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A quaternionic description of kinematics and dynamics universal joint

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Abstract. The purpose of the work is figuring out the kinematic and dynamic equations of u-joint motion in the quaternion parametrization. The quaternion description of u-joint kinematics allowed us to formulate the claims about the crosspiece angular velocity projections. We've also determined the principle point projection acting on the crosspiece by solving the inverse problem from Euler's dynamic equations with the found driving shaft's angular velocity and the inertial crosspiece characteristics.

The modeling results are verified in the computer software - Mathcad and can be used for creating optimal u-joint designs by changing its geometry.

1. Introduction

Cardan-Hooke's joint (or universal joint) – is the main knot of the cardan transmission, which is designed for transferring the rotation between 2 shafts, being at an angle to each other, and consisted of 2 plugs, hard connected with the shafts and with the crosspiece. Picture 1 demonstrates the universal joint with the set of frontal wrenches. The classical results gained from the kinematics study of the u-joint we can see in works [1-5]. The cardan transmissions are widely used in automotive industry, for producing agricultural and street cars, in mechanical equipment for metal production etc. In the background of massive techno «rearmament» there's a need for creating universal joints with a high carrying capacity, with reduced relative dimensions and increased durability. Dynamic equations in quaternion parameterization that are represented here allow us to gain the link in an analytical form of geometric, kinematic and dynamic characteristic of the u-joint and let us model its movement in different operation modes.



Figure 1. The example of a U-joint



2. Known universal joint kinematic relations

As we see in [5], the link between the angular speed of a driven shaft ω_{II} and the angular speed of a main shaft ω_I in the u-joint is set by the ratio

$$\omega_{II} = \frac{\cos \alpha}{1 - \sin^2 \alpha \sin^2 \varphi_1} \omega_I, \quad (1)$$

where φ_1 – is the rotation angle of the main shaft, α – is the angle between shafts' axis.

Traditionally that result was found by the differentiation of an equation, which links the rotation angles of the main shaft φ_1 and the driven shaft φ_{II} , that was found by the geometric method:

$$tg \varphi_{II} = tg \varphi_1 \cos \alpha \quad (2)$$

Beside the formula (1), that includes that at the initial state the plug of main shaft located perpendicularly to the plane of crossing axis shafts often uses yet another equation:

$$\omega_{II} = \frac{\cos \alpha}{1 - \sin^2 \alpha \cos^2 \varphi_1} \omega_I.$$

This formula is figured out for the case when the plug of the main shaft is located in the plane of crossing axis shafts.

The kinematic scheme of the U-joint is shown on figure 2. For forming the crosspiece position of the U-joint the following plane angles are used: ϑ – the pitch angle, ψ – the destination angle, γ – the roll angle (Figure 2). The beginning of a stationary coordinate system is placed in the middle of the crosspiece, axis Ox_2 of the stationary coordinate system is headed vertically. Axis Ox_1 is located in the horizontal plane of the main shaft's plug. The pitch angle ϑ matches the rotation angle of the main shaft around the axis Ox_3 of the stationary coordinate system. The result of the main shaft spinning determines the coordinate system $Ox_1'x_2'x_3'$. While the destination angle ψ is zero. The roll angle γ is found as a result of turning the crosspiece around axis Ox_1' , which is carried out from axis Ox_2' to axis Ox_3' . Consistent executing of those two turns gives us the moving coordinate system $Ox_1''x_2''x_3''$ connected with the crosspiece.

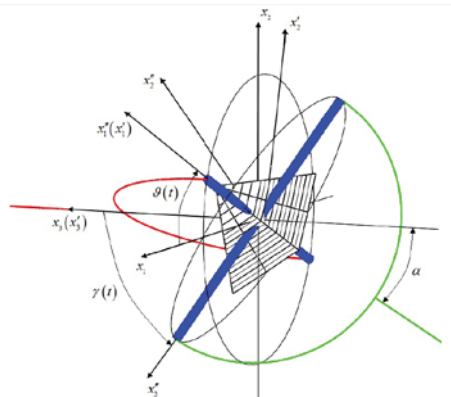


Figure 2. Using plane's angles for describing of u-joint crosspiece position

The U-joint kinematics mathematically submits to the following main statements.

In every fixed motion moment angles ϑ , γ and α , determining the u-joint crosspiece position, are bound by the equations:

$$\operatorname{tg} \alpha \operatorname{tg} \gamma \operatorname{tg} \vartheta = 1.$$

From which follows that when the u-joint crosspiece moves the roll angle γ changes according to the law:

$$\gamma(t) = \frac{\pi}{2} - \operatorname{arctg}(\operatorname{tg} \alpha \cos \vartheta(t)). \quad (3)$$

In every fixed moment for angles φ_1 , φ_{II} and α spins of the main and driven shafts equality holds (1) Considering the range of function values $\operatorname{arctg}(\operatorname{tg} \varphi_1 \cos \alpha)$, the equation (2) doesn't let us write down the law of changing the rotation angle of the driven shaft on the entire time interval and the law of driven shaft spinning is better find using the equality integration method (1). As a result, we get:

$$\varphi_{II} = \int_0^t \frac{\cos \alpha}{1 - \sin^2 \alpha \sin^2 \varphi_1(t)} \dot{\varphi}_1 dt.$$

3. The quaternion model of the universal joint kinematics

These formulated statements let us do the complete description of the u-joint crosspiece spherical movements. To do so we just need to take the advantage of the quaternion model, by which the coordinates of the single orientation quaternion $q = [q_0, q_1, q_2, q_3]$, $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ are linked with the plane angles by the following ratios [6, 7]:

$$\begin{aligned} q_0 &= \cos \frac{\vartheta}{2} \cos \frac{\psi}{2} \cos \frac{\gamma}{2} + \sin \frac{\vartheta}{2} \sin \frac{\psi}{2} \sin \frac{\gamma}{2}, \\ q_1 &= \cos \frac{\vartheta}{2} \cos \frac{\psi}{2} \sin \frac{\gamma}{2} - \sin \frac{\vartheta}{2} \sin \frac{\psi}{2} \cos \frac{\gamma}{2}, \\ q_2 &= \cos \frac{\vartheta}{2} \sin \frac{\psi}{2} \cos \frac{\gamma}{2} + \sin \frac{\vartheta}{2} \cos \frac{\psi}{2} \sin \frac{\gamma}{2}, \\ q_3 &= \sin \frac{\vartheta}{2} \cos \frac{\psi}{2} \cos \frac{\gamma}{2} - \cos \frac{\vartheta}{2} \sin \frac{\psi}{2} \sin \frac{\gamma}{2}. \end{aligned} \quad (4)$$

In case of even spinning of the main shaft with the angle speed ω_1 considering the equations (3), (4) the quaternion coordinate in every fixed time moment can be figured out with such equations:

$$\begin{aligned} q_0 &= \cos \frac{\omega_1 t}{2} \cos \frac{1}{2} \left[\frac{\pi}{2} - \operatorname{arctg}(\operatorname{tg} \alpha \cos \omega_1 t) \right], \\ q_1 &= \cos \frac{\omega_1 t}{2} \sin \frac{1}{2} \left[\frac{\pi}{2} - \operatorname{arctg}(\operatorname{tg} \alpha \cos \omega_1 t) \right], \\ q_2 &= \sin \frac{\omega_1 t}{2} \sin \frac{1}{2} \left[\frac{\pi}{2} - \operatorname{arctg}(\operatorname{tg} \alpha \cos \omega_1 t) \right], \\ q_3 &= \sin \frac{\omega_1 t}{2} \cos \frac{1}{2} \left[\frac{\pi}{2} - \operatorname{arctg}(\operatorname{tg} \alpha \cos \omega_1 t) \right]. \end{aligned}$$

While

$$q_0^2 + q_1^2 = \cos^2 \frac{\omega_1 t}{2}, \quad q_2^2 + q_3^2 = \sin^2 \frac{\omega_1 t}{2}, \quad q_0^2 + q_3^2 = \cos^2 \frac{\gamma}{2}, \quad q_1^2 + q_2^2 = \sin^2 \frac{\gamma}{2}$$

The position of the crosspiece given its initial state can be found by the following matrix equation:

where $R(t) = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$ – is the rotation matrix.

In the computer algebra pack on Mathcad with given values $\omega_1 = 0,5 \frac{\text{rad}}{\text{sec}}$ and $\alpha = \frac{\pi}{6}$ we can get the u-joint crosspiece moving animation. The dependences of changing the roll angle, the rotation angle for the driven shaft and also the quaternion coordinate from time are displayed on figure 3,4, and 5.

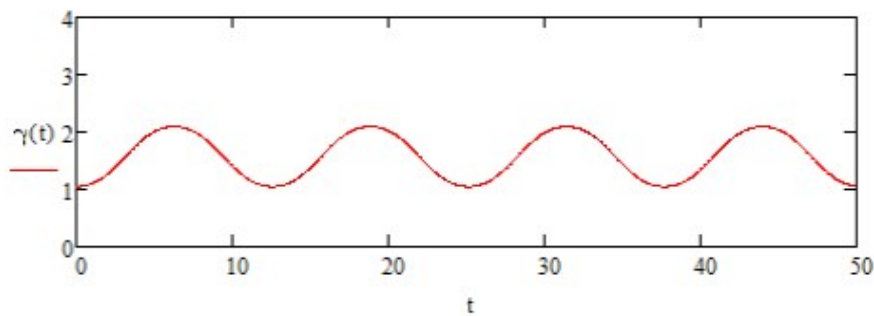


Figure 3. The dependence of changing the roll angle of the u-joint crosspiece from time

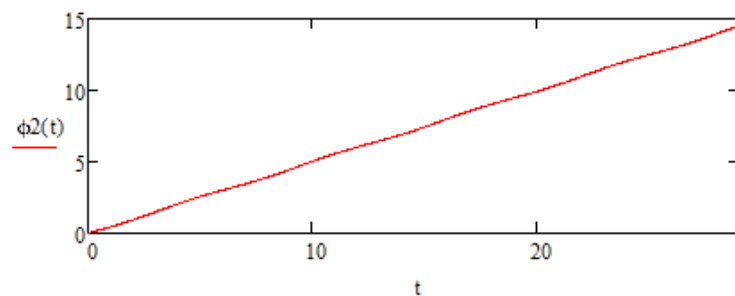


Figure 4. The dependence of changing the rotation angle of the driven shaft from time

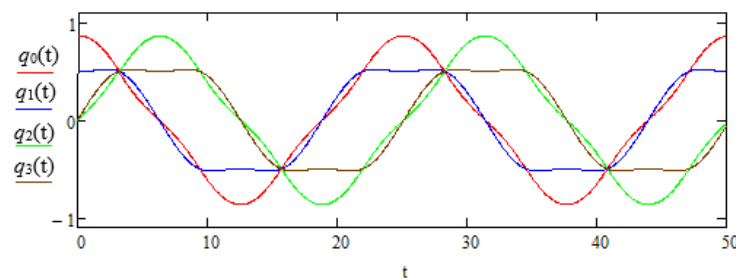


Figure 5. The dependences of changing quaternion orientation coordinates of the u-joint crosspiece from time

The functions of changing quaternion coordinates help us find the projection of the angle speed vector to the connected axis thanks to these kinematic equations [6,7]:

$$\begin{aligned}\Omega_1 &= 2(\dot{q}_1q_0 - \dot{q}_0q_1 - \dot{q}_3q_2 + \dot{q}_2q_3), \\ \Omega_2 &= 2(\dot{q}_2q_0 - \dot{q}_0q_2 - \dot{q}_1q_3 + \dot{q}_3q_1), \\ \Omega_3 &= 2(\dot{q}_3q_0 - \dot{q}_0q_3 - \dot{q}_2q_1 + \dot{q}_1q_2).\end{aligned}$$

At the same time the projections of the u-joint crosspiece angle speed to the moving axis go along with the following differential equations:

$$\dot{\Omega}_3 + \Omega_1 \Omega_2 = 0, \quad \dot{\Omega}_2 - \Omega_3 \Omega_1 \quad (5)$$

Which leads us to when the u-joint crosspiece moves this equation is true:

$$\Omega_2^2 + \Omega_3^2 = const.$$

The quaternionic description of the u-joint kinematics let's us also formulate the following statements relating the projections of the crosspiece angle speed to the moving axis.

In every fixed moment of movement at spinning of the main shaft with the constant angle speed ω_1 this condition is satisfied

$$\Omega_2^2 + \Omega_3^2 = \omega_1^2.$$

Figure 6 displays the dependences of changing the vector coordinates of the angle speed from time with the given values $\omega_1 = 0,5 \frac{\text{rad}}{\text{sec}}$ and $\alpha = \frac{\pi}{6}$

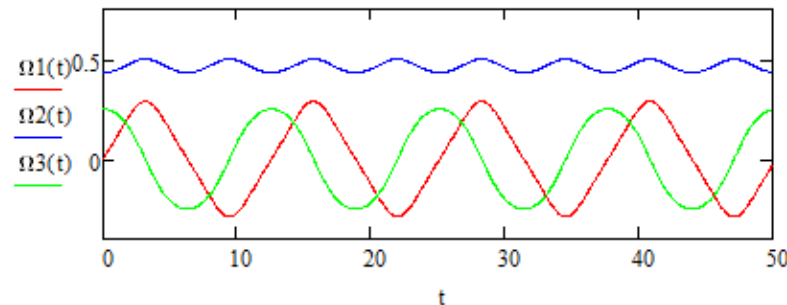


Figure 6. The dependences of changing the vector coordinates of the angle speed from time at the linked axis

The verification of the suggested description for kinematics of the u-joint is made in the computer algebra pack Mathcad [8].

4. The U-joint crosspiece dynamics

With the known angle speed of the main shaft and inertial crosspiece characteristics of the main moment of force projection, action on the crosspiece, can be determined by solving the reverse task from Euler's dynamic equations. Let's say the main crosspiece inertial moments I_1, I_2 и I_3 are set, then we have

$$M_1 = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2),$$

$$M_2 = I_2 \dot{\Omega}_2 + \Omega_3 \Omega_1 (I_1 - I_3),$$

$$M_3 = I_3 \dot{\Omega}_3 + \Omega_1 \Omega_2 (I_2 - I_1),$$

where I_1, I_2, I_3 – are the crosspiece inertial moments relating to the axis Ox_1'', Ox_2'', Ox_3'' respectively.

Considering the equations (5) the dynamic equations of the u-joint crosspiece movement has the form:

$$M_1 = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2),$$

$$M_2 = [I_2 - (I_1 - I_3) \dot{\Omega}_2],$$

$$M_3 = [I_3 - (I_1 - I_3) \dot{\Omega}_3].$$

(6)

In a specific case when $I_3 = I_2$ the moments, acting on the u-joint crosspiece, are figured out with such ratios:

$$\begin{aligned}M_1 &= I_1 \dot{\Omega}_1, \\M_2 &= I_2 \dot{\Omega}_2, \\M_3 &= I_3 \dot{\Omega}_3.\end{aligned}$$

The crosspiece kinetic energy is gained by:

$$T = \frac{1}{2}(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2) \quad (7)$$

5. Simulation results

From equations (6) and (7) follows that by varying the crosspiece's inertial characteristics we can greatly change the force and energy parameters of the spinning crosspiece. In particular, from equations (6) follows that with rationing inertial moments $I_1 = I_3 = \frac{1}{2}I_2$, the external forces moment projection M_3 is zero, regardless of the angle speed of the main shaft spinning value. Figure 7 demonstrates the graphs of the force moments projection, acting on the crosspiece. Picture 8 shows the graphs of changing kinetic energy of the spinning crosspiece with different axis inertial moments ratios with the same central inertial moment $I_0 = \frac{1}{2}(I_1 + I_2 + I_3)$, the same angle speed of the main shaft ω_1 and the same bend angle α .

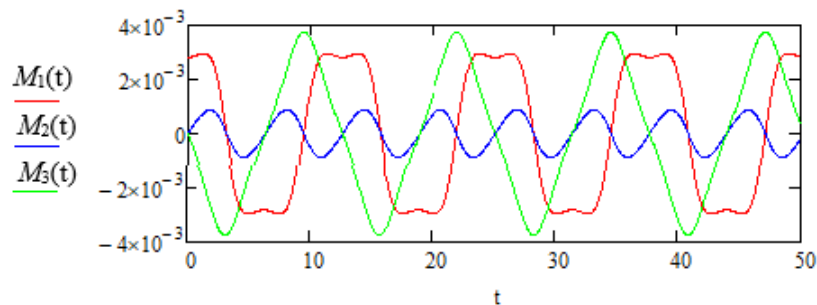


Figure 7. The dependences of force moments projection, acting on the crosspiece from time ($\omega = 0,5 \frac{\text{rad}}{\text{sec}}$, $\alpha = \frac{\pi}{6}$, $I_1 = I_2 = I_3 = 0,026 \text{kgm}^2$)

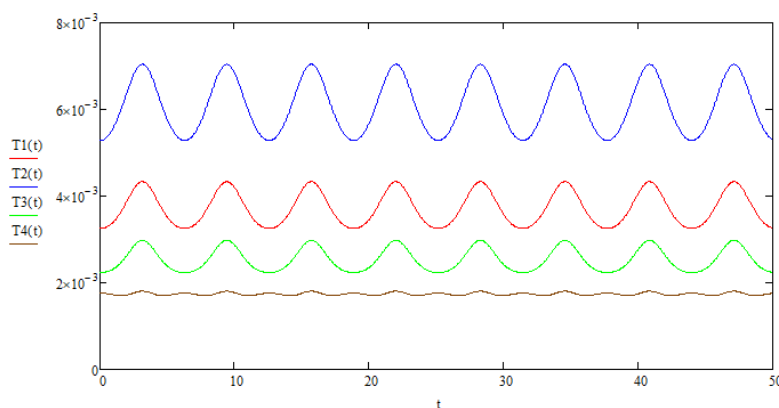


Figure 8. The dependences of changing kinetic energy (J) from time provided

$$\begin{aligned}I_0 &= 0,039 \text{kgm}^2; T_1: I_1 = I_2 = I_3 = \frac{2}{3}I_0; T_2: I_1 = I_3 = \frac{1}{3}I_0, I_2 = \frac{4}{3}I_0; T_3: I_1 = I_3 = \frac{5}{6}I_0, I_2 = \frac{1}{3}I_0; \\T_4: I_1 &= \frac{7}{9}I_0, I_2 = \frac{1}{9}I_0, I_3 = \frac{10}{9}I_0\end{aligned}$$

As we can see on graphics on picture 8, with the constant frequency of spinning of the main shaft ω_1 and the same bend angle α so the changing of the crosspiece kinetic energy is just oscillating. The vibration amplitude value significantly changes as well as an average energy. These proofs the possibility of setting up a task using the optimal mass distribution towards the spinning crosspiece.

6. Conclusion

In that article we examined the quaternion description of the u-joint and also found the kinematic characteristics of its movements. Studied the u-joint dynamics. With the given constant angle speed of the main shaft spinning, the angle between the axis and inertial characteristics of the crosspiece we have found its kinetic energy and force moments for u-joint plugs reactions. These acquired results may be used for creating optimal u-joints by changing its geometry.

7. References

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