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A difference scheme for multidimensional transfer equations with time delay

Svyatoslav I. Solodushkin [a,b,*], Irina F. Yumanova [a], Rob H. De Staelen [c]

[a] Institute of Mathematics and Computer Science, Ural Federal University, Russia
[b] Institute of Mathematics and Mechanics, Ural branch of the RAS, Russia
[c] Department of Mathematical Analysis, Ghent University, Belgium

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ABSTRACT

This paper continues research initiated in Solodushkin et al. (2015). We develop a finite difference scheme for a first order multidimensional partial differential equation including a time delay. This class of equations is used to model different time lapse phenomena, e.g. birds migration, proliferation of viruses or bacteria and transfer of nuclear particles. For the constructed difference schemes the order of approximation, stability and convergence order are substantiated. To conclude we support the obtained results with some test examples.

1. Introduction

First order partial differential equations with time delay – also known as advection equations – with distributed parameters arise for example in the modeling of birds migration, viral or bacterial growth and transfer of nuclear particles [1,2]. When diffusion is more dominant, such as in elasto-plasticity and in the theory of reactive contaminant transport, also time delay can also occur and can be modeled through a convolution term, see e.g. [3,4].

The qualitative theory of partial functional differential equations (PFDE) is developed quite well (see, for example, [5,6] and references therein). Papers which deal with an advection equation with time delay as well as with the retardation of a state variable usually consider questions of existence, uniqueness and global stability; this type of equations have numerous applications in cell dynamics [7–9]. In short, the equation is rewritten as a linear evolution problem in a Banach space and results are formulated in terms of a strongly continuous semigroup of bounded linear operators. Specific systems were analyzed numerically in [10–12], nevertheless numerical algorithms for the equations in general form were not constructed and theorems establishing the convergence were not yet formulated.

Since in most cases one cannot find the explicit (analytical) solution of PFDEs; the elaboration, substantiation, and program realization of numerical methods for these equations are of essential interest. Below we review some approaches to numerically solve such equations, see also [13].

Method of lines [14–16] reduces PFDEs to a system of differential equations with time delay in ordinal derivatives which could be solved by special methods [17–20]; unfortunately after discretization with respect to state variables a stiff system

* Corresponding author at: Institute of Mathematics and Computer Science, Ural Federal University, Russia.
E-mail addresses: solodushkin_s@mail.ru (S.I. Solodushkin), yuirfa@mail.ru (I.F. Yumanova), rob.destaelen@ugent.be (R.H. De Staelen).

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appears.[1] Implicit difference methods for first order PFDEs [21–23] allow to avoid this stiffness by an appropriate choice of the step. However to obtain a solution on each next time layer one must solve high-dimensional nonlinear systems.

To combine the good properties of both approaches (to avoid the stiffness and to get away from the necessity of solving the large-dimensional nonlinear algebraic system) especially effective difference schemes for PFDE of parabolic, hyperbolic and advection type were elaborated in [24–28]. The main idea in these works is a separation principle that consists of distinguishing finite and infinite dimensional components in the structure of PFDE. To take into account the time delay effect, interpolation and extrapolation of discrete prehistory is used. This extrapolation also is needed for the realization of implicit methods and allows the authors to avoid the necessity of solving nonlinear systems; let us explain some more how this is achieved.

For example for the PFDE of parabolic type $\partial_x u(x,t) = a^2\partial_x^2 u(x,t) + f(x,t,u(x,t),u(x,t-\tau))$ authors consider a two-layer difference scheme. To replace a derivative with respect to space by finite differences they use a linear combination of the second order approximations on both time layers: on the current time layer, j, where the approximate solution u^j is already known, and on the next layer, $j+1$, where it should be found. This replacement introduces into the difference scheme the linear terms only.

Note, that the nonlinearity in the difference scheme could appear because of the nonlinear functional f. To preserve a high order of convergence authors involve the approximate value of the function u^{j+1} on the next time layer, when they calculate the value of the functional f. This could lead to the nonlinearity of the difference scheme on the whole, but they, instead of using the value of u^{j+1}, use the result of an extrapolation by two points. Namely, u^{j-1} and u^j for the extrapolation and substitute the extrapolated value into the functional f. Thus the scheme is linear with respect to u^{j+1}.

Our approach is close to [25,28] and is based on a combination of the stability verification methods for two-layer difference schemes [29] and the separation principle mentioned above. The present paper continues the investigation initiated in [30,13] and consists of a multidimensional extension. We first fix some notations and introduce the mathematical problem.

Let $\overline{G} = \prod_{\alpha=1}^{p}[0, X_\alpha]$ be a p-dimensional bar with boundary Γ. We want to find a sufficiently smooth function $u(x,t)$ which satisfies the advection equation with aftereffect

$$\frac{\partial u}{\partial t} + a\sum_{\alpha=1}^{p}\frac{\partial u}{\partial x_\alpha} = f(x,t,u(x,t),u_t(x,\cdot)), \tag{1a}$$

in the cylinder $\overline{G}\times[t_0,\theta]$. Here $x = (x_1,\ldots,x_p)$, $x_\alpha \in [0, X_\alpha]$ are state variables and $t\in[t_0,\theta]$ represents time; $u(x,t)$ is an unknown function; $u_t(x,\cdot) = \{u(x,t+\xi),\ -\tau \le \xi < 0\}$ is a prehistory-function of the unknown function up to the moment t, and $\tau > 0$ is a value of time delay, $a > 0$ is a constant.

Together with the advection equation we have the following initial condition

$$u(x,t) = \varphi(x,t), \quad x\in\overline{G},\ t\in[t_0-\tau, t_0], \tag{1b}$$

and the boundary condition

$$u(x,t) = g(x,t), \quad x\in\Gamma,\ t\in[t_0,\theta]. \tag{1c}$$

Questions of the existence and uniqueness of a solution to the stated initial–boundary value problem (1) were considered in [5] and we assume that the functional f and functions φ and g are such that problem has a unique solution.

We denote by $\mathcal{Q} = \mathcal{Q}[-\tau,0)$ the set of functions $v(\xi)$ that are piecewise continuous on $[-\tau,0)$ with a finite number of points of discontinuity of the first kind and right continuous at the points of discontinuity. We define a norm on $\mathcal{Q}$ by $\|v\|_{\mathcal{Q}} = \sup_{\xi\in[-\tau,0)}|v(\xi)|$. We additionally assume that the functional $f(x,t,u,v(\cdot))$ is given on $\overline{G}\times[t_0,\theta]\times\mathbb{R}\times\mathcal{Q}$ and is Lipschitz in the last two arguments:

$$\exists L_f\in\mathbb{R}\ \forall x\in\overline{G},\ t\in[t_0;\theta],\ u^1\in\mathbb{R},\ u^2\in\mathbb{R},\ v^1\in\mathcal{Q},\ v^2\in\mathcal{Q}:$$
$$\left|f(x,t,u^1,v^1(\cdot)) - f(x,t,u^2,v^2(\cdot))\right| \le L_f\Big(|u^1-u^2| + \|v^1(\cdot)-v^2(\cdot)\|_{\mathcal{Q}}\Big).$$

2. The difference scheme

We consider an equidistant partition of $[0, X_\alpha]$ into parts with step size $h_\alpha = X_\alpha/N_\alpha$. On the set $\overline{G}$ we introduce a rectangular grid $\overline{\omega}_h = \{(i_1h_1,\ldots,i_ph_p),\ i_\alpha = 0,1,\ldots,N_\alpha\}$ which is uniform with respect to each direction. For brevity we denote $x_{(i)} = (x_{i_1},\ldots,x_{i_p}) = (i_1h_1,\ldots,i_ph_p)$, $i_\alpha = 0,1,\ldots,N_\alpha$, where the index i is a p-dimensional vector. Note that $x_\alpha\in\mathbb{R}$ is an α-coordinate of the vector x and $x_{(i)}\in\mathbb{R}^p$ is a particular node of the grid $\overline{\omega}_h$. We also split the time interval

[1] Here we follow C.F. Curtiss and J.O. Hirschfelder who were one of the first that attempted to give a definition of stiff systems in 1952, and whose concept of stiffness is the most useful form the pragmatic point of view. They proposed the following interpretation: stiff equations are equations where certain implicit methods perform better, than using classical explicit ones like Euler or Adams methods.

$[t_0, \theta]$ into M parts with step size Δ and define the grid $\overline{\omega}_\Delta = \{t_j = t_0 + j\Delta,\ j = 0, 1, \dots, M\}$. Without loss of generality and to simplify the narration we assume that the value $\tau/\Delta = m$ is a natural number. In order to replace the differential equation (1a) for a finite difference equation we define a grid on the cylinder $\overline{G} \times [t_0, \theta]$ and consider the inner product $\overline{\omega}_{h\Delta} = \overline{\omega}_h \times \overline{\omega}_\Delta = \{(x_i, t_j),\ x_i \in \overline{\omega}_h,\ t_j \in \overline{\omega}_\Delta\}$.

Denote by u_j^i the approximation of the function value $u(x_{(i)}, t_j)$, $i = (i_1, \dots, i_p)$, $i_\alpha = 0, 1, \dots, N_\alpha$, $j = 0, \dots, M$, at the respective node.

Since functional $f(x_{(i)}, t_j, u(x_{(i)}, t_j), u_{t_j}(x_{(i)}, \cdot))$ may depend on values of the function u between grid nodes one needs to interpolate. For every fixed t_j and time delay $\xi \in [-\tau, 0)$ there are only two possibilities: if $t_j + \xi \le t_0$, interpolation is not needed, we use the initial condition, $u(x_{(i)}, t_j + \xi) = \varphi(x_{(i)}, t_j + \xi)$, otherwise we use the interpolation as described below.

Definition 1. For every fixed node $(x_{(i)}, t_j)$ we introduce its *discrete prehistory* as

$$\{u_l\}_j^i = \{u_l \mid \max\{0, j - m\} \le l \le j\}.$$

A mapping I defined on the set $\mathcal{A}_j^i$ of all admissible discrete prehistories and acting by the rule

$$I : \mathcal{A}_j^i \to \mathcal{Q}[-\min\{\tau, t_j\}, 0] : \left\{u_l^i\right\}_j^i \mapsto v^{i,j}(\cdot) = v^{i,j}(t_j + \xi)$$

is called an *interpolation operator* for the discrete history.

Let us give an example of a concrete interpolation operator, which has the properties required for the numerical method that we are going to construct. For the discrete prehistory $\{u_l\}_j^i$ we define

$$v^{i,j}(t_j + \xi) = \frac{1}{\Delta}\Big((t_{l+1} - t_j - \xi)\, u_l^i + (t_j + \xi - t_l)\, u_{l+1}^i\Big), \quad t_l \le t_j + \xi \le t_{l+1}. \tag{2}$$

We say an interpolation operator has order of error q on the exact solution, if there exist constants C_1 and C_2 such that, for all $i = (i_1, \dots, i_p)$, $i_\alpha = 0, 1, \dots, N_\alpha$, $j = 1, \dots, M$, and $t \in [\max\{0, t_j - \tau\}, t_j]$ the following inequality holds:

$$\left|v^{i,j}(t) - u(x_{(i)}, t)\right| \le C_1 \max_{\max\{0, j-m\} \le l \le j} \left|u_l^i - u(x_{(i)}, t_l)\right| + C_2 \Delta^q.$$

For example, the operator of interpolation (2) is of second order.

Let us consider difference operators which approximate first derivatives with respect to state variables

$$\Omega_\alpha u_j^i = \begin{cases} \dfrac{-4u_j^{i[-1_\alpha]} - \frac{2h_\alpha}{a}\left(f_j^{i[-1_\alpha]} - \sum\limits_{\beta=1,\beta\ne\alpha}^{p} \frac{\partial g(x_{(i)[-1_\alpha]}, t_j)}{\partial x_\beta} - \frac{\partial g(x_{(i)[-1_\alpha]}, t_j)}{\partial t}\right) + 4u_j^i}{2h_\alpha}, & i_\alpha = 1, \\[2ex] \dfrac{u_j^{i[-2_\alpha]} - 4u_j^{i[-1_\alpha]} + 3u_j^i}{2h_\alpha}, & i_\alpha \ge 2; \end{cases}$$

here $u_j^{i[-1_\alpha]}$ is a grid approximation of the function $u(x, t)$ which is evaluated in the node $(x_{(i)[-1_\alpha]}, t_j)$. We use the notation $x_{(i)[-1_\alpha]} = (i_1 h_1, \dots, i_{\alpha-1} h_{\alpha-1}, (i_\alpha - 1)h_\alpha, i_{\alpha+1} h_{\alpha+1}, \dots, i_p h_p)$, that means $x_{(i)[-1_\alpha]}$ is a left neighbor of the node $x_{(i)}$ which is shifted by one step h_α in a corresponding coordinate.

We consider the following family of difference schemes (parametrized by $0 \le s \le 1$), with $j = 0, \dots, M - 1$, and $i = (i_1, \dots, i_p)$, $i_\alpha = 1, \dots, N_\alpha$:

$$\frac{u_{j+1}^i - u_j^i}{\Delta} + a \sum_{\alpha=1}^{p} \left(s\, \Omega_\alpha u_{j+1}^i + (1 - s)\, \Omega_\alpha u_j^i\right) = f_j^i, \tag{3a}$$

with the initial condition

$$u_0^i = \varphi(x_{(i)}, t_0), \qquad v^{i,0}(t) = \varphi(x_{(i)}, t), \quad t < t_0, \text{ for all possible } i, \tag{3b}$$

and boundary condition

$$u_j^i = g(x_{(i)}, t_j), \quad x_{(i)} \in \Gamma,\ j = 0, \dots, M. \tag{3c}$$

Here $f_j^i = f\left(x_{(i)}, t_j, u_j^i, v^{i,j}(\cdot)\right)$ is the value of the functional f, calculated on an approximate solution and $v^{i,j}(\cdot)$ is the result of an interpolation. Namely, we use a piecewise linear interpolation (2). For constructing a numerical method, we additionally assume that $g(x, t)$ is a differentiable function.

2.1. Context and origin of the finite-difference scheme

Let us put the proposed scheme (3) in the context of the existing finite-difference schemes for hPDEs.

So-called FTCS (Forward-Time, Central-Space) method uses the following approximation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} \xrightarrow{\text{FTCS}} \frac{u^i_{j+1} - u^i_j}{\Delta} + a\frac{u^{i+1}_j - u^{i-1}_j}{2h}$$

and therefore has a first-order in time and second-order in space. This scheme is unconditionally unstable for advection equations unless artificial viscosity is included, therefore it is not studied here.

The solution of the FTCS scheme stability problem was proposed by Lax. The main idea is based on replacing in the FTCS formula the term u^i_j with its spatial average $(u^{i+1}_j + u^{i-1}_j)/2$, this guarantees the stability if the Courant condition $c \le h/\Delta$ is fulfilled [31]. The Lax scheme approximates the equation as $O(h^2 + h^2/\Delta + \Delta)$ and therefore is inconsistent. Because of the inconsistency and conditional stability, h and Δ cannot be independent. The more sophisticated Lax–Wendroff method, which could be considered as a multistep method, leads to the accuracy $O(h^2 + \Delta^2)$ and is stable under the same Courant condition. Both of these methods are widely used to solve initial problems when the initial condition $u(x, 0) = u_0,\ x \ge 0$, is defined on the semi-axis, but they are not suitable when the initial condition is defined only on the segment $[0, X]$ coupled with the boundary condition defined on the segment $[0, T]$. This is the main reason why we do not try to generalize this method in the case of hereditary systems.

Widely-known first-order upwind schemes are the particular cases of a running scheme family which is circumstantially studied in [32]. Second-order upwind schemes improve the spatial accuracy of the first-order upwind scheme by including three data points instead of just two and was the basis of the elaborated method (3). Unfortunately these schemes are not directly applicable, they must be modified near the boundary without loss of accuracy. This modification is a feature of our method.

For the advection equation with time delay grid methods were built in [28], the approach is very close to that we use in this paper. These methods are analogs of running scheme families, analogs of the Crank–Nicolson scheme and an approximation method to the middle of the square.

To conclude this subsection let us explain the way in which we have obtained the scheme. The derivative $\partial u/\partial t$ in (1a) is approximated by a finite difference over two nodes. For nodes $(x_{(i)}, t_j)$ the derivative $\partial u/\partial x_\alpha$ is approximated by a finite difference over three nodes on the right edge while $i_\alpha \ge 2$. For $i_\alpha = 1$ this approximation requires to calculate value of grid function in the node $x_{(i)[-2_\alpha]}$ beyond left boundary of the grid $\overline{\omega}_h$. So, for $i_\alpha = 1$ we apply the approximation over three nodes with the double node $x_{(i)[-1_\alpha]}$:

$$\frac{\partial u^i_j}{\partial x_\alpha} \approx \frac{1}{2h_\alpha}\left(-4u^{i[-1_\alpha]}_j - 2h\frac{\partial u^{i[-1_\alpha]}_j}{\partial x_\alpha} + 4u^i_j\right).$$

Because of (1a) we have

$$\frac{\partial u^{i[-1_\alpha]}_j}{\partial x_\alpha} = \frac{1}{a}\left(f^{i[-1_\alpha]}_j - \sum_{\beta=1,\beta\ne\alpha}^{p}\frac{\partial g(x_{(i)}, t_j)}{\partial x_\beta} - \frac{\partial g(x_{(i)}, t_j)}{\partial t}\right).$$

2.2. The residual of the finite-difference scheme

We call the mesh function

$$\Psi^i_j = \frac{u(x_{(i)}, t_{j+1}) - u(x_{(i)}, t_j)}{\Delta} + a\sum_{\alpha=1}^{p}\left(s\,\Omega_\alpha u(x_{(i)}, t_{j+1}) + (1-s)\,\Omega_\alpha u(x_{(i)}, t_j)\right) - \bar{f}^i_j, \tag{4}$$

the residual of method (3). Here $\bar{f}^i_j = f(x_{(i)}, t_j, u(x_{(i)}, t_j), u_{t_j}(x_{(i)}, \cdot))$ is the value of the functional f calculated on the exact solution. We stress that the residual is calculated on the exact solution. We will obtain and prove its order now.

Theorem 1. *Let the exact solution $u(x, t)$ of problem (1) have continuous derivatives with respect to state variables x_α up to third order, continuous derivatives with respect to time t up to second order and all first derivatives of the solution with respect to x_α are continuously differentiable in t. Then the residual of method (3) has order $\sum_{\alpha=1}^{p} h^2_\alpha + \Delta$.*

Proof. Let us expand the function $u(x, t)$ in a Taylor series in a neighborhood of the points $(x_{(i)}, t_j)$ and $(x_{(i)}, t_{j+1})$, $i = 2, \dots, N$. We obtain the following equalities for the values of the function at these points:

$$u(x_{i[-1_\alpha]}, t_j) = u(x_{(i)}, t_j) - h_\alpha\frac{\partial u}{\partial x_\alpha}(x_{(i)}, t_j) + \frac{h^2_\alpha}{2}\frac{\partial^2 u}{\partial x^2_\alpha}u(x_{(i)}, t_j) + O(h^3_\alpha),$$

$$u(x_{i[-2_\alpha]}, t_j) = u(x_{(i)}, t_j) - 2h_\alpha\frac{\partial u}{\partial x_\alpha}(x_{(i)}, t_j) + 2h^2_\alpha\frac{\partial^2 u}{\partial x^2_\alpha}u(x_{(i)}, t_j) + O(h^3_\alpha),$$

$$u(x_{i[-1_\alpha]}, t_{j+1}) = u(x_{(i)}, t_{j+1}) - h_\alpha\frac{\partial u}{\partial x_\alpha}(x_{(i)}, t_{j+1}) + \frac{h^2_\alpha}{2}\frac{\partial^2 u}{\partial x^2_\alpha}u(x_{(i)}, t_{j+1}) + O(h^3_\alpha),$$

$$u(x_{i[-2\alpha]}, t_{j+1}) = u(x_{(i)}, t_{j+1}) - 2h_\alpha \frac{\partial u}{\partial x_\alpha}(x_{(i)}, t_{j+1}) + 2h_\alpha^2 \frac{\partial^2 u}{\partial x_\alpha^2} u(x_{(i)}, t_{j+1}) + O(h_\alpha^3),$$

$$u(x_{(i)}, t_{j+1}) = u(x_{(i)}, t_j) + \frac{\partial u}{\partial t}(x_{(i)}, t_j)\Delta + O(\Delta^2).$$

Substituting these relations into (4) we obtain

$$\Psi_j^i = \frac{\partial u}{\partial t}(x_{(i)}, t_j) + O(\Delta) + a\sum_{\alpha=1}^{p}\left(s\Big(\frac{\partial u}{\partial x_\alpha}(x_{(i)}, t_{j+1}) + O(h^2)\Big) + (1-s)\Big(\frac{\partial u}{\partial x}(x_{(i)}, t_j) + O(h_\alpha^2)\Big)\right) - \bar{f}_j^i.$$

Now we expand the function $\frac{\partial u}{\partial x}(x, t)$ in a Taylor series in a neighborhood of the point $(x_{(i)}, t_{j+1})$, to obtain

$$\frac{\partial u}{\partial x}(x_{(i)}, t_{j+1}) = \frac{\partial u}{\partial x} u(x_{(i)}, t_j) + O(\Delta),$$

which yields

$$\Psi_j^i = \frac{\partial u}{\partial t}(x_{(i)}, t_j) + a\frac{\partial u}{\partial x}(x_{(i)}, t_j) + O\left(\sum_{\alpha=1}^{p} h_\alpha^2 + \Delta\right) - \bar{f}_j^i.$$

Invoking (1a) we arrive at $\Psi_j^i = O(\sum_{\alpha=1}^{p} h_\alpha^2 + \Delta)$. For $i = 1$ this theorem is proved in a similar way. □

Definition 2. Denote $\varepsilon_j^i = u(x_{(i)}, t_j) - u_j^i,\ i = (i_1, \dots, i_p),\ i_\alpha = 1, \dots, N_\alpha,\ j = 0, \dots, M$. We say that method (3) converges if $\varepsilon_j^i \to 0$ when $h_\alpha \to 0,\ \alpha = 1, \dots, p$, and $\Delta \to 0$ for all i and j. We say that it converges with order $\sum_{\alpha=1}^{p} h_\alpha^{q_1} + \Delta^{q_2}$, if there exists a constant C such that $\|\varepsilon_j^i\| \le C\left(\sum_{\alpha=1}^{p} h_\alpha^{q_1} + \Delta^{q_2}\right)$ for all i and j.

In the next section, we address the problem of convergence and stability. The fundamental theorem in the analysis of finite difference methods for the numerical solution of partial differential equations without time delay is the Lax equivalence theorem. This theorem [31] states that for a consistent finite difference method for a well-posed linear initial value problem, the method is convergent if and only if it is stable. In the case of equations with time delay one should deal with infinite-dimensional spaces, where it is difficult to build constructive and effective algorithms. If the difference scheme is finite-dimensional it must contain the delay term, therefore it is impossible to apply the Lax equivalence theorem directly. This problem was solved in [33] where the general difference scheme with aftereffect was elaborated. The principal modification was the introduction of an intermediate interpolation space.

In consideration of the nonlinear dependence of the functional f (and, consequently, F) on the state and its prehistory, the traditional methods of stability verification [29] are not applicable. However, to investigate the convergence of the schemes, as in the case of other evolutionary problems with delay effect, we can apply the technique of abstract schemes with aftereffect developed earlier [33] in the case of function-differential equations with ordinary derivatives. Below we describe the main points of this technique as applied to our case (see also [24]).

3. General difference scheme with aftereffect and its order of convergence

In this section, we reintroduce some of the notation used earlier, for example, τ and Δ. This is done deliberately for simplifying the embedding of the schemes from the previous section.

Let an interval $[t_0, \theta]$ be given, and let $\tau > 0$ be the value of the delay. Define the step of the grid $\Delta > 0$; to simplify the narration we assume that $\tau/\Delta = m$ and $(\theta - t_0)/\Delta = M$ are natural numbers. Denote by $\{\Delta\}$ the set of steps. A (uniform) grid is, by definition, a finite set of numbers $\Sigma_\Delta = \{t_i = t_0 + i\Delta,\ i = -m, \dots, M\}$. We use the notation $\Sigma_\Delta^- = \{t_i \in \Sigma_\Delta,\ i < 0\}$ and $\Sigma_\Delta^+ = \{t_j \in \Sigma_\Delta,\ i \ge 0\}$.

A discrete model is defined as a grid function $t_i \in \Sigma_\Delta \to y(t_i) = y_i \in Y,\ i = -m, \dots, M$, where Y is a q-dimensional normed space with norm $\|\cdot\|_Y$. We will assume that the dimension q of the space Y depends on a number $h > 0$. The set $\{y_i\}_n = \{y_i \in Y,\ i = n - m, \dots, n\}$ will be called the prehistory of the discrete model by the time t_n, $0 \le n \le M$. Let V be a linear normed space with norm $\|\cdot\|_V$, so-called interpolation space. A mapping $I:\ I(\{y_i\}_n) = v \in V$ is, by definition, an operator of the interpolation of the discrete prehistory.

We will say that the interpolation operator satisfies the Lipschitz condition if there exists a constant L_I such that, for all prehistories $\{y_i^1\}_n$ and $\{y_i^2\}_n$ of the discrete model, the following inequality holds:

$$\left\|I(\{y_i^1\}_n) - I(\{y_i^2\}_n)\right\|_V \le L_I \max_{n-m\le i\le n} \left\|y_n^1 - y_n^2\right\|_Y . \tag{5}$$

Starting values of the model are defined by the function acting from Σ_Δ^- to Y:

$$y(t_i) = y_i, \quad i = -m, \dots, 0. \tag{6}$$

The formula of advance of the model by a step is, by definition, the relation

$$y_{n+1} = Sy_n + \Delta\Phi(t_n, I(\{y_i\}_n), \Delta), \quad n = 0, \dots, M-1, \tag{7}$$

where $\Phi : \Sigma_\Delta^+ \times V \times \Delta \to Y$ is the function of advance by a step and the transition operator $S : Y \to Y$ is a linear operator.

Thus, a discrete model (in what follows, simply a method) is defined by starting values (6), formula of advance by a step (7), and an interpolation operator. We assume that the function $\Phi(t_n, v, \Delta)$ in (7) is Lipschitz with respect to the second argument; i.e., there exists a constant L_Φ such that, for all $t_n \in \Sigma_\Delta^+$, $\Delta \in \Delta$, and $v^1, v^2 \in V_n$ the following inequality holds:

$$\|\Phi(t_n, v^1, \Delta) - \Phi(t_n, v^2, \Delta)\|_Y \le L_\Phi \|v^1 - v^2\|_V.$$

The function of exact values is, by definition, the mapping $Z(t_i, \Delta) = z_i \in Y$, $i = -m, \dots, M$. We will say that starting values of the model have order $\Delta^{p_1} + h^{p_2}$ if there exists a constant C independent of z_i, y_i, Δ, and h such that

$$\|z_i - y_i\|_Y \le C(\Delta^{p_1} + h^{p_2}), \quad i = -m, \dots, 0.$$

We will say that the method converges with order $\Delta^{p_1} + h^{p_2}$ if there exists a constant C independent of z_i, y_i, Δ, and h such that for all $n = -m, \dots, M$, the following inequality holds:

$$\|z_n - y_n\|_Y \le C(\Delta^{p_1} + h^{p_2}).$$

In what follows, we will omit subscripts at norms. Method (7) is called stable if $\|S\| \le 1$. An error of approximation with interpolation (a residual) is, by definition, the grid function

$$d_n = (z_{n+1} - Sz_n)/\Delta - \Phi(t_n, I(\{z_i\}_n), \Delta), \quad n = 0, \dots, M-1. \tag{8}$$

We will say that method (7) has order of error of approximation with interpolation $\Delta^{p_1} + h^{p_2}$ if there exists a constant C independent of d_n, Δ, and h such that for all $n = 1, \dots, M$, the following inequality holds:

$$\|d_n\| \le C(\Delta^{p_1} + h^{p_2}).$$

Theorem 2. *Suppose that method (7) is stable, the function Φ satisfies the Lipschitz condition with respect to the second argument, the interpolation operator I satisfies the Lipschitz condition, the starting values have order $\Delta^{p_1} + h^{p_2}$, and the error of approximation with interpolation has order $\Delta^{p_3} + h^{p_4}$, where $p_1 > 0$, $p_2 > 0$, $p_3 > 0$ and $p_4 > 0$. Then, the method converges and the order of the convergence is at least $\Delta^{\min\{p_1,p_3\}} + h^{\min\{p_2,p_4\}}$.*

Proof. Let $\delta_n = z_n - y_n$ for $n = -m, \dots, M$, then, for $n = 0, \dots, M-1$ we have

$$\delta_{n+1} = S\delta_n + \Delta\hat{\delta}_n + \Delta d_n, \tag{9}$$

where $\hat{\delta}_n = \Phi(t_n, I(\{z_i\}_n), \Delta) - \Phi(t_n, I(\{y_i\}_n), \Delta)$. The assumption that the mappings Φ and I are Lipschitz implies that

$$\|\hat{\delta}_n\| \le K \max_{n-m \le i \le n} \{\|\delta_i\|\}, \tag{10}$$

where $K = L_I L_\Phi$. It follows from (9) that

$$\delta_{n+1} = S^{n+1}\delta_0 + \Delta \sum_{j=0}^{n} S^{n-j}\hat{\delta}_j + \Delta \sum_{j=0}^{n} S^{n-j} d_j. \tag{11}$$

From (10) and (11) together with the stability of S we have

$$\|\delta_{n+1}\| \le \|\delta_0\| + K\Delta \sum_{j=0}^{n} \max_{j-m \le i \le j} \{\|\delta_i\|\} + (\theta - t_0) \max_{0 \le i \le N-1} \|d_i\|. \tag{12}$$

Let us denote $R_0 = \max_{-m \le i \le 0}\{\|\delta_i\|\}$, $R = \max_{0 \le i \le N-1}\{\|d_i\|\}$, and $D = R_0 + (\theta - t_0)R$ so we can rewrite estimate (12) in the form

$$\|\delta_{n+1}\| \le K\Delta \sum_{j=0}^{n} \max_{j-m \le i \le j} \{\|\delta_i\|\} + D. \tag{13}$$

Suppose the following estimate

$$\|\delta_n\| \le D(1 + K\Delta)^n, \tag{14}$$

is valid for all $n = 1, \dots, M$. From this we obtain $\|\delta_{n+1}\| \le D\exp(K(\theta - t_0))$, which implies the conclusion of the theorem, as by definition of D, the inequality $D < C\left(\Delta^{\min\{p_1,p_3\}} + h^{\min\{p_2,p_4\}}\right)$ holds. It remains to prove (14) is valid. We proceed by induction.

Induction base. If we set $n = 0$ in (13), then $\|\delta_1\| \le K\Delta\|\delta_0\| + D \le (1 + K\Delta)D$.

Induction step. Let estimate (14) be valid for all indices from 1 to n. Let us show that the estimate is also valid for $n + 1$. Fix $j \le n$, and let $i_0 = i_0(j)$ be an index for which $\max_{j-m \le i \le j}\{\|\delta_i\|\}$ is attained. The following two situations are possible:

- $i_0 \le 0$, then, $\max_{j-m\le i\le j}\{\|\delta_i\|\} = \|\delta_{i_0}\| \le R_0 \le D(1+K\Delta)^j$;
- $1 \le i_0 \le j$, then, by the induction hypothesis

$$\max_{j-m\le i\le j}\{\|\delta_i\|\} = \|\delta_{i_0}\| \le D(1+K\Delta)^{i_0} \le D(1+K\Delta)^j.$$

Thus, the following estimate is valid in any case $\max_{j-m\le i\le j}\{\|\delta_i\|\} \le D(1+K\Delta)^j$. This estimate combined with (13) yields

$$\|\delta_{n+1}\| \le K\Delta \sum_{j=0}^{n} D(1+K\Delta)^j + D = D(1+K\Delta)^{n+1},$$

by which (14) is proved. □

4. Stability and convergence of the method

Due to the nonlinear dependence of the functional f on the state and its prehistory we apply the technique of abstract schemes with aftereffect developed earlier [33] in the case of function-differential equations with ordinary derivatives.

In this section we consider problems with the homogeneous boundary condition $u(0,t) = 0, t \in [t_0, \theta]$. The replacement $\tilde{u}(x,t) = u(x,t) - g(t)$ turns the initial problem into the mentioned one. We embed the schemes from family (3) into the general difference scheme with aftereffect [33]. The idea is close to [30,13] and is based on the method of dimension increase.

Let p be fixed. For every $t_j \in \overline{\omega}_\Delta$ we define the values of the discrete prehistory by the vector $\mathbf{y}_j$, for this we order components of u_j^i in the lexicographical order

$$\mathbf{y}_j = \left(u_j^{(1,\dots,1)}, u_j^{(2,\dots,1)}, \dots, u_j^{(N_1,\dots,1)}, \dots, u_j^{(N_1,\dots,N_p)}\right)^\top \in Y_p,$$

here $j = 0, \dots, M$, the sign $\cdot^\top$ means transposition, Y_p is a linear space, $\dim Y_p = N_1 \times N_2 \times \cdots \times N_p$. For example, when $p = 1$, we have $\mathbf{y}_j = \left(u_j^1, u_j^2, \dots, u_j^{N_1}\right)^\top$.

To build the difference operator $A : Y_p \to Y_p$ that correspond to $\sum_{\alpha=1}^{p} \Omega_\alpha$ we consider a sequence of matrices D_α, $\alpha = 1, \dots, p$, where each next matrix is recursively defined.

We start with the $N_1 \times N_1$-matrix D_1 which corresponds to the difference operator Ω_1 as follows

$$\mathsf{D}_1 = \frac{a}{2h_1}\begin{pmatrix} 4 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -4 & 3 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -4 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & -4 & 3 & \cdots & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 1 & -4 & 3 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -4 & 3 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & -4 & 3 \end{pmatrix}.$$

We define the matrix $\mathsf{E}_1 = \frac{a}{2h_2}\mathsf{I}_1$, where I_1 is identity $N_1 \times N_1$-matrix. Next, we can define the matrix D_2 as a matrix which has a block-3-banded form: each block has size $N_1 \times N_1$ and there are N_2 blocks in the line:

$$\mathsf{D}_2 = \begin{pmatrix} \mathsf{D}_1 + 4\mathsf{E}_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4\mathsf{E}_1 & \mathsf{D}_1 + 3\mathsf{E}_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathsf{E}_1 & -4\mathsf{E}_1 & \mathsf{D}_1 + 3\mathsf{E}_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathsf{E}_1 & -4\mathsf{E}_1 & \mathsf{D}_1 + 3\mathsf{E}_1 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -4\mathsf{E}_1 & \mathsf{D}_1 + 3\mathsf{E}_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \mathsf{E}_1 & -4\mathsf{E}_1 & \mathsf{D}_1 + 3\mathsf{E}_1 \end{pmatrix}$$

and so on... In such a way we can define a linear operator A by the square matrix of size $N_1 \times N_2 \times \cdots \times N_p$:

$$\mathsf{A} = \begin{pmatrix} \mathsf{D}_{p-1} + 4\mathsf{E}_{p-1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4\mathsf{E}_{p-1} & \mathsf{D}_{p-1} + 3\mathsf{E}_{p-1} & 0 & \cdots & 0 & 0 & 0 \\ \mathsf{E}_{p-1} & -4\mathsf{E}_{p-1} & \mathsf{D}_{p-1} + 3\mathsf{E}_{p-1} & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & -4\mathsf{E}_{p-1} & \mathsf{D}_{p-1} + 3\mathsf{E}_{p-1} & 0 \\ 0 & 0 & 0 & \cdots & \mathsf{E}_{p-1} & -4\mathsf{E}_{p-1} & \mathsf{D}_{p-1} + 3\mathsf{E}_{p-1} \end{pmatrix},$$

where $\mathsf{E}_{p-1} = \frac{a}{2h_p}\mathsf{I}_{p-1}$ and I_{p-1} is the identity matrix of dimension $N_1 \times \cdots \times N_{p-1}$.

Proposition 4.1. *For each p, the matrix D_p is positive definite.*

Proof. The matrix D_1 has eigenvalues $\lambda_1(\mathsf{D}_1) = 2a/h,\ \lambda_2(\mathsf{D}_1) = \cdots = \lambda_{N_1}(\mathsf{D}_1) = 3a/2h$ and therefore is positive definite. For each p matrices D_p are lower triangular with positive diagonal elements. All of their principal minors are equal to the product of appropriate diagonal elements, so, the are positive. According to the Sylvester's criterion all matrices D_p are positive definite. $\square$

Now we can rewrite system (3) in the form

$$\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta} + s\mathsf{A}\,\mathbf{y}_{j+1} + (1-s)\mathsf{A}\,\mathbf{y}_j = \mathbf{F}_j. \tag{15}$$

Let us use the obvious identity

$$\mathbf{y}_{j+1} = \mathbf{y}_j + \Delta\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta}$$

and define the linear operator $B = I + s\Delta A$, (I is the identity operator of the appropriate dimension) to rewrite (15) as a two-layer difference scheme in the canonical form [29]

$$\mathsf{B}\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta} + \mathsf{A}\,\mathbf{y}_j = \mathbf{F}_j. \tag{16}$$

The operator A is positive definite therefore B is a positive definite operator. Since B is invertible, we can rewrite (16) in the form

$$\mathbf{y}_{j+1} = \mathsf{S}\,\mathbf{y}_j + \Delta\mathsf{B}^{-1}\mathbf{F}_j,$$

where $S = (I - \Delta B^{-1}A)$ is the transition operator.

In the space Y_p we introduce scalar product and the energy norm

$$(y, u) = h_1\, h_2 \cdots h_p \sum_{i=1}^{N_1\times\cdots\times N_p} y^i u^i, \qquad \|y\|_{Y_p} = \sqrt{(\mathsf{A}y, y)},$$

thereafter we define the corresponding induced operator norm.

Definition 3. The difference scheme (16) is said to be stable, if $\|S\|_{Y_p} < 1$.

Note that the equivalent formalization of stability of two-layer difference scheme is given in [29, p. 324–330].

Theorem 3. *If the condition $s \geq 1/2$ is fulfilled then the difference scheme (16) is stable.*

Proof. Let us consider (16) from the point of view of operator-difference equations and apply methods of the stability verification for a two-layer difference scheme [29] and the separation of finite-dimensional and infinite-dimensional components [18,24].

We symmetrize (16) by multiplying through by A^{-1}

$$(\mathsf{A}^{-1} + s\Delta\mathsf{E})\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta} + \mathsf{E}\,\mathbf{y}_j = \mathsf{A}^{-1}\mathbf{F}_j.$$

Denoting $\hat{B} = A^{-1} + s\Delta E,\ \hat{A} = E$, and $\hat{\mathbf{F}}_j = \mathsf{A}^{-1}\,\mathbf{F}_j$, we obtain

$$\hat{\mathsf{B}}\,\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta} + \hat{\mathsf{A}}\,\mathbf{y}_j = \hat{\mathbf{F}}_j. \tag{17}$$

Method (17) is stable in the energy norm if and only if $2\hat{B} \geq \hat{A}$, see [29, p. 333 Theorem 1]. This requirement is equivalent to $A^{-1} + \Delta E(s - 0.5) \geq 0$. Since A^{-1} is a positive definite operator, the latter inequality is fulfilled for any Δ, as soon as $s \geq 0.5$. $\square$

The peculiar feature of the presented method is that the condition $s \geq 1/2$ does not impose any restriction on the steps sizes like the Courant–Friedrichs–Lewy condition does. Note that in difference schemes for parabolic and hyperbolic equations with time delay [25,24,27,26] such conditions of Courant type are essential.

We define the function of exact values by the relations

$$\mathbf{z}_j = (u(x_{(1,\ldots,1)}, t_j), u(x_{(2,\ldots,1)}, t_j), \ldots, u(x_{(N_1,\ldots,1)}, t_j), \ldots, u(x_{(N_1,\ldots,N_p)}, t_j))^\top \in Y_p.$$

Starting values for the model can be taken equal to the function of exact values

$$\mathbf{y}_j = \mathbf{z}_j = (\varphi(x_{(1,\ldots,1)}, t_j), \varphi(x_{(2,\ldots,1)}, t_j), \ldots, \varphi(x_{(N_1,\ldots,1)}, t_j), \ldots, \varphi(x_{(N_1,\ldots,N_p)}, t_j))^\top, \quad j = -m, \ldots, 0.$$

The definition of the residual without interpolation (4) in the scheme with weights for the advection equation with time delay and retardation of a state variable and the definition of the residual with interpolation (8) in the general scheme are essentially different. However, the following obvious statement connects these two definitions.

Table 1
Numerical results of Experiment 1.

Case	1	2	3	4	5	6	7	8
h	1/5	1/10	1/10	1/20	1/5	1/10	1/20	1/40
Δ	$\pi/20$	$\pi/20$	$\pi/40$	$\pi/40$	$\pi/200$	$\pi/200$	$\pi/200$	$\pi/200$
`diff`	0.0902	0.0919	0.0481	0.0495	0.0530	0.0135	0.0038	0.0031
CPU-time	2.31	9.12	18.0	71.8	21.9	84.2	358	1466

Theorem 4. *Let the conditions of Theorem 3 be satisfied and the interpolation operator (2) is used. Then, the residual with interpolation in the sense of (8) has order* $\sum_{\alpha=1}^{p} h_\alpha^2 + \Delta$.

The embedding of the scheme with weights for the Eq. (1a) into the general scheme has been carried out, thereafter the following statement is true.

Theorem 5. *Let the conditions of Theorem 3 be satisfied and the interpolation operator (2) is used. Then if* $s \geq 0.5$ *method (3) converges with order* $\sum_{\alpha=1}^{p} h_\alpha^2 + \Delta$.

We illustrate our obtained results with two numerical experiments.

5. Numerical experiments

Simulations were done in MATLAB, on a PC ASUS, CPU Intel Core i5-2401M, 2.3 GHz, 4 Gb RAM.

Experiment 1. Let us consider the following first order partial differential equation with discrete and distributed time delays:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = -2(u - \sin t)\frac{x+y}{1+x^2+y^2} + \frac{\pi}{2}u(x,y,t-\pi) - \int_{-\pi/2}^{0} u(x,y,t+\xi)\,d\xi + \left(\frac{\pi}{2}+1\right)\sin t,$$

where $-2 < x < 2,\ -5 < y < 5,\ 0 < t \leq 4\pi$, with initial and boundary conditions

$$u(x,y,t) = \frac{1}{1+x^2+y^2} + \sin t, \quad -2 \leq x \leq 2,\ -5 \leq y \leq 5,\ -\pi \leq t \leq 0,$$
$$u(-2,y,t) = \frac{1}{5+y^2} + \sin t, \qquad u(x,-5,t) = \frac{1}{26+x^2} + \sin t, \quad -2 \leq x \leq 2,\ -5 \leq y \leq 5,\ 0 \leq t \leq 4\pi.$$

This initial–boundary value problem has the exact solution $u(x,y,t) = \frac{1}{1+x^2+y^2} + \sin t$. In Table 1 we report the absolute error `diff` $= \max_{i,j} \left|u_j^i - u(x_i, t_j)\right|$ of the approximate solution calculated by method (3) with $s = 0.8$ from the exact one for different values of h and Δ. For simplicity let us take $h_1 = h_2$ and denote them h.

Since the exact solution may be equal to zero on the considered domain, we do not report relative error here. Nevertheless note that the maximum value of u on the considered domain is equal to 2; this allows one to compare the value of absolute error with the value of function u. We also report the CPU-time in seconds.

In cases nos. 5–7 the error related to the time discretization is small in comparison with the error related to the coordinate discretization; the analysis of the error behavior reveals the square convergence with respect to space variables, i.e., when the step becomes half as much, the error becomes almost four times less. The analysis of the data in the table shows that only the consistent decrease of steps yields the decrease of error. Thus, in cases nos. 7–8 the halving of h does not cause the corresponding decrease of error, because the total error is mostly induced by the time discretization.

By Theorem 3 for $s = 0.8$ scheme (3) is stable with any ratio of steps; however, due to the ill-posedness of the numerical differentiation, the decrease of h makes the approximations of $\partial u/\partial x$ and $\partial u/\partial y$ in (3) more sensitive to the computer rounding error, which leads to the increase of the error. The decrease of Δ consistent with h is a peculiar regularizer which prevents errors from growing and accumulating. Cases nos. 1–4 illustrate this fact.

Remark that the order of numerical integration must be consistent with the order of difference method (3).

Experiment 2. Let us consider an equation where the right-hand part is nonlinear with respect to the prehistory of the unknown function:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = u(x,y,t-3\pi/2) - \left(\int_{-2\pi}^{0} u(x,y,t+\xi)\,d\xi\right)^2 + \sin t + (1+e^y)(x+1+e^y),$$

where $0 < x < 4,\ 0 < y < 2,\ 0 < t \leq 6\pi$, with initial and boundary conditions

$$u(x,y,t) = x(1+e^y) + y\sin t, \quad 0 \leq x \leq 4,\ 0 \leq y \leq 2,\ -2\pi \leq t \leq 0,$$
$$u(0,y,t) = y\,\sin t, \qquad u(x,0,t) = 2x, \quad 0 \leq x \leq 4,\ 0 \leq y \leq 2,\ 0 \leq t \leq 4\pi.$$

Table 2
Numerical results of Experiment 2.

Case	1	2	3	4	5	6	7	8
h	1/5	1/10	1/10	1/20	1/5	1/10	1/20	1/40
Δ	$\pi/20$	$\pi/20$	$\pi/40$	$\pi/40$	$\pi/400$	$\pi/400$	$\pi/400$	$\pi/400$
`diff`	3.1513	2.2446	1.2643	1.1077	1.6438	0.4201	0.1089	0.0341
CPU-time	2.47	9.12	21.5	92.4	56.7	223	870	3520

Table 3
Numerical results of Experiment 3.

Case	1	2	3	4	5
h	0.05	0.025	0.05	0.025	0.0125
Δ	0.05	0.05	0.00625	0.00625	0.00625
`diff`	0.2237	0.1713	0.0983	0.0330	0.0160
`diff_1`	0.3338	0.1991	0.6051	0.3120	0.1609
`diff_2`	0.0633	∞	0.1142	0.0653	0.0366

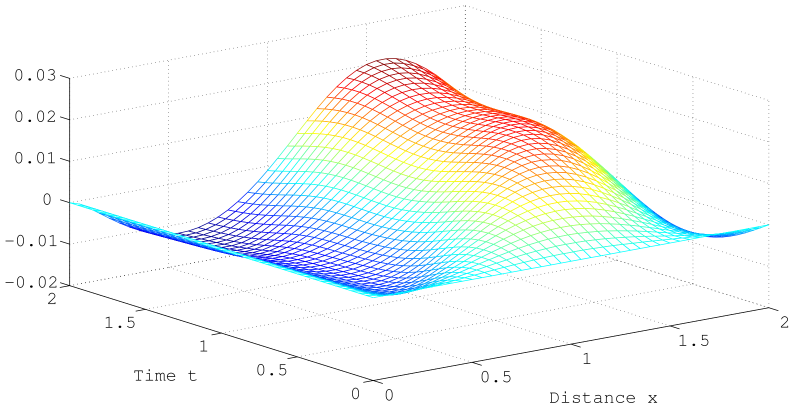

Fig. 1. The difference between the exact solution and its grid approximation of Experiment 3.

This initial–boundary value problem has the exact solution $u(x, y, t) = x(1+e^y)+y \sin t$. In Table 2 we report the deviations `diff` $= \max_{i,j} \left|u_j^i - u(x_i, t_j)\right|$ of the approximate solution calculated by method (3) with $s = 0.8$ from the exact one for different values of h and Δ. Again, for simplicity we take $h_1 = h_2$ and denote them h.

Experiment 3. Numerical algorithms for a one dimensional (in space) advection equation with time delay were elaborated in [28], they are analogs of running schemes and midpoint rules. The first proposed method has an order of convergence $O(h+\Delta)$ and is unconditionally stable. To achieve the second order convergence $O(h^2+\Delta^2)$, some parameters were chosen on the boundary of the stability domain; thereafter the authors reported that at a certain ratio of step sizes the second method may give unsatisfying results precisely for this choice.

Surely, our method allows one to solve a one dimensional (in space) equation, as a particular case. To compare our results let us consider the following test example, which was taken form [28];

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = \sin \pi x + \pi t \cos \pi x - (t-1) \sin \pi x + u(x, t-1),$$

where $0 < x < 2,\ 0 < t < 2$, with initial and boundary conditions

$$u(x, t) = t \sin \pi x, \quad 0 \le x \le 2, \ -1 \le t \le 0,$$
$$u(0, t) = 0, \quad 0 \le t \le 2.$$

This initial–boundary value problem has the exact solution $u(x, t) = t \sin \pi x$. In Table 3 we report the deviations `diff` $= \max_{i,j} \left|u_j^i - u(x_i, t_j)\right|$ of the approximate solution calculated by method (3) with $s = 0.8$ from the exact one and deviations of two methods from [28]. As one can see, the rectangle method from [28] gives better results for big steps, but since the parameters of that method were chosen on the boundary of the stability domain the error may grow indefinitely. Since our method (3) is unconditionally stable we outperform the corresponding results of [28].

The difference between the exact solution $u(x, t)$ and its grid approximation u_j^i is depicted in Fig. 1.

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