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# On some computational problems of constructing piecewise linear approximations of functions with the Delaunay property

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**Abstract.** In the article, we discuss issues related to the construction of a piecewise linear approximation of some function, information about which is limited to its values at given points from some finite set. A wide range of tasks from various fields of knowledge leads to such a formulation. A feature of the approach to the problem under consideration is the application of the Delaunay partition methodology, based on linear programming technologies. The goal is to extend this promising methodology to a more wide class of problems without requiring that the convex hull of the set is bodily.

## 1. Introduction

In the article, we discuss issues related to the construction of a piecewise linear approximation of some given function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ , information about which is limited to its values at some given points from a finite set  $\mathfrak{A} = \{a^1, \dots, a^m\}$ . A wide range of tasks from various fields of knowledge leads to such a formulation. A feature of the approach to the problem under consideration is the application of the Delaunay partition methodology based on linear programming technologies [1–4]. The goal is to extend this promising methodology to a wide class of problems without the assumption that the convex hull of the set  $\mathfrak{A}$  is bodily [5–6]. To do this, it is necessary to resolve a number of issues related to both the ambiguity of the calculations and the emerging need to solve additional problems.

One of such important issues for discussion is the method of finding a solution from the minimum facet of the optimal set of the following auxiliary linear programming (LP) problem

$$\max\{(x, u) + w : (a^i, u) + w \leq \|a^i\|^2, i = 1, \dots, m\}$$

(here  $(\cdot, \cdot)$  denotes the scalar product of vectors). The above problem is dual to the basic one, and the projection of the component  $u$  of its solution from the minimal facet of the optimal set onto the affine hull of the set  $\mathfrak{A}$  determines the center of the Delaunay sphere when finding the estimate  $f(x)$ . Finding a solution from the facet of the smallest dimension of the optimal set is nontrivial here for the reason that this dual problem does not have to have vertices. The standard simplex methodology recommends introducing artificial variables into this task, but the solution obtained with their help will not necessarily have the required property [7]. Below the ways are suggested to solve this question and to discuss other computational problems of this approach.



## 2. The problem setting

Let us consider a slightly more general formulation of the LP problem, which consists in finding a solution belonging to the minimum facet of its feasible set. Let there be a standard problem with constraints in the form of inequalities

$$\max\{(c, x): Ax \leq b\} \quad (1)$$

and a dual one

$$\min\{(b, y): A^T y = c, y \geq 0\}; \quad (2)$$

here  $A_{m \times n}$  is a given real matrix,  $\text{rank}(A) < n$ ,  $c, x \in \mathbb{R}^n$ ;  $b, y \in \mathbb{R}^m$ .

Since the direct problem has no restrictions on the sign of its variables, we can always choose the scale of their measurement so that the condition  $c \geq 0$  is satisfied.

It is well known that solutions to the problem (1) lie on the boundary of its multifaceted feasible set and completely fill one of its facets. Usually, these are the vertices of a feasible set, i.e., facets of zero dimension. However, our assumption about the rank of the matrix  $A$  speaks about their absence. Our goal is to effectively find a solution to the problem (1) which belongs to the facet of the smallest dimension.

To begin we recall some well-known facts [4,7] for the case  $\text{rank}(A) = n$ . Under this assumption, the typical approach to solving problems (1), (2) consists of applying the primal simplex method to the problem (2) because it has the canonical form. The simplex method works with the so-called basic sub-matrices of the constraint matrix of the initial problem and, starting from a given one, leads to such a sub-matrix that allows us to construct the desired solution. The process of calculation is divided into two phases. Phase 1 is designed to find a basic sub-matrix that satisfies certain conditions of feasibility. This phase can start from an arbitrary initial non-degenerate sub-matrix. After finding a feasible basic sub-matrix or in the case when the starting sub-matrix is already feasible, the algorithm switches to phase 2, which is intended to optimize the objective function of the problem.

Each basic sub-matrix consists of  $n$  linearly independent columns of the matrix  $A^T$ . If such a sub-matrix is selected (for example, these are the first  $n$  columns of the matrix  $A^T$ ), the initial data of the dual problem acquire the following block structure:

$y_B^T$	$y_N^T$	
$b_B^T$	$b_N^T$	min
$B$	$N$	$c$

Here  $B$  is the basis sub-matrix of the matrix  $A^T$ ,  $N$  is its non-basis sub-matrix,  $y_B$  and  $b_B$  are the basis variables and coefficients of the objective function,  $y_N$  and  $b_N$  are the non-basic (free) variables and coefficients of the objective function.

With the base sub-matrix  $B$  there is associated a specific plan of the dual problem (also called basic), namely a plan  $y^T = (y_B^T, y_N^T)$ , in which  $y_B = B^{-1}c$ ,  $y_N = 0$ . This plan is feasible if  $B^{-1}c \geq 0$  (conditions for the feasibility of a basis), and is optimal if also  $d^T = b_N^T - b_B^T B^{-1}N \geq 0$  (conditions for its optimality). Moreover, the optimal basis of the problem (2) can also give a solution to the problem dual to the problem (2), that is, a solution to the problem (1) by the simple formula  $\bar{x}^T = b_B^T B^{-1}$ . This solution turns out to be the vertices of the multifaceted domain of the direct problem.

Thus, if problems (1), (2) are solvable and if the assumption  $\text{rank}(A) = n$  is valid, then the simplex method can find the optimal basis of dual problem and the corresponding optimal basic plan, as well as the plan of the primal problem, that is, the plan of the problem (1). However,

its application requires the existence of at least one basic sub-matrix of the matrix  $A^T$ . In our case, this means that the rank of the matrix  $A$  should be equal to  $n$ .

Let us return to our main assumption that  $\text{rank}(A) = r < n$ .

The standard methodology for applying the simplex method to the canonical problem, whose restriction matrix has an incomplete row rank, is also well known [7]. It recommends to introduce the so-called artificial variables into the problem (2), that is, go to the problem

$$\min\{(b, y) + \mu(e, w): A^T y + w = c, y \geq 0, w \geq 0\}, \quad (3)$$

where  $e = (1, \dots, 1)^T \in \mathbb{R}^n$ ,  $w \in \mathbb{R}^m$  are the artificial variables,  $\mu > 0$  is sufficiently large constant.

For the convenience of further investigations, we rewrite the problem (3) in the aggregative form

$$\min\{(p, z): Mz = c, z \geq 0\}; \quad (4)$$

here  $z^T = (y^T, w^T)$ ,  $M = (A^T|E)$ ,  $p^T = (b^T, \mu e^T)$ ,  $E$  is unite matrix of the order  $n$ . Now,  $\text{rank}(M) = n$ , and the simplex method applied to the solvable problem (4) finds its basic solution corresponding to some basic  $(n \times n)$ -sub-matrix of  $B$  of the matrix  $M$ , i.e. a sub-matrix composed of  $n$  of its linearly independent columns and satisfying the conditions of feasibility and optimality given above. Moreover, the vector  $\bar{u}^T = (\bar{u}_B^T, \bar{u}_N^T)$  with components  $\bar{u}_B = B^{-1}c$ ,  $\bar{u}_N = 0$  is optimal for (4), and the vector  $\bar{x}^T = p_B^T B^{-1}$  is optimal for the dual to (4) problem

$$\max\{(c, x): Ax \leq b, x \leq \mu e\}. \quad (5)$$

The latter is obtained from the original problem (1) by introducing upper bounds on the values of its variables. It is easy to see that any solution to the original problem (1) for sufficiently large  $\mu$  will also be a solution to the problem (5). But in this case the optimal sub-matrix will necessarily include a part of the columns with artificial variables.

Therefore, after introducing artificial variables into a dual problem, the direct problem is supplied with additional boundaries on the values of its variables. As a result, its feasible set acquires the vertices, one of which will be found by the simplex method as the optimal one. However, this vertex is usually not unique and may or may not lie on the facet of the minimal dimension of the initial feasible region. Moreover, the presented approach is complicated by the need to select a sufficiently large value of  $\mu$ .

Consider an example illustrating the above reasoning.

Example 1.

In the problem (1), set

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c^T = (0, 1, 0).$$

This example is designed in such a way that the gradient of the objective function coincides with the normal of the second inequality constraint. Therefore, the optimal value of the problem is not higher than 0 (this is the right side of the second constraint). But since the vector of the right-hand sides is non-negative, then the zero vector is feasible for this task. It delivers the objective function a value of 0, which thereby is the desired optimal value of the direct problem (and therefore the task dual to it). Note that the minimal facet of the feasible set here is the edge, consisting of points satisfying both constraints of the problem (1) as equalities. The zero vector does not lie on this facet.

Let us construct the initial data of the dual problem (recall that this is a minimization problem)

$$M = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad p^T = (1, 0, \mu, \mu, \mu).$$

Consider a basis composed of the 2nd, 3rd and 5th variables. The base matrix and its inverse matrix, as well as the non-basis sub-matrix of the matrix  $M$ , have the form

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The chosen basis is optimal for  $\mu > 0$ , since

$$B^{-1}c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \geq 0,$$

$$d^T = (1, \mu) - (0, \mu, \mu) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ -1 & 0 \end{pmatrix} = (2\mu + 1, \mu) \geq 0.$$

Therefore, the application of the simplex method to the dual problem may lead to this basis. Accordingly, a solution to the direct problem may be found as

$$\bar{x}^T = p_B^T B^{-1} = (0, \mu, \mu) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\mu, 0, \mu).$$

This solution is indeed feasible and provides the objective function with an optimal value of 0. But only one of the inequality constraints of the primal problem turns into equality, i.e., this solution does not belong to the facet of the minimal dimension of the initial feasible set.

So, we can see that artificial variables in a dual problem cannot help to find a solution to the problem (1) that belongs to a facet of minimal dimension. The application of the simplex method directly to the primal problem cannot help this search either. Indeed, at first, this task must be reduced to canonical form. To do this, let replace each of its variables with the difference of two new non-negative variables

$$x_i = u_i - v_i, \quad u_i \geq 0, \quad v_i \geq 0 \quad (i = 1, 2, 3).$$

Also, in each inequality of the problem, we must introduce an additional variable  $w_i$  to turn it into an equation.

After all these transformations, the problem (1) of the example 1 will take the form

$$\max\{(q, z): Hz = b, z^T = (u^T, v^T, w^T) \geq 0\},$$

where

$$H = (A| -A|E) = \left( \begin{array}{ccc|ccc|cc} -1 & 1 & -1 & 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array} \right),$$

$$q^T = (0, 1, 0, 0, -1, 0, 0, 0).$$

Now consider a basis of the 2nd and 7th variables. The base matrix and its inverse matrix, as well as the non-basis sub-matrix of the matrix  $H$  are equal

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

The chosen basis is optimal for the maximization problem, since

$$B^{-1}b = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq 0,$$

$$\begin{aligned} d^T &= (0, 0, 0, -1, 0, 0) - (1, 0) \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} = \\ &= (0, 0, 0, 0, 0, -1) \leq 0. \end{aligned}$$

Therefore, the simplex method can lead us to the following basic components of the solution

$$z_B = B^{-1}b = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The complete solution (in it free variables are equal to 0) has the form

$$\bar{z} = (\bar{u}|\bar{v}|\bar{w}) = (0, 0, 0|0, 0, 0|0, 1, 0),$$

or, after returning to the original variables,

$$\bar{x} = (0, 0, 0).$$

Yes, zero vector is an obvious solution to the problem (1). But at the beginning of this example, we already discussed that this solution does not lie on the facet of the minimum dimension of the feasible set of the primal problem.

### 3. Proposed solution

The approach proposed below is based on the expansion of the original variable space of the primal problem (1) into the direct sum of its two linear subspaces, namely, a nontrivial subspace

$$\mathbb{L} = \{x \in \mathbb{R}^n: Ax = 0\}$$

and its orthogonal complement

$$\mathbb{L}^\perp = \{x \in \mathbb{R}^n: (x, y) = 0 \forall y \in \mathbb{L}\}.$$

By virtue of initial assumptions  $\dim \mathbb{L} = n - r$  and  $\dim \mathbb{L}^\perp = r < n$ .

Let  $U$  be a matrix composed of vectors forming the basis of the subspace  $\mathbb{L}$ , and  $V$  be a matrix composed of vectors forming the basis of the subspace  $\mathbb{L}^\perp$ . Any vector  $x \in \mathbb{R}^n$  can be represented as

$$x = Ux_{\mathbf{u}} + Vx_{\mathbf{v}}, \quad (6)$$

where the coordinate vectors  $x_{\mathbf{u}} \in \mathbb{R}^{n-r}$  and  $x_{\mathbf{v}} \in \mathbb{R}^r$  are uniquely determined. The converse is also true: any two vectors  $x_{\mathbf{u}} \in \mathbb{R}^{n-r}$  and  $x_{\mathbf{v}} \in \mathbb{R}^r$  uniquely determine the vector  $x \in \mathbb{R}^n$  by the formula (6).

**Lemma 1.** *The vector  $x \in \mathbb{R}^n$  if and only if it satisfies the system of inequalities from the problem (1) when  $AVx_{\mathbf{v}} \leq b$ , where  $x_{\mathbf{v}}$  is taken from the decomposition (6).*

*P r o o f* is evident, because  $AU = 0$  by definition of the space  $\mathbb{L}$ .

**Lemma 2.** *If the problem (1) is solvable, then the value of its objective function on the vector  $x \in \mathbb{R}^n$  coincides with the value  $\mu(x) = c^T V x_{\mathbf{v}}$ .*

*P r o o f* is evident, because the subspace  $\mathbb{L}$  is a set of recessive directions for a feasible set of the problem (1). Therefore, its solvability requires the equality  $c^T U = 0$ .

Combining lemmas 1, 2, we obtain the main statement of the paper.

**Theorem 1.** *If the problem (1) is solvable, its solution lying on the facet of minimal dimension can be obtained by solving the truncated problem*

$$\max\{(V^T c, x_{\mathbf{v}}): AVx_{\mathbf{v}} \leq b\}. \quad (7)$$

*If  $\bar{x}_{\mathbf{v}}$  is the vertex solution of the reduced problem (7), then  $\bar{x} = V\bar{x}_{\mathbf{v}}$  is the desired solution to the problem (1).*

*P r o o f.* Indeed, it follows from the rank theorem for the product of two matrices that the feasible set of the problem (7) has at least one vertex and, in particular, at least one optimal vertex. The desired vertex  $\bar{x}_{\mathbf{v}}$  can be found by applying the simplex method to the dual problem

$$\min\{(b, y): V^T A^T y = V^T c, y \geq 0\}$$

according to the rules described at the beginning of this work. The found vertex  $\bar{x}_{\mathbf{v}}$  generates one of the solutions of the original problem by the formula  $\bar{x} = V\bar{x}_{\mathbf{v}}$ , and this solution will obviously lie in the facet of the minimal dimension of the original feasible set since  $r$  constraints-inequalities now are satisfied as to the equalities.

Let consider a simple way to find the necessary bases.

To determine the rank of the matrix  $A$ , we can apply the Jordan — Gauss elimination method to it. After some permutations of columns and renaming of variables we get a transformed matrix of the form

$$\bar{A}_{m \times n} = \left( \begin{array}{c|c} E_{r \times r} & Q_{r \times (n-r)} \\ \hline 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{array} \right).$$

As a result, we know the rank value of the matrix  $A$  and its rank structure

$$A = \left( \begin{array}{c|c} B_{r \times r} & N_{r \times (n-r)} \\ \hline S_{(m-r) \times r} & F_{(m-r) \times (n-r)} \end{array} \right);$$

here  $B$  is a non-degenerate  $(r \times r)$ -sub-matrix.

It is important that at the same time we find the matrix  $Q = B^{-1}N$ .

The matrix  $Q$  helps us find the required bases in the form

$$U = \left( \begin{array}{c} Q \\ -E \end{array} \right), \quad V = \left( \begin{array}{c} E \\ Q^T \end{array} \right).$$

Indeed, the matrices  $U$  and  $V$  have the required dimension, their columns are linearly independent, and the equality holds

$$U^T V = (Q^T | -E) \left( \begin{array}{c} E \\ Q^T \end{array} \right) = 0.$$

Now we can specify the entry for the auxiliary task (7). However, since the subspace  $\mathbb{L}^\perp$  coincides with the row space of the matrix  $A$ , we could simply put  $V^T = (B|N)$ .

To illustrate the above reasoning, let us return to example 1.

Recall that in this example

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c^T = (0, 1, 0).$$

Here, as the rank sub-matrix, we can take the sub-matrix from the first two columns of the matrix  $A$ . We have

$$B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Therefore

$$Q = B^{-1}N = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad V = \begin{pmatrix} E \\ -Q^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Further

$$AV = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix},$$

$$c^T V = (0, 1, 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = (0, 1).$$

We substitute this data into the problem (7) and solve it by applying the simplex method to the dual problem according to the technique described at the beginning of this paper. We get its the only vertex solution

$$\bar{x}_V^T = (-1/2, 0).$$

Now we can restore the solution to the original problem

$$\bar{x} = Vx_V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ -1/2 \end{pmatrix}.$$

We have got a solution that turns both constraint inequalities into the primal problem considered in the example, that is, a solution lying on the facet of the minimal dimension of the original feasible set.

Let consider one more example in which two facets of the minimal dimension turn out to be optimal.

Example 2.

Let us set

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad c = (0, 1, 0)^T.$$

As a ranking sub-matrix, we may take  $(2 \times 2)$ -sub-matrix lying in the upper left corner of the matrix  $A$ . We have

$$B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore

$$Q = B^{-1}N = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad V = \left( \frac{E}{Q^T} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Further

$$AV = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix},$$

$$c^T V = (0, 1, 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = (0, 1).$$

Using the graphical method (omitting cumbersome calculations by the simplex algorithm), we find two optimal vertices of the resulting reduced problem, namely,

$$\bar{x}_V^{(1)} = (-1/2, 0)^T \quad \text{and} \quad \bar{x}_V^{(2)} = (1/2, 0)^T.$$

They correspond to two solutions to the original problem

$$\bar{x}^{(1)} = Vx_V^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

and

$$\bar{x}^{(2)} = Vx_V^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}.$$

The first solution turns the first two inequality constraints into the equalities. The second one turns the last two inequality constraints into equalities. Thus, both solutions lie in the facets of minimal dimension.

#### 4. Conclusion

As it is known, to construct a piecewise linear approximation of the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ , given its values at points from a finite set  $\mathfrak{A} = \{a^1, \dots, a^m\}$ , it is useful apply Delaunay partitions in  $\mathbb{R}^n$ . The last may be done by linear programming technologies. This paper was aimed to extend this promising methodology to a wider class of problems without requiring that the convex hull of the set  $\mathfrak{A}$  is bodily. To do this, one must be able to find solutions from the minimal facet of the optimal set of linear programming problem of a special kind, namely,

$$\max\{(x, u) + w : (a^i, u) + w \leq \|a^i\|^2, i = 1, \dots, m\}.$$

To find such a solution from the facet of the smallest dimension of the optimal set of this program is a nontrivial task because the dual problem does not have to have vertices. The standard simplex methodology recommends introducing artificial variables into the task, but the solution obtained with their help will not necessarily have this property. The authors managed to find simple and effective methods for solving the problems posed by decomposing the original space of variables into the direct sum of two auxiliary linear subspaces.

## References

- [1] Rajan V 1991 *Proc. Annual Symp. on Computational Geometry* (North Conway, New Hampshire, United States) pp 357–363
- [2] Bencheikroun B and Falk J 1991 *Computers Math. Applic.* B **21** pp 77–85
- [3] Yepremyan L and Falk J 2005 *Computers & Oper. Research* B **32** pp 793–812
- [4] Eremin I 2002 *Theory of Linear Optimization* (VSP. Utrecht, Boston, Koln, Tokyo)
- [5] Plotnikov S 2016 *Proc. Int. Conf. ICAM'2016* (Ekaterinburg) pp 83–84
- [6] Plotnikov S 2018 *Proc. Int. Conf. ICAM'2018* (Altai Republic, Katun) pp 77–78
- [7] Murtagh B 1981 *Advanced Linear Programming: Computation and Practice*. (McGraw – Hill International Book Company)