal{H}} 0_{\mathcal{G}}$.
Then $x' := x$ and $y' := 0_{\mathcal{G}}$ satisfy (b1) of Proposition 23.
(ii.2) $x \leqslant_{\mathcal{H}} 0_{\mathcal{G}}$.
Then $x \leqslant_{\mathcal{H}} y$, which contradicts $x \parallel_{\mathcal{H}} y$.
(ii.3) $x >_{\mathcal{H}} 0_{\mathcal{G}}$.
Then by (f) of Lemma 21, $x \in (\max\mathcal{G})\lfloor_{\mathcal{H}}$. Let $r := \mathsf{root}_{\mathcal{H}}(x, \max\mathcal{G})$. We have $r \parallel_{\mathcal{G}} y$ (else $r \geqslant_{\mathcal{G}} y$, which contradicts $x \parallel_{\mathcal{H}} y$), so $x' := r$ and $y' := y$ satisfy (b2) of Proposition 23.
(iii) $x, y \notin \mathsf{support}(\mathcal{T},\gamma)$.
Then there are $\mathcal{D}, \mathcal{E} \in \gamma$ such that $x \in \mathsf{implant}\mathcal{D}$ and $y \in \mathsf{implant}\mathcal{E}$.
(iii.1) $\mathcal{D} = \mathcal{E}$.
Then by (a) of Lemma 21, $x' := x$ and $y' := y$ satisfy (b2) of Proposition 23.
(iii.2) $\mathcal{D} \neq \mathcal{E}$ and $0_{\mathcal{D}} \parallel_{\mathcal{H}} 0_{\mathcal{E}}$.
Then $x' := 0_{\mathcal{D}}$ and $y' := 0_{\mathcal{E}}$ satisfy (b1) of Proposition 23.
(iii.3) $\mathcal{D} \neq \mathcal{E}$ and $0_{\mathcal{D}} \nparallel_{\mathcal{H}} 0_{\mathcal{E}}$.
Then by (c) of Definition 17 we may assume without loss of generality that $0_{\mathcal{E}} \in (\max\mathcal{D})\lfloor_{\mathcal{T}}$. We have $x \parallel_{\mathcal{H}} 0_{\mathcal{E}}$ — otherwise $x \leqslant_{\mathcal{H}} 0_{\mathcal{E}}$, which contradicts $x \parallel_{\mathcal{H}} y$, or $x >_{\mathcal{H}} 0_{\mathcal{E}}$, which contradicts $0_{\mathcal{E}} \in (\max\mathcal{D})\lfloor_{\mathcal{T}}$. Let us consider $x_1 := x$ and $y_1 := 0_{\mathcal{E}}$. Then $x_1 \parallel_{\mathcal{H}} y_1$ and $\left|\{x_1,y_1\} \cap \mathsf{support}(\mathcal{T},\gamma)\right| = 1$, so by (ii) there are corresponding $x'_1 \in x_1{\upharpoonleft}_{\mathcal{H}}$ and $y'_1 \in y_1{\upharpoonleft}_{\mathcal{H}}$. Then $x' := x'_1 \in x{\upharpoonleft}_{\mathcal{H}}$ and $y' := y'_1 \in y{\upharpoonleft}_{\mathcal{H}}$ satisfy (b1) or (b2) of Proposition 23.

(c) Suppose $\mathcal{T}$ has the least node. Then $0_{\mathcal{T}} \in \mathsf{support}(\mathcal{T},\gamma)$ by (b) of Lemma 18, therefore $0_{\mathcal{T}}$ is the least node of $\mathcal{H}$ by (b) and (d) of Lemma 21.

(d) Suppose $\max\mathcal{T} = \varnothing$. Let $x \in \mathsf{nodes}\mathcal{H}$. If $x \in \mathsf{support}(\mathcal{T},\gamma) \setminus \{0_{\mathcal{G}} : \mathcal{G} \in \gamma\}$, then $\mathsf{sons}_{\mathcal{H}}(x) \neq \varnothing$ by (a) of Proposition 23 and by (a) of Lemma 6, hence $x \notin \max\mathcal{H}$. If $x \in \{0_{\mathcal{G}}\} \cup \mathsf{implant}\mathcal{G}$ for some $\mathcal{G} \in \gamma$, then $x \notin \max\mathcal{G}$ by (a) of Lemma 18, so $x \notin \max\mathcal{H}$ by (a) of Lemma 21.

(e) Suppose $\mathcal{T}$ is κ-branching and for each $\mathcal{G} \in \gamma$, the $\mathcal{G}$ is κ-branching. Then $\mathcal{H}$ is κ-branching by (a) of Proposition 23 and by (a) of Lemma 6.

(f) Suppose $\mathsf{height}\mathcal{T} \leqslant \omega$ and for each $\mathcal{G} \in \gamma$, we have $\mathsf{height}\mathcal{G} \leqslant \omega$. It is enough to prove that for each $x \in \mathsf{nodes}\mathcal{H}$, the $x{\upharpoonleft}_{\mathcal{H}}$ is finite.

If $x \in \mathsf{support}(\mathcal{T},\gamma)$, then $x{\upharpoonleft}_{\mathcal{H}} \cap \mathsf{support}(\mathcal{T},\gamma) \subseteq x{\upharpoonleft}_{\mathcal{T}}$, so $x{\upharpoonleft}_{\mathcal{H}} \cap \mathsf{support}(\mathcal{T},\gamma)$ is finite. Suppose $G \in \gamma$. If $x{\upharpoonleft}_{\mathcal{H}} \cap \mathsf{implant}\mathcal{G} \neq \varnothing$, then $0_{\mathcal{G}} \in x{\upharpoonleft}_{\mathcal{H}}$ by (c) of Lemma 21, so $0_{\mathcal{G}} \in x{\upharpoonleft}_{\mathcal{T}}$. Then $x \in (\max\mathcal{G})\lfloor_{\mathcal{T}}$ by (d) of Lemma 18, so $x{\upharpoonleft}_{\mathcal{H}} \cap \mathsf{implant}\mathcal{G} \subseteq \big(\mathsf{root}_{\mathcal{T}}(x, \max\mathcal{G})\big){\upharpoonleft}_{\mathcal{G}}$ by (b4) of Definition 19. This means that $x{\upharpoonleft}_{\mathcal{H}} \cap \mathsf{implant}\mathcal{G}$ is finite, since $v{\upharpoonleft}_{\mathcal{G}}$ is finite for every $v \in \mathsf{nodes}\mathcal{G}$. So it is enough to show that the set $\{\mathcal{G} \in \gamma : 0_{\mathcal{G}} \in x{\upharpoonleft}_{\mathcal{T}}\}$ is finite. Since $x{\upharpoonleft}_{\mathcal{T}}$ is finite, the (f) of Lemma 18 implies that this is indeed the case.

If $x \in \mathsf{implant}\mathcal{G}$ for some $\mathcal{G} \in \gamma$, then $x{\upharpoonleft}_{\mathcal{H}} = x{\upharpoonleft}_{\mathcal{G}} \cup (0_{\mathcal{G}}){\upharpoonleft}_{\mathcal{H}}$ by (g) of Lemma 21. Since $0_{\mathcal{G}} \in \mathsf{support}(\mathcal{T},\gamma)$, the $(0_{\mathcal{G}}){\upharpoonleft}_{\mathcal{H}}$ is finite by the above, therefore $x{\upharpoonleft}_{\mathcal{H}}$ is finite. □

Finally we establish two properties of branches in $\mathsf{hybrid}(\mathcal{T},\gamma)$:

Lemma 24. *Suppose that γ is a consistent family of grafts for a tree $\mathcal{T}$ and B is a branch in $\mathsf{hybrid}(\mathcal{T},\gamma)$. Then:*

(a) *If $\mathcal{G} \in \gamma$ and $B \cap \mathsf{nodes}\mathcal{G} \neq \varnothing$, then $B \cap \mathsf{nodes}\mathcal{G}$ is a branch in $\mathcal{G}$.*
(b) *If every graft in γ has bounded chains, then $B \cap \mathsf{support}(\mathcal{T},\gamma)$ is $\mathsf{hybrid}(\mathcal{T},\gamma)$-cofinal in B.*

Proof. Let $\mathcal{H} := \mathsf{hybrid}(\mathcal{T},\gamma)$.

(a) Suppose that $\mathcal{G}\in\gamma$, B is a branch in $\mathcal{H}$, and $x\in C_\mathcal{G} := B\cap\mathsf{nodes}\mathcal{G}$. We must prove that $C_\mathcal{G}$ is a branch in $\mathcal{G}$. We consider two cases:

Case 1. $\exists y\in B\setminus\big((0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup\mathsf{nodes}\mathcal{G}\big)$.

By (f) of Lemma 6, $x\dot{\uparrow}_\mathcal{H}\subseteq B$, so $0_\mathcal{G}\in B$. Then since B is a chain in $\mathcal{H}$ and $y\notin(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup\{0_\mathcal{G}\}$, we have $y>_\mathcal{H}0_\mathcal{G}$. Then $y\in(\max\mathcal{G})\downarrow_\mathcal{H}$ by (f) of Lemma 21. Let $r:=\mathsf{root}_\mathcal{H}(y,\max\mathcal{G})$. We have $y\in B$, so by (f) of Lemma 6 $y\dot{\uparrow}_\mathcal{H}\subseteq B$, hence $r\dot{\uparrow}_\mathcal{H}\subseteq B$. Now $r\dot{\uparrow}_\mathcal{G}\subseteq\mathsf{nodes}\mathcal{G}$ and by (a) of Lemma 21, $r\dot{\uparrow}_\mathcal{G}\subseteq r\dot{\uparrow}_\mathcal{H}$, so $r\dot{\uparrow}_\mathcal{G}\subseteq B\cap\mathsf{nodes}\mathcal{G}=C_\mathcal{G}$. Further, $r\dot{\uparrow}_\mathcal{G}$ is branch in $\mathcal{G}$ by (g) of Lemma 6, $r\dot{\uparrow}_\mathcal{G}\subseteq C_\mathcal{G}$, and $C_\mathcal{G}$ is a chain in $\mathcal{G}$, therefore $C_\mathcal{G}$ is a branch in $\mathcal{G}$.

Case 2. $B\subseteq(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup\mathsf{nodes}\mathcal{G}$.

Since $C_\mathcal{G}$ is a chain in $\mathcal{G}$, then by (c) of Lemma 6 there is $B_\mathcal{G}\in\mathsf{branches}\mathcal{G}$ such that $C_\mathcal{G}\subseteq B_\mathcal{G}$. Now $(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}$ and $B_\mathcal{G}$ are chains in $\mathcal{H}$ and $B_\mathcal{G}\subseteq\mathsf{nodes}\mathcal{G}\subseteq(0_\mathcal{G})\dot{\downarrow}_\mathcal{H}$, therefore $(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup B_\mathcal{G}$ is a chain in $\mathcal{H}$. Furthermore, B is a branch in $\mathcal{H}$, by Case 2

$$B\subseteq(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup\big(B\cap\mathsf{nodes}\mathcal{G}\big)=(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup C_\mathcal{G}\subseteq(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup B_\mathcal{G},$$

and $(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup B_\mathcal{G}$ is a chain in $\mathcal{H}$, so

$$B=(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup C_\mathcal{G}=(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cup B_\mathcal{G}.$$

Then $C_\mathcal{G}=B_\mathcal{G}$ because $(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cap C_\mathcal{G}=\varnothing$ and $(0_\mathcal{G})\mathring{\uparrow}_\mathcal{H}\cap B_\mathcal{G}=\varnothing$, so $C_\mathcal{G}$ is a branch in $\mathcal{G}$.

(b) Suppose that every $\mathcal{G}\in\gamma$ has bounded chains and $B\in\mathsf{branches}\mathcal{H}$. Let $x\in B$ and $C:=B\cap\mathsf{support}(\mathcal{T},\gamma)$. We must prove that $x\in C\lceil_\mathcal{H}$. If $x\in\mathsf{support}(\mathcal{T},\gamma)$, then $x\in C$, so $x\in C\lceil_\mathcal{H}$. If $x\notin\mathsf{support}(\mathcal{T},\gamma)$, then there is $\mathcal{G}\in\gamma$ such that $x\in\mathsf{implant}\mathcal{G}$. We have $B\cap\mathsf{nodes}\mathcal{G}\neq\varnothing$, so by (a), $B_\mathcal{G}:=B\cap\mathsf{nodes}\mathcal{G}$ is a branch in $\mathcal{G}$. Now, by (h) of Lemma 6, there is $m\in\max\mathcal{G}$ such that $B_\mathcal{G}=m\dot{\uparrow}_\mathcal{G}$. Then $x\in B_\mathcal{G}=m\dot{\uparrow}_\mathcal{G}\subseteq m\dot{\uparrow}_\mathcal{H}$ and $m\in\mathsf{support}(\mathcal{T},\gamma)$, whence $m\in C$, so $x\in C\lceil_\mathcal{H}$. □

5. Foliage hybrid operation

In this section we construct the foliage hybrid operation and establish its properties — see Definition 27 and Proposition 29. The foliage hybrid operation modifies a given foliage tree $\mathbf{F}$ with the help of a family φ of special foliage trees, which we call *foliage grafts*. This operation deals with nonincreasing foliage trees and it acts as follows. At first, applying the hybrid operation (see Section 4) to $\mathsf{skeleton}\mathbf{F}$ and $\{\mathsf{skeleton}\mathbf{G}:\mathbf{G}\in\varphi\}$, we obtain a tree. After that we define leaves at nodes of this tree by using leaves of $\mathbf{F}$ and leaves of foliage grafts $\mathcal{G}$, $\mathcal{G}\in\varphi$.

Definition 25. Let $\mathbf{F}$ be a nonincreasing foliage tree. Then a **foliage graft** for $\mathbf{F}$ is a foliage tree $\mathbf{G}$ such that:

(a) $\mathbf{G}$ is nonincreasing;
(b) $\mathsf{skeleton}\mathbf{G}$ is a graft for $\mathsf{skeleton}\mathbf{F}$
(hence $0_\mathbf{G}\in\mathsf{nodes}\mathbf{F}$ and $\max\mathbf{G}\subseteq\mathsf{nodes}\mathbf{F}$);
(c) $\mathbf{G}_{0_\mathbf{G}}\subseteq\mathbf{F}_{0_\mathbf{G}}$;
(d) $\forall m\in\max\mathbf{G}\,[\mathbf{G}_m=\mathbf{F}_m]$.

The set

$$\mathsf{cut}(\mathbf{F}, \mathbf{G}) := \mathbf{F}_{0_\mathbf{G}} \setminus \mathbf{G}_{0_\mathbf{G}}$$

is called the **cut** from $\mathbf{F}$ by $\mathbf{G}$.

Definition 26. Let $\mathbf{F}$ be a nonincreasing foliage tree. Then φ is a **consistent** family of foliage grafts for $\mathbf{F}$ iff

(a) $\forall \mathbf{G} \in \varphi\, [\, \mathbf{G}$ is a foliage graft for $\mathbf{F}\,]$;
(b) $\forall \mathbf{D} \neq \mathbf{E} \in \varphi\, [\, \mathsf{skeleton}\, \mathbf{D} \neq \mathsf{skeleton}\, \mathbf{E}\,]$;
(c) $\{\mathsf{skeleton}\, \mathbf{G} : \mathbf{G} \in \varphi\}$ is a consistent family of grafts for $\mathsf{skeleton}\, \mathbf{F}$.

The set

$$\mathsf{loss}(\mathbf{F}, \varphi) := \bigcup_{\mathbf{G} \in \varphi} \mathsf{cut}(\mathbf{F}, \mathbf{G})$$

is called the **loss** of $\mathbf{F}$ on φ.

Now we define the foliage hybrid operation:

Definition 27. Let φ be a consistent family of foliage grafts for a nonincreasing foliage tree $\mathbf{F}$. Then the **foliage hybrid** of $\mathbf{F}$ and φ — in symbols, $\mathsf{fol.hybr}(\mathbf{F}, \varphi)$ — is a foliage tree $\mathbf{H}$ such that:

(a) $\mathsf{skeleton}\, \mathbf{H} := \mathsf{hybrid}\big(\mathsf{skeleton}\, \mathbf{F}, \{\mathsf{skeleton}\, \mathbf{G} : \mathbf{G} \in \varphi\}\big)$;
(b) $\mathbf{H}_x := \begin{cases} \mathbf{G}_x \setminus \mathsf{loss}(\mathbf{F}, \varphi), & \text{if } x \in \mathsf{implant}\, \mathbf{G} \text{ for some } \mathbf{G} \in \varphi; \\ \mathbf{F}_x \setminus \mathsf{loss}(\mathbf{F}, \varphi), & \text{otherwise } \big(\text{i.e., when } x \in \mathsf{support}(\mathbf{F}, \varphi)\big), \end{cases}$

where

$$\mathsf{support}(\mathbf{F}, \varphi) := \mathsf{support}\big(\mathsf{skeleton}\, \mathbf{F}, \{\mathsf{skeleton}\, \mathbf{G} : \mathbf{G} \in \varphi\}\big).$$

Note that the hybrid of $\mathsf{skeleton}\, \mathbf{F}$ and $\{\mathsf{skeleton}\, \mathbf{G} : \mathbf{G} \in \varphi\}$ is a tree by Proposition 22, so a foliage hybrid is indeed a foliage tree.

Lemma 28. *Suppose that φ is a consistent family of foliage grafts for a nonincreasing foliage tree $\mathbf{F}$ and $\mathbf{H} = \mathsf{fol.hybr}(\mathbf{F}, \varphi)$. Then:*

(a) $\forall \mathbf{G} \in \varphi\ \forall x \in \mathsf{nodes}\, \mathbf{G}\, \big[\, \mathbf{H}_x = \mathbf{G}_x \setminus \mathsf{loss}(\mathbf{F}, \varphi)\,\big]$.
(b) *For any set A: if $\forall \mathbf{G} \in \varphi\, \big[\, A \subseteq \mathbf{G}_{0_\mathbf{G}}$ or $A \cap \mathbf{F}_{0_\mathbf{G}} = \varnothing\,\big]$, then $A \cap \mathsf{loss}(\mathbf{F}, \varphi) = \varnothing$.* □

Now we establish several properties of the foliage hybrid operation:

Proposition 29. *Suppose that φ is a consistent family of foliage grafts for a nonincreasing foliage tree $\mathbf{F}$ and $\mathbf{H} = \mathsf{fol.hybr}(\mathbf{F}, \varphi)$. Then:*

(a) *$\mathbf{H}$ is nonincreasing.*
(b) *If $\mathbf{F}$ and each $\mathbf{G} \in \varphi$ are splittable, then $\mathbf{H}$ is splittable.*
(c) *If $\mathbf{F}$ and each $\mathbf{G} \in \varphi$ are locally strict, then $\mathbf{H}$ is locally strict.*

(d) *If* $\mathbf{F}$ *is complete (has strict branches) and splittable, and each* $\mathbf{G} \in \varphi$ *has bounded chains, then* $\mathbf{H}$ *is complete (has strict branches).*
(e) *If* $\mathbf{F}$ *and each* $\mathbf{G} \in \varphi$ *are open in a space* X*, then* $\mathbf{H}$ *is open in the subspace* $X \setminus \mathsf{loss}(\mathbf{F}, \varphi)$ *of* X*.*

Proof. (a) We must prove that the foliage tree $\mathbf{H}$ is nonincreasing. Suppose $x, y \in \mathsf{nodes}\,\mathbf{H}$ and $x <_{\mathbf{H}} y$. Then one of conditions (b1)–(b5) of Definition 19 holds. For example, if (b4) holds, then there is $\mathbf{G} \in \varphi$ such that

$$x \in \mathsf{implant}\,\mathbf{G}, \quad y \in \mathsf{support}(\mathbf{F}, \varphi), \quad \text{and} \quad x <_{\mathbf{G}} r := \mathsf{root}_{\mathbf{F}}(y, \max \mathbf{G}) \leqslant_{\mathbf{F}} y.$$

Then $\mathbf{G}_x \supseteq \mathbf{G}_r$ and $\mathbf{F}_r \supseteq \mathbf{F}_y$ because $\mathbf{G}$ and $\mathbf{F}$ are nonincreasing by Definition 25. Then $\mathbf{H}_x \supseteq \mathbf{H}_r$ by (a) of Lemma 28 and $\mathbf{H}_r \supseteq \mathbf{H}_y$ by (b) of Definition 27, so $\mathbf{H}_x \supseteq \mathbf{H}_y$. The other cases are similar.

(b) Suppose $\mathbf{F}$ and each $\mathbf{G} \in \varphi$ are splittable; we must prove that $\mathbf{H}$ is also splittable. By (a), $\mathbf{H}$ is nonincreasing. Let $x, y \in \mathsf{nodes}\,\mathbf{H}$ and $x \parallel_{\mathbf{H}} y$. Then by (b) of Proposition 23, there are $x' \in x{\upharpoonleft}_{\mathbf{H}}$ and $y' \in y{\upharpoonleft}_{\mathbf{H}}$ such that

$$\text{either} \qquad x', y' \in \mathsf{support}(\mathbf{F}, \varphi) \ \text{ and } \ x' \parallel_{\mathbf{F}} y' \tag{1}$$

$$\text{or} \qquad \exists \mathbf{G} \in \varphi \big[\, x', y' \in \mathsf{nodes}\,\mathbf{G} \ \text{ and } \ x' \parallel_{\mathbf{G}} y' \,\big]. \tag{2}$$

If (1) holds, then $\mathbf{H}_x \subseteq \mathbf{H}_{x'} \subseteq \mathbf{F}_{x'}$ and $\mathbf{H}_y \subseteq \mathbf{H}_{y'} \subseteq \mathbf{F}_{y'}$ since $\mathbf{H}$ is nonincreasing and by (b) of Definition 27, and $\mathbf{F}_{x'} \cap \mathbf{F}_{y'} = \varnothing$ because $\mathbf{F}$ is splittable, so $\mathbf{H}_x \cap \mathbf{H}_y = \varnothing$. If (2) holds, then $\mathbf{H}_{x'} \subseteq \mathbf{G}_{x'}$ and $\mathbf{H}_{y'} \subseteq \mathbf{G}_{y'}$ by (a) of Lemma 28, and $\mathbf{G}_{x'} \cap \mathbf{G}_{y'} = \varnothing$ since $\mathbf{G}$ is splittable, so $\mathbf{H}_x \cap \mathbf{H}_y = \varnothing$ again.

(c) Suppose that $\mathbf{F}$ and each $\mathbf{G} \in \varphi$ are locally strict; we must prove that $\mathbf{H}$ is also locally strict. Let $x \in \mathsf{nodes}\,\mathbf{H} \setminus \max \mathbf{H}$. Then $\mathsf{sons}_{\mathbf{H}}(x) \neq \varnothing$ by (a) of Lemma 6. We consider two cases:

Case 1. $\exists \mathbf{G} \in \varphi \big[\, x \in \{0_{\mathbf{G}}\} \cup \mathsf{implant}\,\mathbf{G} \,\big]$.

By (a) of Proposition 23 we have $\mathsf{sons}_{\mathbf{G}}(x) = \mathsf{sons}_{\mathbf{H}}(x) \neq \varnothing$, so $x \in \mathsf{nodes}\,\mathbf{G} \setminus \max \mathbf{G}$. Then

$$\mathbf{G}_x \equiv \bigsqcup_{s \in \mathsf{sons}_{\mathbf{G}}(x)} \mathbf{G}_s$$

since $\mathbf{G}$ is locally strict, hence

$$\mathbf{G}_x \setminus \mathsf{loss}(\mathbf{F}, \varphi) \equiv \bigsqcup_{s \in \mathsf{sons}_{\mathbf{H}}(x)} \big(\mathbf{G}_s \setminus \mathsf{loss}(\mathbf{F}, \varphi)\big).$$

Since $x \in \mathsf{nodes}\,\mathbf{G}$ and $\mathsf{sons}_{\mathbf{H}}(x) = \mathsf{sons}_{\mathbf{G}}(x) \subseteq \mathsf{nodes}\,\mathbf{G}$, then by (a) of Lemma 28 we have

$$\mathbf{H}_x \equiv \bigsqcup_{s \in \mathsf{sons}_{\mathbf{H}}(x)} \mathbf{H}_s\,.$$

Case 2. $x \in \mathsf{support}(\mathbf{F}, \varphi) \setminus \{0_{\mathbf{G}} : \mathbf{G} \in \varphi\}$.

By (a) of Proposition 23 we have $\mathsf{sons}_{\mathbf{F}}(x) = \mathsf{sons}_{\mathbf{H}}(x) \neq \varnothing$, so $x \in \mathsf{nodes}\,\mathbf{F} \setminus \max \mathbf{F}$. Then

$$\mathbf{F}_x \equiv \bigsqcup_{s \in \mathsf{sons}_{\mathbf{F}}(x)} \mathbf{F}_s$$

since $\mathbf{F}$ is locally strict, whence

$$\mathbf{F}_x \setminus \mathsf{loss}(\mathbf{F}, \varphi) \equiv \bigsqcup_{s \in \mathsf{sons}_{\mathbf{H}}(x)} \big(\mathbf{F}_s \setminus \mathsf{loss}(\mathbf{F}, \varphi)\big).$$

Since $\mathsf{sons}_{\mathbf{H}}(x) = \mathsf{sons}_{\mathbf{F}}(x) \subseteq \mathsf{nodes}\,\mathbf{F}$ and $\mathsf{sons}_{\mathbf{H}}(x) \subseteq \mathsf{nodes}\,\mathbf{H}$, we have

$$\mathsf{sons}_{\mathbf{H}}(x) \subseteq \mathsf{nodes}\,\mathbf{F} \cap \mathsf{nodes}\,\mathbf{H} = \mathsf{support}(\mathbf{F}, \varphi)$$

by (b) of Lemma 21. Also we have $x \in \mathsf{support}(\mathbf{F}, \varphi)$, so by (b) of Definition 27 we get

$$\mathbf{H}_x \equiv \bigsqcup_{s \in \mathsf{sons}_{\mathbf{H}}(x)} \mathbf{H}_s.$$

(d) First, suppose that $\mathbf{F}$ is complete and splittable, and each $\mathbf{G} \in \varphi$ has bounded chains. We must prove that $\mathbf{H}$ is complete. Since $\mathbf{F}$ is complete, we have $\mathsf{nodes}\,\mathbf{F} \neq \varnothing$, so $\mathsf{nodes}\,\mathbf{H} \neq \varnothing$ because either $\varphi = \varnothing$ and $\mathsf{nodes}\,\mathbf{H} = \mathsf{nodes}\,\mathbf{F}$ or $0_{\mathbf{G}} \in \mathsf{nodes}\,\mathbf{H}$ for some $\mathbf{G} \in \varphi$. Suppose that $B_{\mathbf{H}} \in \mathsf{branches}\,\mathbf{H}$ and $C := B_{\mathbf{H}} \cap \mathsf{support}(\mathbf{F}, \varphi)$. Then it follows by (b) of Lemma 24 that C is $\mathbf{H}$-cofinal in $B_{\mathbf{H}}$, and then $C \neq \varnothing$ since $B_{\mathbf{H}} \neq \varnothing$. By (a), $\mathbf{H}$ is nonincreasing, so by (a) of Lemma 14 we have

$$\mathsf{fruit}_{\mathbf{H}}(B_{\mathbf{H}}) = \mathsf{fruit}_{\mathbf{H}}(C). \tag{3}$$

Since $C \subseteq \mathsf{support}(\mathbf{F}, \varphi)$, then by (b) of Definition 27 we get

$$\mathsf{fruit}_{\mathbf{H}}(C) = \bigcap_{x \in C} \big(\mathbf{F}_x \setminus \mathsf{loss}(\mathbf{F}, \varphi)\big) = \mathsf{fruit}_{\mathbf{F}}(C) \setminus \mathsf{loss}(\mathbf{F}, \varphi). \tag{4}$$

Further, since C is a chain in $\mathbf{H}$ and $C \subseteq \mathsf{support}(\mathbf{F}, \varphi)$, we see by (b) of Lemma 21 that C is a chain in $\mathbf{F}$. Then by (c) of Lemma 6 there is $B_{\mathbf{F}} \in \mathsf{branches}\,\mathbf{F}$ such that $C \subseteq B_{\mathbf{F}}$, so we have

$$\mathsf{fruit}_{\mathbf{F}}(C) \supseteq \mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}}) \neq \varnothing$$

because $\mathbf{F}$ is complete. It follows that

$$\mathsf{fruit}_{\mathbf{H}}(B_{\mathbf{H}}) \supseteq \mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}}) \setminus \mathsf{loss}(\mathbf{F}, \varphi) \quad \text{and} \quad \mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}}) \neq \varnothing,$$

so it is enough to prove

$$\mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}}) \cap \mathsf{loss}(\mathbf{F}, \varphi) = \varnothing.$$

Then, by (b) of Lemma 28, it is enough to show that for each $\mathbf{G} \in \varphi$,

$$\text{either} \qquad \mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}}) \subseteq \mathbf{G}_{0_{\mathbf{G}}} \qquad \text{or} \qquad \mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}}) \cap \mathbf{F}_{0_{\mathbf{G}}} = \varnothing. \tag{5}$$

To show it we consider two cases:

Case 1. $0_{\mathbf{G}} \in B_{\mathbf{F}}$.

First let us prove that $0_{\mathbf{G}} \in B_{\mathbf{H}}$. If not, then by (d) of Lemma 6 there is $b \in B_{\mathbf{H}}$ such that $b \parallel_{\mathbf{H}} 0_{\mathbf{G}}$. Since C is $\mathbf{H}$-cofinal in $B_{\mathbf{H}}$, there is $c \in C$ such that $c \geqslant_{\mathbf{H}} b$, so $c \parallel_{\mathbf{H}} 0_{\mathbf{G}}$ by (b) of Lemma 6. Both c and $0_{\mathbf{G}}$ lie in $\mathsf{support}(\mathbf{F}, \varphi)$, so we have $c \parallel_{\mathbf{F}} 0_{\mathbf{G}}$, but this contradicts $c \in C \subseteq B_{\mathbf{F}} \ni 0_{\mathbf{G}}$.

Now $0_{\mathbf{G}} \in B_{\mathbf{H}}$. Then by (a) of Lemma 24, $B_{\mathbf{H}} \cap \mathsf{nodes}\,\mathbf{G} \in \mathsf{branches}\,\mathbf{G}$, so by (h) of Lemma 6, $B_{\mathbf{H}} \cap \mathsf{nodes}\,\mathbf{G} = m{\upharpoonleft}_{\mathbf{G}}$ for some $m \in \max \mathbf{G}$. Since $\max \mathbf{G} \subseteq \mathsf{support}(\mathbf{F}, \varphi)$, we have $m \in B_{\mathbf{H}} \cap \mathsf{support}(\mathbf{F}, \varphi) = C \subseteq B_{\mathbf{F}}$, that is, $m \in B_{\mathbf{F}}$. Then

$$\mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}}) \subseteq \mathbf{F}_m = \mathbf{G}_m \subseteq \mathbf{G}_{0_{\mathbf{G}}}$$

by (d) of Definition 25 and because $\mathbf{G}$ is nonincreasing, so (5) satisfies.

Case 2. $0_{\mathbf{G}} \notin B_{\mathbf{F}}$.

Then by (d) of Lemma 6, there is $b \in B_{\mathbf{F}}$ such that $b \parallel_{\mathbf{F}} 0_{\mathbf{G}}$. Since $\mathbf{F}$ is splittable, we have $\mathbf{F}_b \cap \mathbf{F}_{0_{\mathbf{G}}} = \varnothing$. Then since $b \in B_{\mathbf{F}}$, we have $\mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}}) \subseteq \mathbf{F}_b$, so (5) satisfies again.

Now suppose that $\mathbf{F}$ is splittable and has strict branches, and each $\mathbf{G} \in \varphi$ has bounded chains. We must prove that $\mathbf{H}$ has strict branches; suppose it does not. Since $\mathbf{F}$ is complete, we already know that $\mathbf{H}$ is also complete, so there is $B_{\mathbf{H}} \in \mathsf{branches}\,\mathbf{H}$ such that $|\mathsf{fruit}_{\mathbf{H}}(B_{\mathbf{H}})| > 1$. Let C and $B_{\mathbf{F}}$ be as above. It follows by (3) and (4) that $|\mathsf{fruit}_{\mathbf{F}}(C)| > 1$, and $|\mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}})| = 1$ since $\mathbf{F}$ has strict branches, so we have $\mathsf{fruit}_{\mathbf{F}}(C) \neq \mathsf{fruit}_{\mathbf{F}}(B_{\mathbf{F}})$. Then, using (a) of Lemma 14, we see that C is not $\mathbf{F}$-cofinal in $B_{\mathbf{F}}$ because $\varnothing \neq C \subseteq B_{\mathbf{F}} \subseteq \mathsf{nodes}\,\mathbf{F}$. Further, since $C \subseteq B_{\mathbf{F}}$, $B_{\mathbf{F}}$ is a chain in $\mathbf{F}$, and C is not $\mathbf{F}$-cofinal in $B_{\mathbf{F}}$, it is not hard to show that there is $x \in B_{\mathbf{F}}$ such that $C \subseteq x\overset{\circ}{\upharpoonleft}_{\mathbf{F}}$. Now we consider two cases:

Case 1. $x \in \mathsf{support}(\mathbf{F}, \varphi)$.

Then $x\overset{\circ}{\upharpoonleft}_{\mathbf{F}} \cap \mathsf{support}(\mathbf{F}, \varphi) \subseteq x\overset{\circ}{\upharpoonleft}_{\mathbf{H}}$. We have $C \subseteq x\overset{\circ}{\upharpoonleft}_{\mathbf{F}}$ and $C \subseteq \mathsf{support}(\mathbf{F}, \varphi)$, so $C \subseteq x\overset{\circ}{\upharpoonleft}_{\mathbf{H}}$. Then $C\urcorner_{\mathbf{H}} \subseteq x\overset{\circ}{\upharpoonleft}_{\mathbf{H}}$, so $B_{\mathbf{H}} \subseteq x\overset{\circ}{\upharpoonleft}_{\mathbf{H}}$ because C is $\mathbf{H}$-cofinal in $B_{\mathbf{H}}$, whence $B_{\mathbf{H}} \subset x\overset{\bullet}{\upharpoonleft}_{\mathbf{H}}$. This contradicts $B_{\mathbf{H}} \in \mathsf{branches}\,\mathbf{H}$, since $x\overset{\bullet}{\upharpoonleft}_{\mathbf{H}}$ is a chain in $\mathbf{H}$.

Case 2. $x \notin \mathsf{support}(\mathbf{F}, \varphi)$.

We have $x \in \mathsf{nodes}\,\mathbf{F} \setminus \mathsf{support}(\mathbf{F}, \varphi)$, so by definition of $\mathsf{support}(\mathbf{F}, \varphi)$ there is $\mathbf{G} \in \varphi$ such that $x \in \mathsf{explant}(\mathbf{F}, \mathbf{G})$. Then (e) of Lemma 18 implies

$$x\overset{\circ}{\upharpoonleft}_{\mathbf{F}} \cap \mathsf{support}(\mathbf{F}, \varphi) \subseteq (0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{F}},$$

so $C \subseteq (0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{F}}$. Since $0_{\mathbf{G}} \in \mathsf{support}(\mathbf{F}, \varphi)$, we have

$$(0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{F}} \cap \mathsf{support}(\mathbf{F}, \varphi) \subseteq (0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{H}},$$

whence $C \subseteq (0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{H}}$. This implies $C\urcorner_{\mathbf{H}} \subseteq (0_{\mathcal{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{H}}$, so $B_{\mathbf{H}} \subseteq (0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{H}}$ because C is $\mathbf{H}$-cofinal in $B_{\mathbf{H}}$. Then $B_{\mathbf{H}} = (0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{H}}$, since $B_{\mathbf{H}}$ is a branch in $\mathbf{H}$ and $(0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{H}}$ is a chain in $\mathbf{H}$. Thus we have $0_{\mathbf{G}} \in B_{\mathbf{H}}$. Now (a) of Lemma 24 with (h) of Lemma 6 implies that $B_{\mathbf{H}} \cap \mathsf{nodes}\,\mathbf{G} = m\overset{\bullet}{\upharpoonleft}_{\mathbf{G}}$ for some $m \in \max \mathbf{G}$. Then, since $\max \mathbf{G} \subseteq \mathsf{support}(\mathbf{F}, \varphi)$, we have $m \in B_{\mathbf{H}} \cap \mathsf{support}(\mathbf{F}, \varphi) = C$. So $m \in C$ and $m >_{\mathbf{F}} 0_{\mathbf{G}}$ by (d) of Definition 15. This contradicts $C \subseteq (0_{\mathbf{G}})\overset{\bullet}{\upharpoonleft}_{\mathbf{F}}$.

(e) Suppose that $\mathbf{F}$ and each $\mathbf{G} \in \varphi$ are open in a space X. Then $\mathbf{H}$ is open in the subspace $X \setminus \mathsf{loss}(\mathbf{F}, \varphi)$ of X by (b) of Definition 27. □

6. Application of the foliage hybrid operation

We will apply the foliage hybrid operation to a π-tree $\mathbf{F}$ of a space X in such a way that the $\mathsf{fol.hybr}(\mathbf{F}, \varphi)$ will be a π-tree on a subspace Y of X. To carry out this construction we need to answer (that is, to find some sufficient conditions) the following questions:

(i) When the $\mathsf{fol.hybr}(\mathbf{F}, \varphi)$ is a Baire foliage tree on Y?
(ii) When the $\mathsf{fol.hybr}(\mathbf{F}, \varphi)$ grows into Y?

The answer to question (i) is given in the following lemma:

Lemma 30. *Suppose that* $\mathbf{F}$ *is a Baire foliage tree on a space* X *and* φ *is a consistent family of foliage grafts for* $\mathbf{F}$ *such that every* $\mathbf{G}$ *in* φ *is* $\aleph_0$*-branching, locally strict, open in* X*, has bounded chains, and has* $\mathsf{height}\,\mathbf{G} \leqslant \omega$*. Then the* $\mathsf{fol.hybr}(\mathbf{F}, \varphi)$ *is a Baire foliage tree on* $X \setminus \mathsf{loss}(\mathbf{F}, \varphi)$*.*

Proof. Let $\mathbf{H} := \mathsf{fol.hybr}(\mathbf{F}, \varphi)$. It follows from (c)–(f) of Proposition 23 and (i) of Lemma 6 that $\mathbf{H}$ is a foliage $\omega, \aleph_0$-tree and $0_{\mathbf{H}} = 0_{\mathbf{F}}$, so $\mathbf{H}_{0_{\mathbf{H}}} = \mathbf{F}_{0_{\mathbf{F}}} \setminus \mathsf{loss}(\mathbf{F}, \varphi) = X \setminus \mathsf{loss}(\mathbf{F}, \varphi)$. By (b) of Lemma 14, $\mathbf{F}$ is splittable, and then $\mathbf{H}$ is open in $X \setminus \mathsf{loss}(\mathbf{F}, \varphi)$, locally strict, and has strict branches by (c)–(e) of Proposition 29. □

The answer to question (ii) is given in Lemma 32, and this answer raises another question: When the $\mathsf{fol.hybr}(\mathbf{F}, \varphi)$ shoots into $\mathbf{F}$? The answer to this question is given in Lemma 34.

Definition 31. Let $\mathbf{H}$ and $\mathbf{F}$ be foliage trees. Then

✎ **H shoots into F** $\;:\longleftrightarrow\;\; \forall p \in \mathsf{flesh}\,\mathbf{H}\ \forall y \in \mathsf{scope}_{\mathbf{F}}(p)\ \exists x \in \mathsf{scope}_{\mathbf{H}}(p)\ \big[\mathsf{shoot}_{\mathbf{H}}(x) \gg \mathsf{shoot}_{\mathbf{F}}(y)\big].$

Lemma 32. *Suppose that a foliage tree* $\mathbf{H}$ *shoots into a foliage tree* $\mathbf{F}$ *and* $\mathbf{F}$ *grows into a space* X*. Then* $\mathbf{H}$ *grows into the subspace* $X \cap \mathsf{flesh}\,\mathbf{H}$ *of* X*.*

Proof. Let $Y := X \cap \mathsf{flesh}\,\mathbf{H}$, $p \in Y$, and $U \in \mathsf{nbhds}(p, Y)$. Then there is $V \in \mathsf{nbhds}(p, X)$ such that $U = V \cap Y$, and there is $y \in \mathsf{scope}_{\mathbf{F}}(p)$ such that $\mathsf{shoot}_{\mathbf{F}}(y) \gg \{V\}$ because $\mathbf{F}$ grows into X. Since $\mathbf{H}$ shoots into $\mathbf{F}$, there is $x \in \mathsf{scope}_{\mathbf{H}}(p)$ with the property $\mathsf{shoot}_{\mathbf{H}}(x) \gg \mathsf{shoot}_{\mathbf{F}}(y)$. It follows that there is $G \in \mathsf{shoot}_{\mathbf{H}}(x) \setminus \{\varnothing\}$ such that $G \subseteq V$. Since $G \subseteq \mathsf{flesh}\,\mathbf{H}$, then $G \subseteq V \cap \mathsf{flesh}\,\mathbf{H} \subseteq X \cap \mathsf{flesh}\,\mathbf{H} = Y$, so $G \subseteq V \cap Y = U$. Therefore we have found $x \in \mathsf{scope}_{\mathbf{H}}(p)$ such that $\mathsf{shoot}_{\mathbf{H}}(x) \gg \{U\}$. □

Definition 33. Let $\mathbf{F}$ be a nonincreasing foliage tree and let $\mathbf{G}$ be a foliage graft for $\mathbf{F}$. Then $\mathbf{G}$ **preserves shoots** of $\mathbf{F}$ iff

➢ for each $p \in \mathsf{flesh}\,\mathbf{G}$ and for each $y \in \mathsf{scope}_{\mathbf{F}}(p) \cap \big(\{0_{\mathbf{G}}\} \cup \mathsf{explant}(\mathbf{F}, \mathbf{G})\big)$
➢ there is $x \in \mathsf{scope}_{\mathbf{G}}(p) \cap \big(\{0_{\mathbf{G}}\} \cup \mathsf{implant}\,\mathbf{G}\big)$ such that
✓ $\mathsf{shoot}_{\mathbf{G}}(x) \gg \mathsf{shoot}_{\mathbf{F}}(y)$.

Lemma 34. *Suppose that*

➢ $\mathbf{F}$ *is a nonincreasing foliage tree,*
➢ φ *is a consistent family of foliage grafts for* $\mathbf{F}$*,*
➢ *the foliage hybrid of* $\mathbf{F}$ *and* φ *has nonempty leaves, and*
➢ *each* $\mathbf{G} \in \varphi$ *preserves shoots of* $\mathbf{F}$*.*

Then the foliage hybrid of $\mathbf{F}$ *and* φ *shoots into* $\mathbf{F}$*.*

Proof. Let $\mathbf{H} := \mathsf{fol.hybr}(\mathbf{F}, \varphi)$, $p \in \mathsf{flesh}\,\mathbf{H}$, and $y \in \mathsf{scope}_{\mathbf{F}}(p)$. We consider two cases:

Case 1. $y \in \mathsf{support}(\mathbf{F}, \varphi) \setminus \{0_{\mathbf{G}} : \mathbf{G} \in \varphi\}$.

By (a) of Proposition 23 we have $\mathsf{sons}_{\mathbf{H}}(y) = \mathsf{sons}_{\mathbf{F}}(y)$, so

$$\mathsf{sons}_{\mathbf{H}}(y) \subseteq \mathsf{nodes}\,\mathbf{F} \cap \mathsf{nodes}\,\mathbf{H} = \mathsf{support}(\mathbf{F}, \varphi)$$

by (b) of Lemma 21. Then by (b) of Definition 27 we have

$$\mathbf{H}_y = \mathbf{F}_y \setminus \mathsf{loss}(\mathbf{F},\varphi) \quad \text{and} \quad \forall s \in \mathsf{sons}_{\mathbf{H}}(y)\left[\mathbf{H}_s = \mathbf{F}_s \setminus \mathsf{loss}(\mathbf{F},\varphi)\right].$$

Further, $p \in \mathbf{F}_y$ and $p \in \mathsf{flesh}\,\mathbf{H}$, so $p \notin \mathsf{loss}(\mathbf{F},\varphi)$, whence $p \in \mathbf{H}_y$, that is, $y \in \mathsf{scope}_{\mathbf{H}}(p)$. Now, for $x := y$ and for each $s \in \mathsf{sons}_{\mathbf{H}}(x) = \mathsf{sons}_{\mathbf{H}}(y) = \mathsf{sons}_{\mathbf{F}}(y)$, we have $\varnothing \neq \mathbf{H}_s \subseteq \mathbf{F}_s$. This implies $\mathsf{shoot}_{\mathbf{H}}(x) \gg \mathsf{shoot}_{\mathbf{F}}(y)$.

Case 2. $\exists \mathbf{G} \in \varphi\left[\, y \in \{0_{\mathbf{G}}\} \cup \mathsf{explant}(\mathbf{F},\mathbf{G})\right]$.

The foliage tree $\mathbf{F}$ is nonincreasing, $p \in \mathbf{F}_y$, and $y \geqslant_{\mathbf{F}} 0_{\mathbf{G}}$, so $p \in \mathbf{F}_{0_{\mathbf{G}}}$. We have $p \in \mathsf{flesh}\,\mathbf{H}$, so $p \notin \mathsf{loss}(\mathbf{F},\varphi)$, hence $p \in \mathbf{F}_{0_{\mathbf{G}}}$ implies $p \in \mathbf{G}_{0_{\mathbf{G}}}$. Then $p \in \mathsf{flesh}\,\mathbf{G}$ and

$$y \in \mathsf{scope}_{\mathbf{F}}(p) \cap \left(\{0_{\mathbf{G}}\} \cup \mathsf{explant}(\mathbf{F},\mathbf{G})\right),$$

so, since $\mathbf{G}$ preserves shoots $\mathbf{F}$, there is

$$x \in \mathsf{scope}_{\mathbf{G}}(p) \cap \left(\{0_{\mathbf{G}}\} \cup \mathsf{implant}\,\mathbf{G}\right)$$

such that $\mathsf{shoot}_{\mathbf{G}}(x) \gg \mathsf{shoot}_{\mathbf{F}}(y)$. Again, by (a) of Proposition 23 we have $\mathsf{sons}_{\mathbf{H}}(x) = \mathsf{sons}_{\mathbf{G}}(x)$, so $\mathsf{sons}_{\mathbf{H}}(x) \subseteq \mathsf{nodes}\,\mathbf{G}$. Then by (a) of Lemma 28 we have

$$\mathbf{H}_x = \mathbf{G}_x \setminus \mathsf{loss}(\mathbf{F},\varphi) \quad \text{and} \quad \forall s \in \mathsf{sons}_{\mathbf{H}}(x)\left[\mathbf{H}_s = \mathbf{G}_s \setminus \mathsf{loss}(\mathbf{F},\varphi)\right].$$

We have $p \in \mathbf{G}_x$ and $p \notin \mathsf{loss}(\mathbf{F},\varphi)$, so $p \in \mathbf{H}_x$, that is, $x \in \mathsf{scope}_{\mathbf{H}}(p)$. Now, for each $s \in \mathsf{sons}_{\mathbf{H}}(x) = \mathsf{sons}_{\mathbf{G}}(x)$, we have $\varnothing \neq \mathbf{H}_s \subseteq \mathbf{G}_s$. This implies $\mathsf{shoot}_{\mathbf{H}}(x) \gg \mathsf{shoot}_{\mathbf{G}}(x)$, so $\mathsf{shoot}_{\mathbf{H}}(x) \gg \mathsf{shoot}_{\mathbf{F}}(y)$ because $\gg$ is transitive. □

7. Main construction

In this section we prove Theorem 37, which can be viewed as the main technical result of this paper. This theorem is a statement about the Baire space $\mathcal{N}$ and the standard foliage tree of ${}^{\omega}\omega$, which we denote by $\mathbf{S}$ — see Notation 12. The connection between $\mathcal{N}$ with $\mathbf{S}$ on the one hand and a space X with a π-tree on the other hand is explained by Lemma 13.

Notation 35. Let $A \subseteq {}^{\omega}\omega$ and $x \in {}^{<\omega}\omega$. Recall that $\mathbf{S}_x = \{p \in {}^{\omega}\omega : x \subseteq p\}$. Then

✎ A is **π-dense** at x $\;:\!\longleftrightarrow\;$ $\forall y \in {}^{<\omega}\omega\left[\, y \supseteq x \;\rightarrow\; \left|\{n \in \omega : \mathbf{S}_{y^\frown\langle n\rangle} \subseteq A\}\right| = \aleph_0\right]$;
✎ A is **π-dense** in the Baire space $\;:\!\longleftrightarrow\;$ $\forall y \in {}^{<\omega}\omega\left[\,\left|\{n \in \omega : \mathbf{S}_{y^\frown\langle n\rangle} \subseteq A\}\right| = \aleph_0\right]$.

Remark 36.

(a) If K is a compact subset of $\mathcal{N}$, then ${}^{\omega}\omega \setminus K$ is an open π-dense subset of $\mathcal{N}$.
(b) If a set D is π-dense in $\mathcal{N}$, then D is dense in $\mathcal{N}$. □

Theorem 37. *Suppose that $Y = \bigcap_{n\in\omega} U_n$, where each U_n is an open π-dense subset of the Baire space. Then there is a Baire foliage tree on Y that shoots into the standard foliage tree of ${}^{\omega}\omega$* (see Definitions 10, 12, 31, and 35).

Question 38. Does Theorem 37 remain true if we replace “π-dense” by “dense”?

We will build this Baire foliage tree on Y which shoots into $\mathbf{S}$ by applying the foliage hybrid operation to $\mathbf{S}$ and φ, where φ is a consistent family of foliage grafts for $\mathbf{S}$. We construct the family φ in the proof of Theorem 37, see below. The construction of a single foliage graft $\mathbf{G}$ (that will be a member of φ) is described in the following lemma:

Lemma 39. *Suppose that $\mathbf{v} \in {}^{<\omega}\omega$ and $O \subset \mathbf{S_v}$ is open in the Baire space and is π-dense at $\mathbf{v}$. Then there is a foliage tree $\mathbf{G}$ such that*

(a1) $0_{\mathbf{G}} = \mathbf{v}$,
(a2) $\mathsf{height}\,\mathbf{G} \leqslant \omega$,
(a3) $\mathbf{G}$ *is* $\aleph_0$*-branching*,
(a4) $\mathbf{G}$ *has bounded chains*,
(a5) $\mathbf{G}$ *is locally strict*,
(a6) $\mathbf{G}$ *is open in the Baire space*,
(a7) $\mathbf{G}$ *is a foliage graft for* $\mathbf{S}$,
(a8) $\mathbf{G}$ *preserves shoots of* $\mathbf{S}$,
(a9) $\mathsf{implant}\,\mathbf{G} \neq \varnothing$,
(a10) $\mathsf{cut}(\mathbf{S}, \mathbf{G}) = \mathbf{S_v} \setminus O$, *and*
(a11) $O \equiv \bigsqcup_{z \in \mathsf{max}\,\mathbf{G}} \mathbf{S}_z$.

In the proof of Lemma 39 (see below) we verify clause (a8), which says that $\mathbf{G}$ preserves shoots of $\mathbf{S}$. We do this by using the following lemma:

Lemma 40. *Suppose that $\mathbf{A}$, $\mathbf{B}$ are foliage trees with nonempty leaves, $x \in \mathsf{nodes}\,\mathbf{A}$, and $y \in \mathsf{nodes}\,\mathbf{B}$. Assume that $\left|\mathsf{sons}_{\mathbf{A}}(x)\right| \geqslant \aleph_0$ and that there is finite F such that*

$$\forall s \in \mathsf{sons}_{\mathbf{A}}(x) \setminus F \left[\, s \in \mathsf{sons}_{\mathbf{B}}(y) \ \text{ and } \ \mathbf{A}_s \subseteq \mathbf{B}_s \,\right].$$

Then $\mathsf{shoot}_{\mathbf{A}}(x) \gg \mathsf{shoot}_{\mathbf{B}}(y)$. □

Proof of Lemma 39. Let

$$\Omega := \{z \in \mathbf{v}{\downarrow_{\mathbf{S}}} : \mathbf{S}_z \subseteq O\}, \quad \Delta := \mathbf{v}{\downarrow_{\mathbf{S}}} \setminus \Omega, \quad \text{and} \quad \mathsf{MAX} := \min(\Omega, <_{\mathbf{S}}).$$

Then we have

(b1) $\mathbf{v} \in \Delta$ and $|\Delta| = \aleph_0$;
(b2) $\Delta = (\Delta{\uparrow_{\mathbf{S}}}) \cap (\mathbf{v}{\downarrow_{\mathbf{S}}})$;
(b3) MAX is an antichain in $\mathbf{S}$;
(b4) $\mathsf{MAX}{\downarrow_{\mathbf{S}}} = \Omega$;
(b5) $O \equiv \bigsqcup_{z \in \mathsf{MAX}} \mathbf{S}_z$.

For each $x \in \Delta$, define

$$\Delta_x := \Delta \cap (x{\downarrow_{\mathbf{S}}}) \quad \text{and} \quad \Omega_x := \mathsf{sons}_{\mathbf{S}}(x) \cap \Omega.$$

Then

(c1) $\forall x \in \Delta\, [\, x \in \Delta_x$ and $|\Delta_x| = \aleph_0\,]$;
(c2) $\forall x \in \Delta\, [\, \Omega_x \subseteq \mathsf{MAX}$ and $|\Omega_x| = \aleph_0\,]$;
(c3) $\mathsf{MAX} \equiv \bigsqcup_{x\in\Delta} \Omega_x$.

Now for each $x \in \Delta$ and all $d \in \Delta_x$, we can find infinite sets $\Omega_{x,d} \subseteq \Omega_x$ in such a way that

$$\forall x \in \Delta\, \big[\, \Omega_x \equiv \textstyle\bigsqcup_{d\in\Delta_x} \Omega_{x,d}\, \big]. \tag{6}$$

Put

$$\mathsf{IMP} := \Big\{ \mathsf{node}_x^l \,:\, x \in \Delta \text{ and } l \in \big\{0, \dots, l(x)\big\}\Big\}$$

where

$$l(x) := \mathsf{length}\, x - \mathsf{length}\, \mathbf{v}$$

and node_x^l are different new nodes for the skeleton of the foliage tree $\mathbf{G}$ such that $\mathsf{IMP} \cap \mathsf{nodes}\,\mathbf{S} = \varnothing$. Put

$$\mathsf{NOD} := \{\mathbf{v}\} \cup \mathsf{MAX} \cup \mathsf{IMP}$$

(we intend to have $\mathsf{nodes}\,\mathbf{G} = \mathsf{NOD}$, $0_{\mathbf{G}} = \mathbf{v}$, $\max \mathbf{G} = \mathsf{MAX}$, and $\mathsf{implant}\,\mathbf{G} = \mathsf{IMP}$).

For $x \in {}^{<\omega}\omega$ and $l \in \{0, \dots, \mathsf{length}\, x\}$, define

$$x_{-\,l} \;:=\; x \restriction \big((\mathsf{length}\, x) - l\big)$$

— that is, $x_{-\,l} = \langle x_0, \dots, x_{(\mathsf{length}\, x)-\,l-1}\rangle \in \omega^{(\mathsf{length}\, x)-\,l}$, $x_{-\,0} = x$, and if $x \in \mathbf{v}{\downarrow}_{\mathbf{S}}$, then $x_{-\,l(x)} = \mathbf{v}$. Using (b2) we have

(d1) $\forall x \in \Delta\, \forall l \in \big\{0, \dots, l(x)\big\}\, \big[\, x_{-\,l} \in \Delta$ and $x \in \Delta_{x_{-l}}\, \big]$;
(d2) $\Big\{\big(x_{-\,l}, x\big) : x \in \Delta$ and $l \in \big\{0, \dots, l(x)\big\}\Big\} = \big\{(z, d) : z \in \Delta$ and $d \in \Delta_z\big\}$.

Now we build a tree $(\mathsf{NOD}, <)$, which will be a skeleton for the foliage tree $\mathbf{G}$. First we define a relation $\lessdot$ on the set NOD as the relation that satisfies exactly the following:

- ➢ for each $x \in \Delta$,
  $\mathbf{v} \lessdot \mathsf{node}_x^{l(x)} \lessdot \mathsf{node}_x^{l(x)-1} \lessdot \dots \lessdot \mathsf{node}_x^1 \lessdot \mathsf{node}_x^0$;
- ➢ for each $x \in \Delta$ and each $l \in \big\{0, \dots, l(x)\big\}$,
  $\mathsf{node}_x^l \lessdot z$ for all $z \in \Omega_{x_{-l},\, x}$.

Note that the last clause is correct by (d1). Then let relation $<$ be the transitive closure of relation $\lessdot$. That is, for each $a, b \in \mathsf{NOD}$,

$$a < b \quad :\longleftrightarrow \quad \exists\, n \in \omega\; \exists\, z_0, \dots, z_{n+1} \in \mathsf{NOD}\; \big[\, a = z_0 \lessdot z_1 \lessdot \dots \lessdot z_{n+1} = b\, \big].$$

Let $\mathcal{T} := (\mathsf{NOD}, <)$. Then it is not hard to show the following:

(e1) $\mathsf{sons}_{\mathcal{T}}(\mathsf{node}_x^0) = \Omega_{x_{-0},\, x}$ for all $x \in \Delta$;
$\mathsf{sons}_{\mathcal{T}}(\mathsf{node}_x^l) = \Omega_{x_{-l},\, x} \cup \big\{\mathsf{node}_x^{l-1}\big\}$ for all $x \in \Delta$ and $l \in \big\{1, \dots, l(x)\big\}$;
$\mathsf{sons}_{\mathcal{T}}(\mathbf{v}) = \big\{\mathsf{node}_x^{l(x)} : x \in \Delta\big\}$.

(e2) $\forall x \in \Delta\, [\, \mathbf{v} \sqsubset_{\mathcal{T}} \mathsf{node}_x^{l(x)} \sqsubset_{\mathcal{T}} \mathsf{node}_x^{l(x)-1} \sqsubset_{\mathcal{T}} \ldots \sqsubset_{\mathcal{T}} \mathsf{node}_x^{1} \sqsubset_{\mathcal{T}} \mathsf{node}_x^{0}\,]$;
in particular, $\mathbf{v} \sqsubset_{\mathcal{T}} \mathsf{node}_{\mathbf{v}}^{l(\mathbf{v})} = \mathsf{node}_{\mathbf{v}}^{0}$.

(e3) $\max \mathcal{T} = \mathsf{MAX}$.
Indeed, using (d2), (c3), and (6), we get

$$\max \mathcal{T} = \bigcup \Big\{ \Omega_{x-l,\,x} : x \in \Delta \text{ and } l \in \big\{0, \ldots, l(x)\big\} \Big\} = \bigcup \big\{ \Omega_{z,d} : z \in \Delta \text{ and } d \in \Delta_z \big\} = \mathsf{MAX}.$$

(e4) $\mathcal{T}$ is an $\aleph_0$-branching tree with the least node and $0_{\mathcal{T}} = \mathbf{v}$.

(e5) $\mathcal{T}$ has bounded chains and $\mathsf{height}\,\mathcal{T} \leqslant \omega$.
To prove (e5) it is enough to show that each chain in $\mathcal{T}$ is finite. If C is a chain in $\mathcal{T}$, then by (c) of Lemma 6, there is $B \in \mathsf{branches}\,\mathcal{T}$ such that $C \subseteq B$, and it follows using (e) of Lemma 6 that there exists some s in $B \cap \mathsf{sons}_{\mathcal{T}}(0_{\mathcal{T}})$. Then $s = \mathsf{node}_x^{l(x)}$ for some $x \in \Delta$, so $|B| \leqslant l(x) + 3$.

(e6) $\mathcal{T}$ is a graft for $\mathbf{S}$ and $\mathsf{implant}\,\mathcal{T} = \mathsf{IMP}$.

(e7) $\mathsf{explant}(\mathbf{S}, \mathcal{T}) = \Delta \setminus \{\mathbf{v}\}$.
Indeed, using (b4), we have

$$\mathsf{explant}(\mathbf{S}, \mathcal{T}) := (0_{\mathcal{T}})\!\downarrow^{\circ}_{\mathbf{S}} \setminus (\max \mathcal{T})\!\downarrow_{\mathbf{S}} = \mathbf{v}\!\downarrow^{\circ}_{\mathbf{S}} \setminus \mathsf{MAX}\!\downarrow_{\mathbf{S}} =$$
$$\big(\mathbf{v}\!\downarrow_{\mathbf{S}} \setminus \{\mathbf{v}\}\big) \setminus \Omega = \big(\mathbf{v}\!\downarrow_{\mathbf{S}} \setminus \Omega\big) \setminus \{\mathbf{v}\} = \Delta \setminus \{\mathbf{v}\}.$$

Now we build a foliage tree $\mathbf{G}$ with $\mathsf{skeleton}\,\mathbf{G} = \mathcal{T}$ as follows:

- ➢ $\mathbf{G}_z := \mathbf{S}_z$ for all $z \in \mathsf{MAX}$;
- ➢ $\mathbf{G}_{\mathsf{node}_x^0} := \bigcup \big\{ \mathbf{S}_z : z \in \Omega_{x-0,\,x} \big\}$ for all $x \in \Delta$;
- ➢ $\mathbf{G}_{\mathsf{node}_x^l} := \mathbf{G}_{\mathsf{node}_x^{l-1}} \cup \bigcup \big\{ \mathbf{S}_z : z \in \Omega_{x-l,\,x} \big\}$ for all $x \in \Delta$ and $l \in \big\{1, \ldots, l(x)\big\}$ (by recursion on l);
- ➢ $\mathbf{G}_{\mathbf{v}} := \bigcup \big\{ \mathbf{G}_{\mathsf{node}_x^{l(x)}} : x \in \Delta \big\}$.

Then (e1), (c3), (6), and disjointness of the union from (b5) imply that $\mathbf{G}$ is locally strict. Also it is not hard to show that $\mathbf{G}$ is nonincreasing, $0_{\mathbf{G}} = \mathbf{v}$, $\mathsf{height}\,\mathbf{G} \leqslant \omega$, $\mathbf{G}$ is $\aleph_0$-branching, $\mathbf{G}$ has bounded chains, $\mathbf{G}$ is open in the Baire space, $\mathbf{G}$ is a foliage graft for $\mathbf{S}$, $\mathsf{implant}\,\mathbf{G} \neq \varnothing$, and $O \equiv \bigsqcup_{z \in \max \mathbf{G}} \mathbf{S}_z$. To prove that $\mathsf{cut}(\mathbf{S}, \mathbf{G}) = \mathbf{S}_{\mathbf{v}} \setminus O$ we must show that $\mathbf{G}_{0_{\mathbf{G}}} = O$. Since $\mathbf{G}$ is nonincreasing, we have $\mathbf{G}_{0_{\mathbf{G}}} = \mathsf{flesh}\,\mathbf{G}$, so using (b) of Lemma 14, (h) of Lemma 6, and (b5) we have

$$\mathbf{G}_{0_{\mathbf{G}}} = \mathsf{flesh}\,\mathbf{G} = \mathsf{yield}\,\mathbf{G} = \bigcup \big\{ \mathsf{fruit}_{\mathbf{G}}(B) : B \in \mathsf{branches}\mathbf{G} \big\} =$$
$$\bigcup \big\{ \mathsf{fruit}_{\mathbf{G}}(z\!\uparrow_{\mathbf{G}}) : z \in \max \mathbf{G} \big\} = \bigcup \{ \mathbf{G}_z : z \in \max \mathbf{G} \} = \bigcup \{ \mathbf{S}_z : z \in \mathsf{MAX} \} = O.$$

It remains to prove that $\mathbf{G}$ preserves shoots of $\mathbf{S}$. Suppose

$$p \in \mathsf{flesh}\,\mathbf{G} = \mathbf{G}_{0_{\mathbf{G}}} = O, \quad y \in \{0_{\mathbf{G}}\} \cup \mathsf{explant}(\mathbf{S}, \mathbf{G}) = \Delta, \quad \text{and} \quad \mathbf{S}_y \ni p.$$

We must find $x \in \{\mathbf{v}\} \cup \mathsf{IMP}$ such that $\mathbf{G}_x \ni p$ and $\mathsf{shoot}_{\mathbf{G}}(x) \gg \mathsf{shoot}_{\mathbf{S}}(y)$. Note that $\mathbf{G}$ has nonempty leaves and $|\mathsf{sons}_{\mathbf{G}}(x)| \geqslant \aleph_0$ for all $x \in \{\mathbf{v}\} \cup \mathsf{IMP}$ since $\mathbf{G}$ is $\aleph_0$-branching and $\{\mathbf{v}\} \cup \mathsf{IMP} \subseteq \mathsf{nodes}\,\mathbf{G} \setminus \max \mathbf{G}$. Lemma 40 says that if there is finite F such that

$$\forall s \in \mathsf{sons}_{\mathbf{G}}(x) \setminus F \,\big[\, s \in \mathsf{sons}_{\mathbf{S}}(y) \text{ and } \mathbf{G}_s \subseteq \mathbf{S}_s \,\big],$$

then $\mathsf{shoot}_{\mathbf{G}}(x) \gg \mathsf{shoot}_{\mathbf{S}}(y)$. If $x \in \mathsf{IMP}$, then by (e1), (c3), and (6) there is finite L such that $\mathsf{sons}_{\mathbf{G}}(x) \setminus L \subseteq \mathsf{MAX}$, so for all $s \in \mathsf{sons}_{\mathbf{G}}(x) \setminus L$ we have $\mathbf{G}_s = \mathbf{S}_s$.

Summarizing the above reasoning we come to the following. Suppose $y \in \Delta$ and $p \in O \cap \mathbf{S}_y$. Then to finish the proof it is enough to find $x \in \mathsf{IMP}$ and finite F such that

$$\mathbf{G}_x \ni p \quad \text{and} \quad \mathsf{sons}_{\mathbf{G}}(x) \setminus F \subseteq \mathsf{sons}_{\mathbf{S}}(y). \tag{7}$$

Since $p \in O$, then by (b5) there is $\dot{z} \in \mathsf{MAX}$ such that $p \in \mathbf{S}_{\dot{z}}$. Then $\mathbf{S}_{\dot{z}} \cap \mathbf{S}_y \neq \varnothing$, so either $y \geqslant_{\mathbf{S}} \dot{z}$ or $y <_{\mathbf{S}} \dot{z}$ since $\mathbf{S}$ is splittable. If $y \geqslant_{\mathbf{S}} \dot{z}$, then by (b4) $y \in \Omega$, which contradicts $y \in \Delta$, so $y <_{\mathbf{S}} \dot{z}$. Let $w := \dot{z}_{-1}$. Then we have

$$\mathbf{v} \leqslant_{\mathbf{S}} y \leqslant_{\mathbf{S}} w \sqsubset_{\mathbf{S}} \dot{z} \in \mathsf{MAX} = \min(\Omega, <_{\mathbf{S}}) \subseteq \Omega,$$

which implies $w \in \Delta$ and $\dot{z} \in \Omega_w$. Then it follows by (6) that there is $d \in \Delta_w$ such that $\dot{z} \in \Omega_{w,d}$. Now we have

$$\mathbf{v} \leqslant_{\mathbf{S}} y \leqslant_{\mathbf{S}} w \leqslant_{\mathbf{S}} d \in \Delta, \quad \dot{z} \in \Omega_{w,d}, \quad \text{and} \quad p \in \mathbf{S}_{\dot{z}}.$$

Let $l := \mathsf{length}\, d - \mathsf{length}\, y$ and $m := \mathsf{length}\, d - \mathsf{length}\, w$. Then $d_{-l} = y$, $d_{-m} = w$, and $0 \leqslant m \leqslant l \leqslant l(d)$, so we may consider nodes node_d^l and node_d^m in IMP. Then $x := \mathsf{node}_d^l$ satisfies condition (7). Indeed, $\mathsf{node}_d^l \leqslant_{\mathbf{G}} \mathsf{node}_d^m$ by (e2) and $\mathbf{G}$ is nonincreasing, so

$$\mathbf{G}_x = \mathbf{G}_{\mathsf{node}_d^l} \supseteq \mathbf{G}_{\mathsf{node}_d^m} \supseteq \bigcup\{\mathbf{S}_z : z \in \Omega_{d_{-m},d}\} = \bigcup\{\mathbf{S}_z : z \in \Omega_{w,d}\} \supseteq \mathbf{S}_{\dot{z}} \ni p.$$

Finally, by (e1) there is finite F such that

$$\mathsf{sons}_{\mathbf{G}}(x) \setminus F = \mathsf{sons}_{\mathbf{G}}(\mathsf{node}_d^l) \setminus F = \Omega_{d_{-l},d} = \Omega_{y,d} \subseteq \Omega_y \subseteq \mathsf{sons}_{\mathbf{S}}(y). \qquad \square$$

Proof of Theorem 37. Let $\mathbf{v} \in {}^{<\omega}\omega$ and $n \in \omega$. Put $O := U_n \cap \mathbf{S}_{\mathbf{v}}$ and assume that $O \neq \mathbf{S}_{\mathbf{v}}$. Then there is a foliage tree $\mathbf{G}$ that satisfies conditions (a1)–(a11) of Lemma 39. Let us denote this foliage tree $\mathbf{G}$ by $\mathbf{G}(\mathbf{v}, n)$. Using this notation, we construct sequences $(Z_n)_{n\in\omega}$, $(\psi_n)_{n\in\omega}$, and $(M_n)_{n\in\{-1\}\cup\omega}$ by recursion on n as follows:

(f1) $M_{-1} := \{0_{\mathbf{S}}\}$;
(f2) $Z_n := \{x \in M_{n-1} : U_n \cap \mathbf{S}_x \neq \mathbf{S}_x\}$;
(f3) $\psi_n := \{\mathbf{G}(x, n) : x \in Z_n\}$;
(f4) $M_n := (M_{n-1} \setminus Z_n) \cup \bigcup_{\mathbf{G}\in\psi_n} \max \mathbf{G}$.

For each $n \in \omega$, we will prove the following:

(g1) $Z_n = \{0_{\mathbf{G}} : \mathbf{G} \in \psi_n\}$;
(g2) M_n is an antichain in $\mathbf{S}$;
(g3) $(M_n)\!\downarrow_{\mathbf{S}} \cap \bigcup_{i\leqslant n} Z_i = \varnothing$;
(g4) $\bigcup_{i\leqslant n} \psi_i$ is a consistent family of foliage grafts for $\mathbf{S}$;
(g5) $\bigcup_{y\in M_n} \mathbf{S}_y = \bigcap_{i\leqslant n} U_i$;
(g6) $\bigcup\{\mathsf{cut}(\mathbf{S}, \mathbf{G}) : \mathbf{G} \in \bigcup_{i\leqslant n} \psi_i\} = {}^{\omega}\omega \setminus \bigcap_{i\leqslant n} U_i$.

Let us first show that (g1)–(g6) yield the conclusion of the theorem. Put $\varphi := \bigcup_{n\in\omega} \psi_n$, so (g4) implies that φ is a consistent family of foliage grafts for $\mathbf{S}$. Then $\mathbf{H} := \mathsf{fol.hybr}(\mathbf{S}, \varphi)$ satisfies the requirements of the theorem. Indeed, (g6) implies that $\mathsf{loss}(\mathbf{S}, \varphi) = {}^{\omega}\omega \setminus \bigcap_{n\in\omega} U_n$, so it follows from Lemma 30, (a) of Lemma 13, and (a2)–(a6) that $\mathbf{H}$ is a Baire foliage tree on Y. Then $\mathbf{H}$ has nonempty leaves, therefore $\mathbf{H}$ shoots into $\mathbf{S}$ by (a8) and Lemma 34.

It remains to prove that (g1)–(g6) hold for all $n \in \omega$. Condition (g1) easily follows from definitions of Z_n and ψ_n: if $x \in Z_n$, then $x = 0_{\mathbf{G}(x,n)}$ by (a1), so $Z_n = \{0_{\mathbf{G}(x,n)} : x \in Z_n\} = \{0_{\mathbf{G}} : \mathbf{G} \in \psi_n\}$. Conditions (g2)–(g6) will be proved by induction. Using (a1)–(a11), and (d)–(e) of Definition 15, it is not hard to show that (g2)–(g6) are satisfied when $n = 0$. Assume as induction hypothesis that (g2)–(g6) hold for all $n \leqslant k$. We must prove that (g2)–(g6) hold for $n = k + 1$.

(g2) We prove that M_{k+1} is an antichain in $\mathbf{S}$. Suppose $v \neq w \in M_{k+1}$. We consider several cases:

(i) $v, w \in M_k$.
Then $v \parallel_{\mathbf{S}} w$ by the induction hypothesis.

(ii) $v, w \in M_{k+1} \setminus M_k$.
It follows by (f4) that there are $\mathbf{D}, \mathbf{E} \in \psi_{k+1}$ such that $v \in \max \mathbf{D}$ and $w \in \max \mathbf{E}$.

(ii.1) $0_{\mathbf{D}} \neq 0_{\mathbf{E}}$.
We have $0_{\mathbf{D}}, 0_{\mathbf{E}} \in Z_{k+1} \subseteq M_k$ by (g1) and (f2), so $0_{\mathbf{D}} \parallel_{\mathbf{S}} 0_{\mathbf{E}}$ by the induction hypothesis. Then $v \parallel_{\mathbf{S}} w$ by using (b) of Lemma 6 twice.

(ii.2) $0_{\mathbf{D}} = 0_{\mathbf{E}}$.
It follows from (f3) and (a1) that $\mathbf{D} = \mathbf{G}(0_{\mathbf{D}}, k+1) = \mathbf{G}(0_{\mathbf{E}}, k+1) = \mathbf{E}$, so we have $v, w \in \max \mathbf{D}$. Consequently $v \parallel_{\mathbf{S}} w$ by (a7) and (e) of Definition 15.

(iii) $\big|\{v, w\} \cap M_k\big| = 1$.
We may assume without lost of generality that $v \in M_{k+1} \setminus M_k$ and $w \in M_{k+1} \cap M_k$. Again, as in (ii), there is $\mathbf{D} \in \psi_{k+1}$ such that $v \in \max \mathbf{D}$, and then $v >_{\mathbf{S}} 0_{\mathbf{D}}$ and $0_{\mathbf{D}} \in Z_{k+1} \subseteq M_k$.

(iii.1) $0_{\mathbf{D}} \neq w$.
We have $v >_{\mathbf{S}} 0_{\mathbf{D}}$ and $0_{\mathbf{D}} \parallel_{\mathbf{S}} w$ the induction hypothesis (because $0_{\mathbf{D}}, w \in M_k$), so $v \parallel_{\mathbf{S}} w$ by (b) of Lemma 6.

(iii.2) $0_{\mathbf{D}} = w$.
We have $w \in M_{k+1}$ and $w = 0_{\mathbf{D}} \in Z_{k+1}$, so it follows from (f4) that there is $\mathbf{F} \in \psi_{k+1}$ such that $w \in \max \mathbf{F}$. Consequently, $w >_{\mathbf{S}} 0_{\mathbf{F}}$ and $0_{\mathbf{F}} \in Z_{k+1}$. This contradicts the induction hypothesis because $w, 0_{\mathbf{F}} \in Z_{k+1} \subseteq M_k$.

(g3) We must prove that $(M_{k+1})\lfloor_{\mathbf{S}} \cap \bigcup_{i \leqslant k+1} Z_i = \varnothing$. Suppose on the contrary that there is some $x \in (M_{k+1})\lfloor_{\mathbf{S}} \cap \bigcup_{i \leqslant k+1} Z_i$. Since by (g2) with $n = k+1$ (which is already proved), M_{k+1} is an antichain in $\mathbf{S}$, then we may consider

$$r := \mathsf{roots}_{\mathbf{S}}(x, M_{k+1}) \in M_{k+1} = (M_k \setminus Z_{k+1}) \cup \bigcup_{\mathbf{G} \in \psi_{k+1}} \max \mathbf{G}.$$

(i) $\exists \mathbf{G} \in \psi_{k+1} \, [\, r \in \max \mathbf{G}\,]$.
Then $x \geqslant_{\mathbf{S}} r >_{\mathbf{S}} 0_{\mathbf{G}} \in Z_{k+1} \subseteq M_k$, therefore $x \in (M_k)\lfloor_{\mathbf{S}}$, so $x \notin \bigcup_{i \leqslant k} Z_i$ by (g3) with $n = k$, and hence $x \in Z_{k+1} \subseteq M_k$. Now we have $x >_{\mathbf{S}} 0_{\mathbf{G}}$ and $x, 0_{\mathbf{G}} \in M_k$, which contradicts (g2) with $n = k$.

(ii) $r \in M_k \setminus Z_{k+1}$.
Then we have $x \geqslant_{\mathbf{S}} r \in M_k$, therefore as in (i) we get $x \in (M_k)\lfloor_{\mathbf{S}}$, $x \notin \bigcup_{i \leqslant k} Z_i$, and $x \in Z_{k+1} \subseteq M_k$. Also we have $x \neq r$ because $r \notin Z_{k+1}$, consequently $x >_{\mathbf{S}} r$ and $x, r \in M_k$, which again contradicts (g2) with $n = k$.

(g4) We must prove that $\bigcup_{i \leqslant k+1} \psi_i$ is a consistent family of foliage grafts for $\mathbf{S}$. Every $\mathbf{G} \in \bigcup_{i \leqslant k+1} \psi_i$ is a foliage graft for $\mathbf{S}$ by (a7). Suppose $\mathbf{D} \neq \mathbf{E} \in \bigcup_{i \leqslant k+1} \psi_i$. We may assume that $\mathsf{implant}\,\mathbf{D} \cap \mathsf{implant}\,\mathbf{E} = \varnothing$ by construction, and then $\mathsf{skeleton}\,\mathbf{D} \neq \mathsf{skeleton}\,\mathbf{E}$ because implants of $\mathbf{D}$ and $\mathbf{E}$ are nonempty by (a9). It remains to check clause (c) of Definition 17. We consider several cases:

(i) $\mathbf{D}, \mathbf{E} \in \bigcup_{i \leqslant k} \psi_i$.
Then (c) of Definition 17 is satisfied by the induction hypothesis.

(ii) $\mathbf{D}, \mathbf{E} \in \psi_{k+1}$.
Then by (f3) $\mathbf{D} = \mathbf{G}(x, k+1)$ and $\mathbf{E} = \mathbf{G}(y, k+1)$ for some $x \neq y \in Z_{k+1}$, so it follows by using (f2), (g1), and (a1) that

$$M_k \supseteq Z_{k+1} \ni 0_{\mathbf{D}} = 0_{\mathbf{G}(x,k+1)} = x \neq y = 0_{\mathbf{G}(y,k+1)} = 0_{\mathbf{E}} \in Z_{k+1} \subseteq M_k.$$

Consequently, $0_{\mathbf{D}} \parallel_{\mathbf{S}} 0_{\mathbf{E}}$ by (g2) with $n = k$.

(iii) $\left|\{\mathbf{D}, \mathbf{E}\} \cap \psi_{k+1}\right| = 1$.
Suppose without lost of generality that $\mathbf{D} \in \bigcup_{i \leqslant k} \psi_i$ and $\mathbf{E} \in \psi_{k+1}$. Then by (g1) $0_{\mathbf{D}} \in \bigcup_{i \leqslant k} Z_i$ and $0_{\mathbf{E}} \in Z_{k+1} \subseteq M_k \subseteq (M_k){\downarrow_{\mathbf{S}}}$, so it follows by using (g3) with $n = k$ that $0_{\mathbf{D}} \neq 0_{\mathbf{E}}$. If $0_{\mathbf{D}} \parallel_{\mathbf{S}} 0_{\mathbf{E}}$, then clause (c) of Definition 17 holds. It remains to consider the following two cases:

(iii.1) $0_{\mathbf{D}} >_{\mathbf{S}} 0_{\mathbf{E}}$.
We have $0_{\mathbf{D}} \in \bigcup_{i \leqslant k} Z_i$ and $0_{\mathbf{D}} >_{\mathbf{S}} 0_{\mathbf{E}} \in Z_{k+1} \subseteq M_k$, so $0_{\mathbf{D}} \in (M_k){\downarrow_{\mathbf{S}}}$. This contradicts (g3) with $n = k$.

(iii.2) $0_{\mathbf{E}} >_{\mathbf{S}} 0_{\mathbf{D}}$.
Now, $\mathbf{D} \in \bigcup_{i \leqslant k} \psi_i$ and $\bigcup_{i \leqslant k} \psi_i$ is a consistent family of foliage grafts for $\mathbf{S}$ by the induction hypothesis. Further, $0_{\mathbf{E}} \in Z_{k+1} \subseteq M_k$ and it is not hard to show that

$$M_k \subseteq \{0_{\mathbf{S}}\} \cup \bigcup\{\max \mathbf{G} : \mathbf{G} \in \bigcup_{i \leqslant k} \psi_i\}$$

by induction on k. Then it follows from (b) of Lemma 18 that $0_{\mathbf{E}} \in \mathsf{support}(\mathbf{S}, \bigcup_{i \leqslant k} \psi_i)$. Furthermore, (d) of Lemma 18 says that $0_{\mathbf{E}} \in \mathsf{support}(\mathbf{S}, \bigcup_{i \leqslant k} \psi_i)$ plus $0_{\mathbf{E}} >_{\mathbf{S}} 0_{\mathbf{D}}$ imply $0_{\mathbf{E}} \in (\max \mathbf{D}){\downarrow_{\mathbf{S}}}$, so (c) of Definition 17 holds.

(g5) We must prove that $\bigcup_{y \in M_{k+1}} \mathbf{S}_y = \bigcap_{i \leqslant k+1} U_i$. Put $B := \bigcup_{y \in M_k \setminus Z_{k+1}} \mathbf{S}_y$. Then (f2) implies

$$B = \bigcup\{U_{k+1} \cap \mathbf{S}_y : y \in M_k \setminus Z_{k+1}\}. \tag{8}$$

Now, using (f4), (f3), (a11), (8), and (g5) with $n = k$, we have

$$\bigcup_{y \in M_{k+1}} \mathbf{S}_y = B \cup \bigcup\left\{\mathbf{S}_y : y \in \bigcup_{\mathbf{G} \in \psi_{k+1}} \max \mathbf{G}\right\} = B \cup \bigcup\left\{\mathbf{S}_y : y \in \bigcup_{x \in Z_{k+1}} \max \mathbf{G}(x, k+1)\right\} =$$
$$B \cup \bigcup_{x \in Z_{k+1}} \left(\bigcup\left\{\mathbf{S}_y : y \in \max \mathbf{G}(x, k+1)\right\}\right) = B \cup \bigcup_{x \in Z_{k+1}} (U_{k+1} \cap \mathbf{S}_x) =$$
$$\bigcup_{x \in M_k} (U_{k+1} \cap \mathbf{S}_x) = U_{k+1} \cap \bigcup_{x \in M_k} \mathbf{S}_x = U_{k+1} \cap \bigcap_{i \leqslant k} U_i = \bigcap_{i \leqslant k+1} U_i.$$

(g6) We must prove that $\bigcup\{\mathsf{cut}(\mathbf{S}, \mathbf{G}) : \mathbf{G} \in \bigcup_{i \leqslant k+1} \psi_i\} = {}^\omega\omega \setminus \bigcap_{i \leqslant k+1} U_i$. Put $A := \bigcap_{i \leqslant k} U_i$, so that the induction hypothesis asserts

$$\bigcup\{\mathsf{cut}(\mathbf{S}, \mathbf{G}) : \mathbf{G} \in \bigcup_{i \leqslant k} \psi_i\} = {}^\omega\omega \setminus A. \tag{9}$$

Then using (9), (f3), (a10), (f2), and (g5) with $n = k$, we have

$$\bigcup\{\mathsf{cut}(\mathbf{S}, \mathbf{G}) : \mathbf{G} \in \bigcup_{i \leqslant k+1} \psi_i\} = ({}^\omega\omega \setminus A) \cup \bigcup_{\mathbf{G} \in \psi_{k+1}} \mathsf{cut}(\mathbf{S}, \mathbf{G}) = ({}^\omega\omega \setminus A) \cup \bigcup_{x \in Z_{k+1}} \mathsf{cut}\big(\mathbf{S}, \mathbf{G}(x, k+1)\big) \cup \varnothing =$$
$$({}^\omega\omega \setminus A) \cup \bigcup_{x \in Z_{k+1}} \big(\mathbf{S}_x \setminus (U_{k+1} \cap \mathbf{S}_x)\big) \cup \bigcup_{x \in M_k \setminus Z_{k+1}} \varnothing = ({}^\omega\omega \setminus A) \cup \bigcup_{x \in Z_{k+1}} (\mathbf{S}_x \setminus U_{k+1}) \cup \bigcup_{x \in M_k \setminus Z_{k+1}} (\mathbf{S}_x \setminus U_{k+1}) =$$

$$({}^{\omega}\omega \setminus A) \cup \bigcup_{x \in M_k} (\mathbf{S}_x \setminus U_{k+1}) = ({}^{\omega}\omega \setminus A) \cup \Big(\big(\bigcup_{x \in M_k} \mathbf{S}_x\big) \setminus U_{k+1} \Big) =$$
$$({}^{\omega}\omega \setminus A) \cup (A \setminus U_{k+1}) = {}^{\omega}\omega \setminus (A \cap U_{k+1}) = {}^{\omega}\omega \setminus \bigcap_{i \leqslant k+1} U_i. \quad \square$$

8. Main results

In this section we prove theorems that allow to construct π-trees for subspaces of a space that already has a π-tree. Recall that $\mathbf{S}$ is the standard foliage tree of ${}^{\omega}\omega$, see Notation 12.

Theorem 41. *Suppose that $\mathbf{S}$ is a π-tree on a space $({}^{\omega}\omega, \tau)$. Let $Y = \bigcap_{n\in\omega} U_n$, where each U_n is an open π-dense*[2] *subset of the Baire space $({}^{\omega}\omega, \tau_{\mathcal{N}})$. Then Y as a subspace of $({}^{\omega}\omega, \tau)$ has a π-tree.*

Using Lemma 13, we can apply this theorem not only to a space of the form $({}^{\omega}\omega, \tau)$, but to an arbitrary space with a π-tree.

Question 42. Does Theorem 41 remain true if we replace "π-dense" by "dense"?

Proof of Theorem 41. Let $\rho(\tau)$ and $\rho(\tau_{\mathcal{N}})$ be topologies on Y inherited from τ and $\tau_{\mathcal{N}}$ respectively. Theorem 37 asserts that there is a Baire foliage tree $\mathbf{H}$ on $\big(Y, \rho(\tau_{\mathcal{N}})\big)$ that shoots into $\mathbf{S}$. Then $\mathbf{H}$ is a Baire foliage tree on $\big(Y, \rho(\tau)\big)$ because $\rho(\tau) \supseteq \rho(\tau_{\mathcal{N}})$ by (b) Lemma 13, and $\mathbf{H}$ grows into $\big(Y, \rho(\tau)\big)$ by Lemma 32 because $\mathsf{flesh}\,\mathbf{H} = Y$. Therefore $\mathbf{H}$ is a π-tree on $\big(Y, \rho(\tau)\big)$. $\square$

Remark 43. The construction of a π-tree in the proof of Theorem 41 does not depend on topology τ.

Theorem 44. *Suppose that a space X has a π-tree and $Y = X \setminus F$, where F is a σ-compact subset of X. Then Y also has a π-tree.*

Corollary 45. *Suppose that a space X has a π-tree and $Y = X \setminus C$, where C is at most countable. Then Y also has a π-tree.* $\square$

Proof of Theorem 44. Let $F = \bigcup_{n\in\omega} K_n$, where each K_n is a compact subset of X. By (d) of Lemma 13, there is a homeomorphism f from X onto a space $({}^{\omega}\omega, \tau)$ such that $\mathbf{S}$ is a π-tree on $({}^{\omega}\omega, \tau)$. Also it follows from (b) of Lemma 13 that each $f(K_n)$ is a compact subset of the Baire space $({}^{\omega}\omega, \tau_{\mathcal{N}})$. Then every $U_n := {}^{\omega}\omega \setminus f(K_n)$ is an open π-dense subset of the Baire space by Remark 36, so the subspace ${}^{\omega}\omega \setminus \bigcup_{n\in\omega} f(K_n) = \bigcap_{n\in\omega} U_n$ of $({}^{\omega}\omega, \tau)$ has a π-tree by Theorem 41. $\square$

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[2] See Definition 35.