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On sequential separability of functional spaces

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ABSTRACT

In this paper, we give necessary and sufficient conditions for the space $B_1(X)$ of first Baire class functions on a Tychonoff space X, with pointwise topology, to be (strongly) sequentially separable. Also we claim that there are spaces X such that $B_1(X)$ is not sequentially separable space, but $C_p(X)$ is sequentially separable (the Sorgenfrey line, the Niemytzki plane).

1. Introduction

In [2,3] were given necessary and sufficient conditions for the space $C_p(X)$ of continuous real-valued functions on a space X, with pointwise topology, to be sequentially separable. Also in [3] was given necessary and sufficient condition for the space $C_p(X)$ to be strongly sequentially separable.

In this paper, we give necessary and sufficient conditions for the space $B_1(X)$ of first Baire class functions on a space X, with pointwise topology, to be sequentially separable and strongly sequentially separable.

2. Main definitions and notation

Throughout this article all topological spaces are considered Tychonoff. As usually, we will be denoted by $C_p(X)$ ($B_1(X)$) a set of all real-valued continuous functions $C(X)$ (a set of all first Baire class functions

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$B_1(X)$ i.e., pointwise limits of continuous functions) defined on X provided with the pointwise convergence topology. If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. If D is a countable sequentially dense subset of X then X call sequentially separable space.

Call X strongly sequentially separable if X is separable and every countable dense subset of X is sequentially dense.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set. If a set $Z = \bigcup_i Z_i$ where Z_i is a zero-set of X for any $i \in \omega$ then Z is called Z_σ-set of X. Note that if a space X is a perfect normal space, then class of Z_σ-sets of X coincides with class of F_σ-sets of X.

It is well known [5], that $f \in B_1(X)$ if and only if $f^{-1}(G)$ — Z_σ-set for any open set G of real line $\mathbb{R}$.

Further we use the following theorems.

Theorem 2.1. *([2]). A space $C_p(X)$ is sequentially separable if and only if there exist a condensation (one-to-one continuous map) $f : X \mapsto Y$ from a space X on a separable metric space Y, such that $f(U)$ — F_σ-set of Y for any cozero-set U of X.*

Theorem 2.2. *([2]). A space $B_1(X)$ is sequentially separable for any separable metric space X.*

Note that proof of this theorem gives more, namely there exists a countable subset $S \subset C(X)$, such that $[S]_{seq} = B_1(X)$.

3. Sequentially separable of $B_1(X)$

The main result of this paper is a next theorem.

Theorem 3.1. *A space $B_1(X)$ is sequentially separable if and only if there exists a bijection $\varphi : X \mapsto Y$ from a space X onto a separable metrizable space Y, such that*

1. *$\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y;*
2. *$\varphi(T)$ — F_σ-set of Y for any zero-set T of X.*

Proof. (1) ⇒ (2). Let $B_1(X)$ be a sequentially separable space, and S be a countable sequentially dense subset of $B_1(X)$. Consider a topology τ generated by the family $\mathcal{P} = \{f^{-1}(G) : G$ is an open set of $\mathbb{R}$ and $f \in S\}$. A space $Y = (X, \tau)$ is a separable metrizable space because S is a countable dense subset of $B_1(X)$. Note that a function $f \in S$, considered as map from Y to $\mathbb{R}$, is a continuous function. Let φ be the identity map from X on Y.

We claim that $\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y. Note that class of Z_σ-sets is closed under a countable unions and finite intersections of its elements. It follows that it is sufficient to prove for any $P \in \mathcal{P}$. But $\varphi^{-1}(P)$ — Z_σ-set for any $P \in \mathcal{P}$ because $f \in S \subset B_1(X)$.

Let T be a zero-set of X and h be a characteristic function of T. Since T is a zero-set of X, $h \in B_1(X)$. There are $\{f_n\}_{n\in\omega} \subset S$ such that $\{f_n\}_{n\in\omega}$ converges to h. Since $S \subset C_p(Y)$, $h \in B_1(Y)$ and, hence, $h^{-1}(\frac{1}{2}, \frac{3}{2}) = T$ is a Z_σ-set of Y.

(2) ⇒ (1). Let φ be a bijection from X on Y satisfying the conditions of theorem. Then $h = f \circ \varphi \in B_1(X)$ for any $f \in C(Y)$ ($h^{-1}(G) = \varphi^{-1}(f^{-1}(G))$ — Z_σ-set of X for any open set G of $\mathbb{R}$). Moreover $g = f \circ \varphi^{-1} \in B_1(Y)$ for any $f \in B_1(X)$ because of $\varphi(Z)$ is a Z_σ-set of Y for any a Z_σ-set Z of X. Define a map $F : B_1(X) \mapsto B_1(Y)$ by $F(f) = f \circ \varphi^{-1}$. Since φ is a bijection, $C(Y)$ embeds in $F(B_1(X))$ i.e., $C(Y) \subset F(B_1(X))$. By Theorem 2.2, each subspace D such that $C(Y) \subset D \subset B_1(Y)$ is sequentially separable. Thus $B_1(X)$ (homeomorphic to $F(B_1(X))$) is sequentially separable. □

Corollary 3.2. *A space $B_1(X)$ is sequentially separable for any regular space X with a countable network.*

Proof. Let X be a regular space X with a countable network. Then there are a countable network $\{F_n : n \in \omega\}$ of X where F_n is closed subset of X for $n \in \omega$. Consider a topology τ on X generated by the family $\mathcal{P} = \{F_n, X \setminus F_n : n \in \omega\}$. Let $Z = (X, \tau)$. The space Z is zero-dimensional space with countable base and, hence, it is metrizable space. Let $h : Z \mapsto X$ be the identity map. Note that any element of $\mathcal{P}$ is a F_σ-set of X. Hence $h(U)$ is a F_σ-set of X for any open set U of Z. Since h is a condensation, $h^{-1}(T)$ is a zero-set of Z for any zero-set T of X. Consider $\varphi = h^{-1} : X \mapsto Z$. Then φ satisfies all the conditions of the Theorem 3.1. □

In [2], Velichko was proved that identity maps of Sorgenfrey line $\mathcal{S}$ onto $\mathbb{R}$ and of Niemytzki Plane $\mathcal{N}$ onto the closed upper half-plane $\mathbb{R}^2_+$ satisfies the condition of the Theorem 2.1. Hence $C_p(\mathcal{S})$ and $C_p(\mathcal{N})$ are sequentially separable spaces.

We claim that $B_1(\mathcal{S})$ and $B_1(\mathcal{N})$ are not sequentially separable spaces.

We recall some concepts and facts related to the space of first Baire class functions.

- A map $f : X \mapsto Y$ be called Z_σ-map, if $f^{-1}(Z)$ is a Z_σ-set of X for any zero-set Z of Y. Note that a continuous map is a Z_σ-map.
- A space X be called an analytic space if there is a continuous map φ from $\mathbb{P}$ (space of irrational numbers) onto X.
- A space X be called an K-analytic space if there are Čech-complete Lindelëof space Y and a continuous map f from Y onto X.

An analytic space or a compact space are K-analytic space [5].

If in definition of K-analytic space the continuous map is replace by Z_σ-map we get a wider class of spaces — class of K_σ-analytic spaces [7].

Note that if X is a K_σ-analytic space then X^n is Lindelëof space for each $n \in \omega$ [7].

Example 3.3. There is a space X such that $C_p(X)$ is sequentially separable space but $B_1(X)$ is not sequentially separable.

Proof. We claim that $B_1(\mathcal{S})$ is not sequentially separable space.

Consider $B_1(\mathcal{S})$. Let $p : \mathcal{S} \mapsto \mathbb{R}$ be the identity map. Then $p(T)$ is a F_σ-set of $\mathbb{R}$ for any cozero-set T of S [2].

Suppose that $B_1(\mathcal{S})$ is sequentially separable space. By Theorem 3.1, there are separable metrizable space Y and a bijection $\varphi : \mathcal{S} \mapsto Y$ from $\mathcal{S}$ onto Y such that $\varphi(T)$ is a F_σ-set of Y for any closed subset T of $\mathcal{S}$, and $\varphi^{-1}(U)$ is a F_σ-set of $\mathcal{S}$ for any open subset U of Y. Consider the map $\varphi \circ p^{-1} : \mathbb{R} \mapsto Y$. Since $\varphi \circ p^{-1}$ is a Borel function, the space Y is an analytic separable metrizable space [5]. Thus the map $\varphi^{-1} : Y \mapsto \mathcal{S}$ is a Z_σ-bijection of analytic separable metrizable space Y. Then $\mathcal{S}$ is a K_σ-analytic space and, hence, $\mathcal{S}^2$ is a Lindelöf space, a contradiction. □

The same proof remains valid for the space $B_1(\mathcal{N})$.

4. Strongly sequentially separable of $B_1(X)$

From [6], we note that $B_1(X)$ is separable if and only if X has a coarser second countable topology.

In [3], Gartside, Lo and Marsh were given necessary and sufficient conditions for spaces $C_p(X)$ to be strongly sequentially separable. The following definition and theorems are relevant. For a proof of Theorem 4.2 see [4], Theorem 4.3 see [3], and more information on the property γ see [4].

Definition 4.1. A family α of subsets of X is called an ω-cover of X if for every finite $F \subset X$ there is a $U \in \alpha$ such that $F \subset U$.

Theorem 4.2. *([4]). The following are equivalent:*

1. *$C_p(X)$ is Frechet–Urysohn;*
2. *X has the property γ: for any open ω-cover α of X there is a sequence $\beta \subset \alpha$ such that* $\liminf \beta = X$.

Theorem 4.3. *([3]). The space $C_p(X)$ is strongly sequentially separable if and only if X has a coarser second countable topology, and every coarser second countable topology for X has the property γ.*

We give necessary and sufficient conditions for the space $B_1(X)$ to be strongly sequentially separable.

Theorem 4.4. *The function space $B_1(X)$ is strongly sequentially separable if and only if X has a coarser second countable topology, and for any bijection φ from a space X onto a separable metrizable space Y, such that $\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y, the space Y has the property γ.*

Proof. ($\Rightarrow$). Assume that $B_1(X)$ is strongly sequentially separable. Let φ be a bijection from a space X onto a separable metrizable space Y, such that $\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y. Then for any $f \in C(Y)$, $h = f \circ \varphi \in B_1(X)$. Since φ is a bijection, a map $F : C(Y) \mapsto B_1(X)$ (define as $F(f) = f \circ \varphi$) is an embedding in $B_1(X)$ i.e., $F(C_p(Y)) \subset B_1(X)$. Note that $F(C_p(Y))$ is dense separable subset of $B_1(X)$ and, hence, $C_p(Y)$ is strongly sequentially separable. By Theorem 4.3, the space Y has the property γ.

($\Leftarrow$). Assume that X has a coarser second countable topology. Then $B_1(X)$ is separable (Theorem 1 in [6]).

Let A be a countable dense subset of $B_1(X)$. We wish to show that for any $f \in B_1(X)$ there is some sequence $\{f_i : i \in \omega\} \subset A$ such that f is the limit of the sequence.

Consider a topology τ generated by the family

$\mathcal{P} = \{g^{-1}(G) : G$ is an open set of $\mathbb{R}$ and $g \in A \bigcup \{f\}\}$. A space $Y = (X, \tau)$ is a separable metrizable space because A is a countable dense subset of $B_1(X)$. Note that a function $g \in A \bigcup \{f\}$, considered as map from Y to $\mathbb{R}$, is a continuous function. Let φ be the identity map from X on Y.

We claim that $\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y. Note that class of Z_σ-sets is closed under a countable unions and finite intersections of its elements. It follows that it is sufficient to prove for any $P \in \mathcal{P}$. But $\varphi^{-1}(P)$ — Z_σ-set for any $P \in \mathcal{P}$ because $g \in A \bigcup \{f\} \subset B_1(X)$. It follows that the space Y has the property γ and, hence, $C_p(Y)$ is strongly sequentially separable. Since A is countable dense subset of $B_1(X)$, the set A (where a map in A considered as map from Y to $\mathbb{R}$) is a countable dense subset of $C_p(Y)$. Hence there is some sequence $\{f_i : i \in \omega\} \subset A$ such that f is the limit of the sequence. □

It is consistent and independent for arbitrary X, that $C_p(X)$ is strongly sequentially separable if and only if $C_p(X)$ is Frechet–Urysohn.

Note that the Theorem 4.2 implies the following proposition (Corollary 17 in [3]).

Proposition 4.5. *(Cons (ZFC)) The following are equivalent:*

1. *$C_p(X)$ is strongly sequentially separable;*
2. *X is countable;*
3. *$C_p(X)$ is separable and Frechet–Urysohn.*

For a space $B_1(X)$ we have a following

Corollary 4.6. *(Cons (ZFC)) The following are equivalent:*

1. $B_1(X)$ *is strongly sequentially separable;*
2. X *is countable;*
3. $B_1(X)$ *is separable and Frechet–Urysohn.*

Recall that the cardinal $\mathfrak{p}$ is the smallest cardinal so that there is a collection of $\mathfrak{p}$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$. (See [1] for more on small cardinals including $\mathfrak{p}$.)

Example 4.7. Assume that $\omega_1 < \mathfrak{p}$. There is uncountable space X such that $B_1(X)$ is strongly sequentially separable but not Frechet–Urysohn.

Proof. Let X be ω_1 with the discrete topology. Clearly, that X is separable submetrizable space and so $B_1(X)$ is a separable, dense subspace of $\mathbb{R}^X$. We know (from Theorem 11 in [3]) that $\mathbb{R}^X$ must be strongly sequentially separable space and so $B_1(X)$ is strongly sequentially separable. Note that $\mathbb{R}^{\omega_1}$ is not Frechet–Urysohn. □

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