SPECIAL ELEMENTS OF THE LATTICE OF EPIGROUP VARIETIES

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ABSTRACT. We study special elements of eight types (namely, neutral, standard, costandard, distributive, codistributive, modular, lower-modular and upper-modular elements) in the lattice **EPI** of all epigroup varieties. Neutral, standard, costandard, distributive and lower-modular elements are completely determined. A strong necessary condition and a sufficient condition for modular elements are found. Modular elements are completely classified within the class of commutative varieties, while codistributive and upper-modular elements are completely determined within the wider class of strongly permutative varieties. It is verified that an element of **EPI** is costandard if and only if it is neutral; is standard if and only if it is distributive; is modular whenever it is lower-modular; is neutral if and only if it is lower-modular and upper-modular simultaneously. We found also an application of results concerning neutral and lower-modular elements of **EPI** for studying of definable sets of epigroup varieties.

1. INTRODUCTION AND SUMMARY

1.1. **Semigroup pre-history.** The main object we examine in this article is the lattice of all epigroup varieties. But our considerations are motivated by some earlier investigations of the lattice of semigroup varieties and closely related with these investigations. To make clearer a context and motivations of our considerations, we start with a brief explanation of the 'semigroup pre-history' of the present work.

One of the main branches of the theory of semigroup varieties is an examination of lattices of semigroup varieties (see the survey [17]). If $\mathcal{V}$ is a variety then $L(\mathcal{V})$ stands for the subvariety lattice of $\mathcal{V}$ under the natural order (the class-theoretical inclusion). The lattice operations in $L(\mathcal{V})$ are the (class-theoretical) intersection denoted by $\mathcal{X} \wedge \mathcal{Y}$ and the join $\mathcal{X} \vee \mathcal{Y}$, i.e., the least subvariety of $\mathcal{V}$ containing both $\mathcal{X}$ and $\mathcal{Y}$.

There are a number of articles devoted to an examination of identities (first of all, the distributive, modular or Arguesian laws) and some related restrictions (such as semimodularity or semidistributivity) in lattices of semigroup

varieties, and many considerable results are obtained here. In particular, semigroup varieties with modular, Arguesian or semimodular subvariety lattice were completely classified and deep results concerning semigroup varieties with distributive subvariety lattices (related to a description of such varieties modulo group ones) were obtained. An overview of all these results may be found in [17, Section 11].

The results mentioned above specify, so to say, 'globally' modular or distributive parts of the lattice of semigroup varieties. The following natural step is to examine varieties that guarantee modularity or distributivity, so to say, in their 'neighborhood'. Saying so, we take in mind special elements in the lattice of semigroup varieties. There are many types of special elements that are considered in lattice theory. Recall definitions of some of them. An element x of a lattice $\langle L; \vee, \wedge \rangle$ is called

neutral if $(x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ for all $y, z \in L$;

standard if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $y, z \in L$;

distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $y, z \in L$;

modular if $(x \vee y) \wedge z = (x \wedge z) \vee y$ for all $y, z \in L$ with $y \leq z$;

upper-modular if $(z \vee y) \wedge x = (z \wedge x) \vee y$ for all $y, z \in L$ with $y \leq x$.

Costandard, codistributive and *lower-modular* elements are defined dually to standard, distributive and upper-modular ones. There is a number of interrelations between types of elements we consider. It is evident that a neutral element is both standard and costandard; a standard or costandard element is modular; a [co]distributive element is lower-modular [upper-modular]. It is well known also that a [co]standard element is [co]distributive (see [2, Theorem 253], for instance). So, eight types of elements defined above form a partially ordered set under class-theoretical inclusion pictured on Fig. 1.
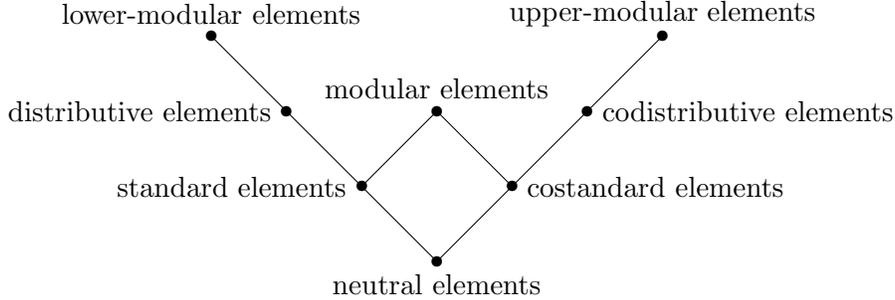
FIGURE 1. Special elements in abstract lattices

Note that special elements play an essential role in the abstract lattice theory (see [2, Section III.2] or [18, Sections 2.1 and 2.2], for instance). For instance, if an element x of a lattice L is neutral then L is decomposed into a subdirect product of its intervals $(x] = \{y \in L \mid y \leq x\}$ and $[x) = \{y \in L \mid x \leq y\}$ (see [2, Theorem 254]). Thus, the knowledge of what elements of a lattice are neutral, gives the important information on a structure of the lattice as a whole.

All types of elements mentioned above are intensively and successfully studied with respect to the lattice **SEM** of all semigroup varieties. For brevity, a semigroup variety that is a neutral element of the lattice **SEM** is called a *neutral in* **SEM** variety. Analogous convention is applied for all other types of special elements. Results about special elements in **SEM** are overviewed in the recent survey [29]. In particular,

- neutral in **SEM** varieties were completely determined in [37];
- it is proved that a semigroup variety is costandard in **SEM** if and only if it is neutral in **SEM** [27]; thus, in view of the previous result, costandard in **SEM** varieties are completely classified;
- distributive in **SEM** varieties were completely classified in [30];
- in fact, standard in **SEM** varieties are completely described in [30] too because the results of this work readily imply that a semigroup variety is standard in **SEM** if and only if it is distributive in **SEM** (see comments after Theorem 3.3 in the survey [29]);
- a strong necessary conditions for modular in **SEM** varieties were discovered in [7][1] and [22] (these results are reproved in a simpler way in [12]);
- a sufficient condition for modular in **SEM** varieties was found in [32] and rediscovered in [7];
- commutative modular in **SEM** varieties were completely determined in [22];
- lower-modular in **SEM** varieties were completely classified in [14];
- commutative upper-modular in **SEM** varieties were completely classified in [25]; it is noted in [24] that this result may be expanded on wider class of strongly permutative varieties without any change (a definition of strongly permutative varieties see in Subsection 1.2 below);
- strongly permutative codistributive varieties were completely described in [27].

Note that the articles [26, 27] contain some other results concerning upper-modular and codistributive elements in **SEM**.

1.2. **Epigroups.** A considerable attention in the semigroup theory is devoted to semigroups equipped by an additional unary operation. Such algebras are said to be *unary semigroups*. As concrete types of unary semigroups, we mention completely regular semigroups (see [9]), inverse semigroups (see [8]), semigroups with involution etc.

One more natural type of unary semigroups is epigroups. A semigroup S is called an *epigroup* if, for any element x of S, there is a natural n such that x^n is a *group element* (this means that x^n lies in some subgroup of S). Extensive information about epigroups may be found in the fundamental work [15] by L. N. Shevrin and the survey [16] by the same author. The class of epigroups is very wide. In particular, it includes all periodic semigroups (because some power of each element in such a semigroup lies in some its finite cyclic subgroup)

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[1]Note that paper [7] deals with the lattice of equational theories of semigroups, that is, the dual of **SEM** rather than the lattice **SEM** itself. When reproducing results from [7], we adapt them to the terminology of the present article.

and all completely regular semigroups (in which all elements are group ones). The unary operation on an epigroup is defined by the following way. If S is an epigroup and $x \in S$ then some power of x lies in a maximal subgroup of S. We denote this subgroup by G_x. The unit element of G_x is denoted by x^ω. It is well known (see [15], for instance) that the element x^ω is well defined and $xx^\omega = x^\omega x \in G_x$. We denote the element inverse to xx^ω in G_x by $\overline{x}$. The map $x \longmapsto \overline{x}$ is just the mentioned unary operation on an epigroup S. The element $\overline{x}$ is called *pseudoinverse* to x. Throughout this paper, we consider epigroups as algebras with the operations of multiplication and pseudoinversion. In particular, this allows us to consider varieties of epigroups as algebras with the two mentioned operations. An idea to examine epigroups in the framework of the theory of varieties was promoted by L. N. Shevrin in [15] (see also [16]). An overview of first results obtained here may be found in [17, Section 2].

If S is a *completely regular* semigroup (i. e., the union of groups) and $x \in S$ then $\overline{x}$ is the element inverse to x in the maximal subgroup containing x. Thus, the operation of pseudoinversion on a completely regular semigroup coincides with the unary operation traditionally considered on completely regular semigroups. We see that varieties of completely regular semigroups (considered as unary semigroups) are varieties of epigroups in the sense defined above. Further, it is well known and may be easily checked that in every periodic epigroup the operation of pseudoinversion may be expressed in terms of multiplication (see [15], for instance). This means that periodic varieties of epigroups may be identified with periodic varieties of semigroups.

It seems to be very natural to examine all restrictions on semigroup varieties mentioned in Subsection 1.1 for epigroup varieties. This is realized in [31, 34] for identities and related restrictions to subvariety lattice. In particular, epigroup varieties with modular, Arguesian or semimodular subvariety lattice are completely classified and epigroup analogs of results concerning semigroup varieties with distributive subvariety lattice are obtained there. In the present article, we start with an examination of special elements in the lattice **EPI** of all epigroup varieties. We consider here elements of all mentioned above eight types in **EPI**. For brevity, we call an epigroup variety *neutral* if it is a neutral element of the lattice **EPI**. Analogous convention will be applied for all other types of special elements. Our main results give:

- a complete description of neutral, standard, costandard, distributive and lower-modular varieties;
- a strong necessary condition and a sufficient condition for modular varieties;
- a description of commutative modular varieties, strongly permutative codistributive varieties and strongly permutative upper-modular varieties.

One can start with formulations of results. We denote by $\mathcal{T}$, $\mathcal{SL}$ and $\mathcal{ZM}$ the trivial variety, the variety of all semilattices and the variety of all semigroups with zero multiplication respectively. Our first result is the following

Theorem 1.1. *For an epigroup variety $\mathcal{V}$, the following are equivalent:*

   a) *$\mathcal{V}$ is a neutral element of the lattice **EPI***;

b) $\mathcal{V}$ *is a costandard element of the lattice* **EPI***;*
c) $\mathcal{V}$ *is simultaneously a lower-modular and upper-modular element of the lattice* **EPI***;*
d) $\mathcal{V}$ *coincides with one of the varieties* $\mathcal{T}$*,* $\mathcal{SL}$*,* $\mathcal{ZM}$ *or* $\mathcal{SL} \vee \mathcal{ZM}$*.*

Thus, there are only a few neutral elements in the lattice **EPI**. In contrast, we note that the lattice of completely regular semigroup varieties contains infinitely many neutral elements including all band varieties, the varieties of all groups, all completely simple semigroups, all orthodox semigroups and some other (this readily follows from [20, Corollary 2.9]).

Let Σ be an identity system written in the language of one associative binary operation and one unary operation. The class of all epigroups that satisfy Σ (where the unary operation is treated as pseudoinversion) is denoted by K_Σ. The class K_Σ is not obliged to be a variety because it maybe not closed under taking of (infinite) direct product (see [16, Subsection 2.3], for instance). Note that identity systems Σ with the property that K_Σ is a variety are completely determined in [4, Proposition 2.15] (see Lemma 2.4 below). If K_Σ is a variety then we use for this variety the standard notation var Σ. It is evident that if the class K_Σ consists of periodic epigroups then it is a periodic semigroup variety and therefore, is an epigroup variety. Whence, the notation var Σ is correct in this case. A pair of identities $wx = xw = w$ where the letter x does not occur in the word w is usually written as the symbolic identity $w = 0$. This notation is justified because a semigroup with such identities has a zero element and all values of the word w in this semigroup are equal to zero. We will refer to the expression $w = 0$ as to a single identity. Such identities and varieties given by them are called 0-*reduced*. Put

$$\mathcal{Q} = \text{var} \{x^2 y = xyx = yx^2 = 0\},$$
$$\mathcal{Q}_n = \text{var} \{x^2 y = xyx = yx^2 = x_1 x_2 \cdots x_n = 0\},$$
$$\mathcal{R} = \text{var} \{x^2 = xyx = 0\},$$
$$\mathcal{R}_n = \text{var} \{x^2 = xyx = x_1 x_2 \cdots x_n = 0\}$$

where n is a natural number. We note that $\mathcal{Q}_1 = \mathcal{R}_1 = \mathcal{T}$. Our second result is the following

Theorem 1.2. *For an epigroup variety* $\mathcal{V}$*, the following are equivalent:*

a) $\mathcal{V}$ *is a distributive element of the lattice* **EPI***;*
b) $\mathcal{V}$ *is a standard element of the lattice* **EPI***;*
c) $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ *where* $\mathcal{M}$ *is one of the varieties* $\mathcal{T}$ *or* $\mathcal{SL}$*, and* $\mathcal{N}$ *is one of the varieties* $\mathcal{Q}$*,* $\mathcal{Q}_n$*,* $\mathcal{R}$ *or* $\mathcal{R}_n$*.*

The following result gives a complete classification of lower-modular varieties.

Theorem 1.3. *An epigroup variety* $\mathcal{V}$ *is a lower-modular element of the lattice* **EPI** *if and only if* $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ *where* $\mathcal{M}$ *is one of the varieties* $\mathcal{T}$ *or* $\mathcal{SL}$*, while* $\mathcal{N}$ *is a 0-reduced variety.*

We follow the agreement that an adjective indicating a property shared by all semigroups of a given variety is applied to the variety itself; the expressions like

'completely regular variety', 'periodic variety', 'nilvariety', etc. are understood in this sense. Recall that an identity of the form

$$x_1 x_2 \cdots x_n = x_{1\pi} x_{2\pi} \cdots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$ is called *permutative*; if $1\pi \neq 1$ and $n\pi \neq n$ then this identity is said to be *strongly permutative*. A semigroup or a variety that satisfies [strongly] permutative identity also is called [*strongly*] *permutative*. The following two results describe codistributive and upper-modular varieties in the strongly permutative case.

Theorem 1.4. *A strongly permutative epigroup variety $\mathcal{V}$ is a codistributive element of the lattice* **EPI** *if and only if $\mathcal{V} = \mathcal{G} \vee \mathcal{X}$ where $\mathcal{G}$ is an Abelian group variety, while $\mathcal{X}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$ or $\mathcal{SL} \vee \mathcal{ZM}$.*

Put $\mathcal{C}_m = \mathrm{var}\{x^m = x^{m+1}, xy = yx\}$ for arbitrary natural m. In particular, $\mathcal{C}_1 = \mathcal{SL}$. It will be convenient for us also to assume that $\mathcal{C}_0 = \mathcal{T}$.

Theorem 1.5. *A strongly permutative epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **EPI** *if and only if one of the following holds:*

(i) *$\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and $\mathcal{N}$ is a nilvariety satisfying the commutative law and the identity*

$$(1.1) \qquad\qquad\qquad x^2 y = xy^2;$$

(ii) *$\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an Abelian group variety, $0 \leq m \leq 2$ and $\mathcal{N}$ satisfies the commutative law and the identity*

$$(1.2) \qquad\qquad\qquad x^2 y = 0.$$

The remaining three results devoted to modular varieties. A word written in the language of multiplication and pseudoinversion is called a *semigroup word* if it does not include the operation of pseudoinversion. An identity is called a *semigroup identity* if both its parts are semigroup words. A semigroup identity $u = v$ is said to be *substitutive* if u and v depend on the same letters and v may be obtained from u by renaming of letters. The following result gives a strong necessary condition for modular varieties.

Theorem 1.6. *If an epigroup variety $\mathcal{V}$ is a modular element of the lattice* **EPI** *then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and $\mathcal{N}$ is a nilvariety given by 0-reduced and substitutive identities only.*

It is easy to see that this theorem completely reduces the problem of description of modular varieties to nilvarieties defined by 0-reduced and substitutive identities only (see Corollary 3.3 below).

We provide the following sufficient condition for modular varieties.

Theorem 1.7. *A 0-reduced semigroup variety is a modular element of the lattice* **EPI***.*

Theorems 1.6 and 1.7 show that in order to describe modular varieties we need to examine nil-varieties satisfying substitutive identities. A natural partial case of substitutive identities are permutative ones, while the strongest permutative identity is the commutative law. Modular varieties satisfying this law possess a complete description.

Theorem 1.8. *A commutative epigroup variety $\mathcal{V}$ is a modular element of the lattice* **EPI** *if and only if $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety that satisfies the commutative law and the identity* (1.2).

Theorems 1.6 and 1.7 provide a necessary and a sufficient condition for an epigroup variety to be modular respectively. The gap between these conditions seems to be not very large. But the necessary condition is not a sufficient one, while the sufficient condition is not a necessary one. This follows from Theorem 1.8. Indeed, this theorem shows that the variety $\mathrm{var}\{x^3 = 0,\ xy = yx\}$ is not modular although it is given by 0-reduced and substitutive identities only, while the variety $\mathrm{var}\{x^2 y = 0,\ xy = yx\}$ is modular although it is not 0-reduced.

The article is structured as follows. It consists of eleven sections. In Section 2 we collect definitions, notation and auxiliary results used in what follows. In Section 3 we verify two special cases of Theorem 1.1, namely the claims that the varieties $\mathcal{SL}$ and $\mathcal{ZM}$ are neutral. These facts are used in each of Sections 4–10. After that we prove Theorem 1.5 in Section 4, Theorem 1.4 in Section 5, Theorems 1.6–1.8 in Section 6 and Theorem 1.3 in Section 7. In Section 8 we discuss an interesting application of Theorems 1.1 and 1.3 concerning so-called definable sets of varieties (a corresponding definition see in Section 8). In Sections 9 and 10 we prove Theorems 1.1 and 1.2 respectively. Sections 4–7, 9 and 10 contain also a number of corollaries of main results. In particular, we note that an epigroup variety is modular whenever it is lower-modular (Corollary 7.3). Finally, in Section 11 we formulate several open questions.

2. Preliminaries

2.1. Some properties of the operation of pseudoinversion.
The following three lemmas are well known and may be easily checked.

Lemma 2.1. *The identity*

$$(2.1) \qquad\qquad x = \overline{\overline{x}}$$

holds in an epigroup S if and only if S is completely regular.   □

It is well known (see [15, 16], for instance) that if S is an epigroup and $x \in S$ then $x\,\overline{x} = \overline{x}\,x = x^{\omega}$. This permits to write in epigroup identities expressions of the form u^{ω} rather than $u\,\overline{u}$, for brevity.

Lemma 2.2. *If an epigroup variety $\mathcal{V}$ satisfies the identity $x^m = x^{m+1}$ then the identities $x^{\omega} = \overline{x} = \overline{\overline{x}} = x^m$ hold in $\mathcal{V}$.*   □

Lemma 2.3. *The identity*

$$(2.2) \qquad\qquad \overline{x} = 0$$

holds in an epigroup S if and only if S is a nil-semigroup.   □

2.2. When K_Σ is a variety?
An identity is called *mixed* if one of its parts is a semigroup word but another one is not. Further, a semigroup identity is called *balanced* if each letter occurs in both its parts the same number of times.

Lemma 2.4 ([4, Proposition 2.15]). *The class K_Σ is an epigroup variety if and only if Σ contains either a semigroup non-balanced identity or a mixed identity.*   □

2.3. **Identities of certain varieties.** We denote by F the free unary semigroup over a countably infinite alphabet (with the operations $\cdot$ and $^{-}$). Elements of F are called *words*. If $w \in F$ then we denote by $c(w)$ the set of all letters occurring in w and by $t(w)$ the last letter of w. A letter is called *simple* [*multiple*] in a word w if it occurs in w ones [at least twice]. Put

$$\mathcal{P} = \text{var}\{xy = x^2y,\ x^2y^2 = y^2x^2\} \quad \text{and} \quad \overleftarrow{\mathcal{P}} = \text{var}\{xy = xy^2,\ x^2y^2 = y^2x^2\}.$$

The symbol $\equiv$ stands for the equality relation on the unary semigroup F. The first two claims of the following lemma are well-known and may be easily verified, the third one was proved in [1, Lemma 7].

Lemma 2.5. *A non-trivial semigroup identity $v = w$ holds:*
   (i) *in the variety $\mathcal{SL}$ if and only if $c(v) = c(w)$;*
   (ii) *in the variety $\mathcal{C}_2$ if and only if $c(v) = c(w)$ and every letter from $c(v)$ is either simple both in v and w or multiple both in v and w;*
   (iii) *in the variety $\mathcal{P}$ if and only if $c(v) = c(w)$ and either the letters $t(v)$ and $t(w)$ are multiple in v and w respectively or $t(v) \equiv t(w)$ and the letter $t(v)$ is simple both in v and w.* □

If w is a semigroup word then $\ell(w)$ stands for the length of w; otherwise, we put $\ell(w) = \infty$. We need the following three remarks about identities of nil-semigroups.

Lemma 2.6. *Let $\mathcal{V}$ be a nilvariety.*
   (i) *If the variety $\mathcal{V}$ satisfies an identity $u = v$ with $c(u) \neq c(v)$ then $\mathcal{V}$ satisfies also the identity $u = 0$.*
   (ii) *If the variety $\mathcal{V}$ satisfies an identity of the form $u = vuw$ where the word vw is non-empty then $\mathcal{V}$ satisfies also the identity $u = 0$.*
   (iii) *If the variety $\mathcal{V}$ satisfies an identity of the form $x_1x_2 \cdots x_n = v$ and $\ell(v) \neq n$ then $\mathcal{V}$ satisfies also the identity $x_1x_2 \cdots x_n = 0$.*

Proof. The claims (i) and (ii) are well known and easily verified.

(iii) If v is a non-semigroup word, it suffices to refer to Lemma 2.3. Let now v be a semigroup word. If $\ell(v) < n$ then $c(v) \neq \{x_1, x_2, \ldots, x_n\}$, and the desired conclusion follows from the claim (i). Finally, if $\ell(v) > n$ then the claim we prove readily follows from [10, Lemma 1]. □

2.4. **Decomposition of some varieties into the join of subvarieties.** As usual, we denote by $\text{Gr}\,S$ the set of all group elements of an epigroup S. For an arbitrary epigroup variety $\mathcal{X}$, we put $\text{Gr}(\mathcal{X}) = \mathcal{X} \wedge \mathcal{GR}$ where $\mathcal{GR}$ is the variety of all groups. The variety generated by an epigroup S is denoted by $\text{var}\,S$. Put

$$\mathcal{LZ} = \text{var}\{xy = x\} \quad \text{and} \quad \mathcal{RZ} = \text{var}\{xy = y\}.$$

The following two facts play an important role in the proof of Theorem 1.5. 'Semigroup prototypes' of Proposition 2.7 and Lemma 2.8 were given in [35, Proposition 1] and [5] respectively.

Proposition 2.7. *If $\mathcal{V}$ is an epigroup variety and $\mathcal{V}$ does not contain the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is a variety generated by a monoid, and $\mathcal{N}$ is a nilvariety.*

Proof. It is verified in [35, Lemma 2] that if a semigroup variety does not contain the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ then $\mathcal{V}$ satisfies the quasiidentity

$$(2.3) \qquad e^2 = e \longrightarrow ex = xe.$$

Repeating literally the proof of this claim (with using a term 'subepigroup' rather than 'subsemigroup'), one can establish that the similar claim is true for epigroup varieties. Thus, $\mathcal{V}$ satisfies the quasiidentity (2.3). The rest of the proof is quite similar to the proof of Proposition 1 in [35].

Let S be an epigroup that generates the variety $\mathcal{V}$, $x \in S$ and E the set of all idempotents from S. In view of (2.3), ES is an ideal in S. By the definition of an epigroup, there is a natural n such that $x^n \in \mathrm{Gr}\, S$. Then $x^n = x^\omega x^n$ and $x^\omega \in E$. We see that $x^n \in ES$. Therefore, the Rees quotient semigroup S/ES is a nil-semigroup and therefore, is an epigroup. The natural homomorphism ρ from S onto S/ES separates elements from $S \setminus ES$.

Let now $e \in E$. In view of (2.3), we have that eS is a subsemigroup in S. It is well known that every epigroup satisfies the identity $\overline{x} = x\,\overline{x}^2$ (see [15, 16], for instance). Hence the equality $\overline{ex} = ex(\overline{ex})^2$ holds. We have verified that, for any $e \in E$, the set eS is a subepigroup in S. Put $S^* = \prod_{e \in E} eS$. Then S^* is an epigroup with unit $(\ldots, e, \ldots)_{e \in E}$. It follows from (2.3) that the map ε from S into S^* given by the rule $\varepsilon(x) = (\ldots, ex, \ldots)_{e \in E}$ is a semigroup homomorphism. As is well known (see [15, 16], for instance), an arbitrary semigroup homomorphism ξ from an epigroup S_1 into an epigroup S_2 is also an epigroup homomorphism (i. e., $\xi(\overline{a}) = \overline{\xi(a)}$ for any $a \in S_1$). Therefore, ε is an epigroup homomorphism from S into S^*. One can verify that ε separates elements of ES. Let $e, f \in E$, $x, y \in S$ and $\varepsilon(ex) = \varepsilon(fy)$. Then $e \cdot ex = e \cdot fy$ and $f \cdot ex = f \cdot fy$. Since $e, f \in E$, we have

$$(2.4) \qquad ex = efy \quad \text{and} \quad fex = fy.$$

Therefore,

$$ex \overset{(2.4)}{=} efy \overset{(2.4)}{=} efex \overset{(2.3)}{=} feex \overset{e \in E}{=} fex \overset{(2.4)}{=} fy.$$

We see that $ex = fy$ whenever $\varepsilon(ex) = \varepsilon(fy)$. This means that ε separates elements of ES.

Thus, ε and ρ are homomorphisms from S into S^* and S/ES respectively, and the intersection of kernels of these homomorphisms is trivial. Therefore, the epigroup S is decomposable into a subdirect product of the epigroups S^* and S/ES, whence $\mathcal{V} \subseteq \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M} = \mathrm{var}\, S^*$ is a variety generated by a monoid and $\mathcal{N} = \mathrm{var}(S/ES)$ is a nilvariety. On the other hand $S^*, S/ES \in \mathcal{V}$, whence $\mathcal{M} \vee \mathcal{N} \subseteq \mathcal{V}$. We have proved that $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$. $\square$

Recall that an epigroup is called *combinatorial* if all its subgroups are trivial.

Lemma 2.8. *If an epigroup variety $\mathcal{M}$ is generated by a commutative epigroup with unit then $\mathcal{M} = \mathcal{G} \vee \mathcal{C}_m$ for some Abelian group variety $\mathcal{G}$ and some $m \geq 0$.*

Proof. It is well known that the variety of all Abelian groups is the least non-periodic epigroup variety. This variety evidently contains the infinite cyclic group. Further, for each natural m, let G_m denote the cyclic group of order m.

It is evident that if $\mathcal{M}$ is periodic then the set $\{m \in \mathbb{N} \mid G_m \in \mathcal{M}\}$ has the greatest element. We denote by G the infinite cyclic group whenever the variety $\mathcal{M}$ is non-periodic, and the finite cyclic group of the greatest order among all cyclic groups in $\mathcal{M}$ otherwise. In both the cases $G \in \mathcal{M}$. Further, let D_m be the finite cyclic combinatorial epigroup of order m and d_m is a generator of D_m. Put $X = \{m \in \mathbb{N} \mid D_m \in \mathcal{M}\}$. If the set X has not the greatest element then the semigroup $\prod_{m \in X} D_m$ is not an epigroup since, for example, no power of the element $(\ldots, d_m, \ldots)_{m \in X}$ belongs to a subgroup. Therefore, the set of numbers X contains the greatest number. We denote this number by n. Repeating literally arguments from the proof of Theorem 1 in [5], we have that every epigroup from $\mathcal{M}$ is a homomorphic image of some subepigroup of the epigroup $G \times D_n$. Therefore, $\mathcal{M} = \mathcal{G} \vee \mathcal{D}$ where $\mathcal{G} = \operatorname{var} G$ and $\mathcal{D} = \operatorname{var} D_n$. Clearly, $\mathcal{G}$ is a variety of Abelian groups. The variety $\mathcal{D}$ is generated by a finite epigroup, whence it may be considered as a semigroup variety. It is well known and may be easily verified that $(m + 1)$-element combinatorial cyclic monoid generates the variety $\mathcal{C}_m$. Therefore, $\mathcal{D} = \mathcal{C}_m$ for some $m \geq 0$. $\qquad\square$

It is evident that a strongly permutative variety does not contain the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$. Besides that, every monoid satisfying a permutative identity is commutative. Thus, we have the following corollary of Proposition 2.7 and Lemma 2.8.

Corollary 2.9. *If $\mathcal{V}$ is a strongly permutative epigroup variety then $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an Abelian group variety, $m \geq 0$ and $\mathcal{N}$ is a nilvariety.* $\qquad\square$

2.5. **A direct decomposition of one varietal lattice.** We denote by $\mathcal{AG}$ the variety of all Abelian groups. The aim of this subsection is to prove the following

Proposition 2.10. *The lattice $L(\mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{Q})$ is isomorphic to the direct product of the lattices $L(\mathcal{AG})$ and $L(\mathcal{C}_2 \vee \mathcal{Q})$.*

Proof. We need the following auxiliary statement.

Lemma 2.11. *If $\mathcal{X} \subseteq \mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{Q}$ then $\mathcal{X} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is some Abelian group variety, $0 \leq m \leq 2$ and $\mathcal{N} \subseteq \mathcal{Q}$.*

Proof. Being a subvariety of the variety $\mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{Q}$, the variety $\mathcal{X}$ satisfies the identity $x^2 y = y x^2$. It is evident that this identity fails in the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$. Further, Lemma 2.5(iii) and the dual statement imply that this identity is false in the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ as well. Therefore, none of the four mentioned varieties is contained in $\mathcal{X}$. Besides that, the variety $\mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{Q}$ (and therefore, $\mathcal{X}$) satisfies the identity $x^2 yz = x^2 zy$. Substituting 1 for x, we have that all monoids in $\mathcal{X}$ are commutative. Proposition 2.7 and Lemma 2.8 imply now that $\mathcal{X} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ for some Abelian group variety $\mathcal{G}$, some $m \geq 0$ and some nilvariety $\mathcal{N}$. It is evident that $\mathcal{G} \subseteq \mathcal{AG}$. Lemmas 2.1, 2.2 and 2.3 imply that $\mathcal{AG}$, $\mathcal{C}_m$ and $\mathcal{Q}$ satisfy the identities (2.1), $\overline{\overline{x}} = x^m$ and (2.2) respectively. Therefore, the identity $x^2 y = \overline{\overline{x}}^2 y$ holds in the variety $\mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{Q}$. But this identity is false in the variety $\mathcal{C}_m$ with $m > 2$. Hence $m \leq 2$. This implies

that the variety $\mathcal{AG} \vee \mathcal{C}_m \vee \mathcal{Q}$ satisfies the identities $x^2y = yx^2 = \overline{\overline{x}}^2y$ and $xyx = xy\overline{\overline{x}}$. Since $\mathcal{N} \subseteq \mathcal{X} \subseteq \mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{Q}$, Lemma 2.3 implies that the variety $\mathcal{N}$ satisfies the identities $x^2y = xyx = yx^2 = 0$, whence $\mathcal{N} \subseteq \mathcal{Q}$. $\qquad\square$

A semigroup word w is called *linear* if every letter from $c(w)$ is simple in w. For convenience of references, we formulate the following observation that will be helpful also in Section 10.

Lemma 2.12. *All non-semigroup words and all non-linear semigroup words except x^2 equal to 0 in the variety $\mathcal{Q}$.*

Proof. If u is a non-semigroup word then $u = 0$ in $\mathcal{Q}$ by Lemma 2.3. The claim that a non-linear semigroup word differ from x^2 equal to 0 in $\mathcal{Q}$ is evident. $\quad\square$

Now we start with the direct proof of Proposition 2.10. Let $\mathcal{V} \subseteq \mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{Q}$. In view of Lemma 2.11, $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ for some Abelian group variety $\mathcal{G}$, some $0 \leq m \leq 2$ and some variety $\mathcal{N}$ with $\mathcal{N} \subseteq \mathcal{Q}$. Put $\mathcal{U} = \mathcal{C}_m \vee \mathcal{N}$. We have that $\mathcal{V} = \mathcal{G} \vee \mathcal{U}$ where $\mathcal{G} \subseteq \mathcal{AG}$ and $\mathcal{U} \subseteq \mathcal{C}_2 \vee \mathcal{Q}$. It remains to establish that this decomposition of the variety $\mathcal{V}$ into the join of some subvariety of the variety $\mathcal{AG}$ and some subvariety of the variety $\mathcal{C}_2 \vee \mathcal{Q}$ is unique.

Let $\mathcal{V} = \mathcal{G}' \vee \mathcal{U}'$ where $\mathcal{G}' \subseteq \mathcal{AG}$ and $\mathcal{U}' \subseteq \mathcal{C}_2 \vee \mathcal{Q}$. We need to verify that $\mathcal{G} = \mathcal{G}'$ and $\mathcal{U} = \mathcal{U}'$. Let $u = v$ be an arbitrary identity satisfied by $\mathcal{G}$. The variety $\mathcal{U}$ satisfies the identity $x^3 = x^4$. Then the identity $u^4v^3 = u^3v^4$ holds in the variety $\mathcal{G} \vee \mathcal{U}$. Let us cancel this identity on u^3 from the left and on v^3 from the right, thus concluding that $u = v$ holds in $\mathrm{Gr}\,(\mathcal{G} \vee \mathcal{U})$. Therefore, $\mathrm{Gr}\,(\mathcal{G} \vee \mathcal{U}) \subseteq \mathcal{G}$. The opposite inclusion is evident. Thus, $\mathrm{Gr}\,(\mathcal{G} \vee \mathcal{U}) = \mathcal{G}$. Analogously, $\mathrm{Gr}\,(\mathcal{G}' \vee \mathcal{U}') = \mathcal{G}'$. We see that

$$\mathcal{G} = \mathrm{Gr}\,(\mathcal{G} \vee \mathcal{U}) = \mathrm{Gr}\,(\mathcal{V}) = \mathrm{Gr}\,(\mathcal{G}' \vee \mathcal{U}') = \mathcal{G}'.$$

It remains to check that $\mathcal{U} = \mathcal{U}'$. Recall that $\mathcal{U} = \mathcal{C}_m \vee \mathcal{N}$ where $0 \leq m \leq 2$ and $\mathcal{N} \subseteq \mathcal{Q}$, while $\mathcal{U}' \subseteq \mathcal{C}_2 \vee \mathcal{Q}$. It is evident that the variety $\mathcal{C}_2 \vee \mathcal{Q}$ (and therefore, $\mathcal{U}'$) is combinatorial. Therefore, Lemma 2.11 implies that $\mathcal{U}' = \mathcal{C}_k \vee \mathcal{N}'$ for some $0 \leq k \leq 2$ and some variety $\mathcal{N}'$ with $\mathcal{N}' \subseteq \mathcal{Q}$.

Suppose that $m \neq k$. We may assume without any loss that $m < k$, i.e., either $m = 0$, $1 \leq k \leq 2$ or $m = 1$, $k = 2$. Suppose at first that $m = 0$ and $1 \leq k \leq 2$. It is evident that any group satisfies the identity $x^\omega = y^\omega$. Lemma 2.3 implies that this identity holds in $\mathcal{N}$ and therefore, in $\mathcal{V} = \mathcal{G} \vee \mathcal{T} \vee \mathcal{N} = \mathcal{G} \vee \mathcal{N}$. But Lemma 2.5(ii) implies that this identity fails in the variety $\mathcal{SL}$. However, this is impossible because $\mathcal{SL} \subseteq \mathcal{G}' \vee \mathcal{C}_k \vee \mathcal{N}' = \mathcal{V}$. Suppose now that $m = 1$ and $k = 2$. Then Lemmas 2.1, 2.2 and 2.3 imply that the identity $x^2y = x^2\overline{\overline{y}}$ holds in the variety $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N} = \mathcal{G} \vee \mathcal{SL} \vee \mathcal{N}$. But this identity is false in the variety $\mathcal{C}_2$ (and therefore, in $\mathcal{G}' \vee \mathcal{C}_k \vee \mathcal{N}' = \mathcal{V}$) by Lemmas 2.2 and 2.5(ii). A contradiction shows that $m = k$.

Note that if $\mathcal{X}$ is an arbitrary epigroup variety then the class of all epigroups in $\mathcal{X}$ satisfying the identity (2.2) is the greatest nilsubvariety of $\mathcal{X}$. We denote this subvariety by $\mathrm{Nil}(\mathcal{X})$. Put $\overline{\mathcal{N}} = \mathrm{Nil}\,(\mathcal{U})$ and $\overline{\mathcal{N}'} = \mathrm{Nil}\,(\mathcal{U}')$. It suffices to verify that $\overline{\mathcal{N}} = \overline{\mathcal{N}'}$ because

$$\mathcal{U} = \mathcal{C}_m \vee \mathcal{N} = \mathcal{C}_m \vee \overline{\mathcal{N}} = \mathcal{C}_k \vee \overline{\mathcal{N}'} = \mathcal{C}_k \vee \mathcal{N}' = \mathcal{U}'$$

in this case. Suppose that $\overline{\mathcal{N}} \neq \overline{\mathcal{N}'}$. It suffices to verify that

$$(2.5) \qquad \mathcal{G} \vee \mathcal{C}_m \vee \overline{\mathcal{N}} \neq \mathcal{G}' \vee \mathcal{C}_k \vee \overline{\mathcal{N}'}$$

because this contradicts the equalities

$$\mathcal{G} \vee \mathcal{C}_m \vee \overline{\mathcal{N}} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N} = \mathcal{V} = \mathcal{G}' \vee \mathcal{C}_k \vee \mathcal{N}' = \mathcal{G}' \vee \mathcal{C}_k \vee \overline{\mathcal{N}'}.$$

We will say that varieties $\mathcal{X}_1$ and $\mathcal{X}_2$ *differ with an identity* $u = v$ if this identity holds in one of the varieties $\mathcal{X}_1$ or $\mathcal{X}_2$ but fails in another one. Since $\overline{\mathcal{N}} \neq \overline{\mathcal{N}'}$, the varieties $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}'}$ differ with some identity. We may assume without loss of generality that this identity holds in $\overline{\mathcal{N}}$ but fails in $\overline{\mathcal{N}'}$. Suppose at first that the identity we mention is a 0-reduced identity $u = 0$. Since $\overline{\mathcal{N}'} \subseteq \mathcal{Q}$, this identity fails in $\mathcal{Q}$. In view of Lemma 2.12, either u is a linear word or $u \equiv x^2$. Suppose that $u \equiv x_1 x_2 \cdots x_n$ for some n. Then the identity $x_1 x_2 \ldots x_n = 0$ holds in $\overline{\mathcal{N}}$ but fails in $\overline{\mathcal{N}'}$. If $m = 2$ then the variety $\overline{\mathcal{N}}$ contains the variety $\mathrm{Nil}\,(\mathcal{C}_2) = \mathrm{var}\{x^2 = 0,\ xy = yx\}$. The latest variety does not satisfy the identity $x_1 x_2 \ldots x_n = 0$. Therefore, $m \leq 1$. Then the variety $\mathcal{G} \vee \mathcal{C}_m$ is completely regular. Lemmas 2.1 and 2.3 imply now that the variety $\mathcal{G} \vee \mathcal{C}_m \vee \overline{\mathcal{N}}$ satisfies the identity $x_1 x_2 \cdots x_n = \overline{\overline{x_1}} x_2 \cdots x_n$. But Lemma 2.3 implies that this identity is false in $\overline{\mathcal{N}'}$ and therefore, in $\mathcal{G}' \vee \mathcal{C}_k \vee \overline{\mathcal{N}'}$. Thus, (2.5) holds. Let now $u \equiv x^2$. Then Lemmas 2.1, 2.2 and 2.3 imply that the identity $x^2 = \overline{\overline{x}}^{\,2}$ holds in the variety $\mathcal{G} \vee \mathcal{C}_m \vee \overline{\mathcal{N}}$ but is false in the variety $\mathcal{G}' \vee \mathcal{C}_k \vee \overline{\mathcal{N}'}$. We see again that the inequality (2.5) holds.

It remains to consider the case when $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}'}$ differ with some non-0-reduced identity $u = v$. Suppose that $c(u) \neq c(v)$. Lemma 2.6(i) then implies that the variety $\overline{\mathcal{N}}$ satisfies both the identities $u = 0$ and $v = 0$. Then the variety $\overline{\mathcal{N}'}$ does not satisfy at least one of them because $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}'}$ do not differ with $u = v$ otherwise. We see that $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}'}$ differ with some 0-reduced identity, and we go to the situation considered in the previous paragraph. Let now $c(u) = c(v)$. Suppose that the identity $u = 0$ holds in $\mathcal{Q}$. Since $\overline{\mathcal{N}}, \overline{\mathcal{N}'} \subseteq \mathcal{Q}$, we have that the identity $u = 0$ holds in both the varieties $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}'}$. Then $\overline{\mathcal{N}}$ satisfies also the identity $v = 0$. But $v = 0$ fails in $\overline{\mathcal{N}'}$ because $u = v$ holds in $\overline{\mathcal{N}'}$ otherwise. We see that the varieties $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}'}$ differ with some 0-reduced identity. This case has been already considered in the previous paragraph. Thus the identity $u = 0$ fails in $\mathcal{Q}$. Analogously, $v = 0$ fails in $\mathcal{Q}$. Since the identity $u = v$ is non-trivial and $c(u) = c(v)$, Lemma 2.12 implies that the words u and v are linear. Using the fact that $c(u) = c(v)$ again, we have that the identity $u = v$ is permutative. Then it is evident that this identity holds in $\mathcal{G} \vee \mathcal{C}_m \vee \overline{\mathcal{N}}$ but fails in $\mathcal{G}' \vee \mathcal{C}_k \vee \overline{\mathcal{N}'}$. We prove that the inequality (2.5) holds.

Thus, (2.5) fulfills always, whence we are done.   □

Analog of Proposition 2.10 for semigroup varieties was proved in [21, Proposition 2a] (namely it was checked there that $L(\mathcal{G} \vee \mathcal{C}_2 \vee \mathcal{Q}) \cong L(\mathcal{G}) \times L(\mathcal{C}_2 \vee \mathcal{Q})$ where $\mathcal{G}$ is an Abelian periodic group variety). The proof of Proposition 2.10 given above is quite similar to the proof of the mentioned result from [21]. But results of [21] was not used directly above. Therefore, the mentioned result from [21] may be considered now as a consequence of Proposition 2.10.

2.6. **Varieties of finite degree.** If n is a natural number then a variety $\mathcal{X}$ is called a *variety of degree n* if all nil-semigroups in $\mathcal{X}$ are nilpotent of degree $\leq n$ and n is the least number with such a property. If $\mathcal{X}$ is not a variety of degree $\leq n$, we will say that $\mathcal{X}$ is a variety of *degree $> n$*. A variety is said to be a *variety of finite degree* if it is a variety of degree n for some n. If $\mathcal{V}$ is a variety of finite degree, we denote the degree of $\mathcal{V}$ by $\deg(\mathcal{V})$; otherwise we put $\deg(\mathcal{V}) = \infty$. We need the following

Proposition 2.13 ([4, Corollary 1.3]). *Let n be an arbitrary natural number. For an epigroup variety $\mathcal{V}$, the following are equivalent:*

1) $\deg(\mathcal{V}) \leq n$;
2) $\mathcal{V} \not\supseteq \mathrm{var}\{x^2 = x_1 x_2 \cdots x_{n+1} = 0,\ xy = yx\}$;
3) $\mathcal{V}$ *satisfies an identity of the form*

$$(2.6) \qquad x_1 \cdots x_n = x_1 \cdots x_{i-1} \cdot \overline{\overline{x_i \cdots x_j}} \cdot x_{j+1} \cdots x_n$$

    for some i and j with $1 \leq i \leq j \leq n$. $\qquad\square$

Proposition 2.13 readily implies

Corollary 2.14. $\deg(\mathcal{X} \wedge \mathcal{Y}) = \min\{\deg(\mathcal{X}), \deg(\mathcal{Y})\}$ *for arbitrary epigroup varieties $\mathcal{X}$ and $\mathcal{Y}$.* $\qquad\square$

The following corollary may be proved quite analogously to Corollary 2.13 of [25] with referring to Proposition 2.13 rather than Proposition 2.11 of [25].

Corollary 2.15. *If $\mathcal{V}$ is an arbitrary epigroup variety and $\mathcal{N}$ is a nilvariety then $\deg(\mathcal{V} \vee \mathcal{N}) = \max\{\deg(\mathcal{V}), \deg(\mathcal{N})\}$.* $\qquad\square$

Note that the analog of Corollary 2.15 for arbitrary epigroup varieties is wrong even in the periodic case. For instance, it is easy to deduce from Lemma 2.5(iii), the dual fact and Proposition 2.13 that $\deg(\mathcal{P}) = \deg(\overleftarrow{\mathcal{P}}) = 2$ but $\deg(\mathcal{P} \vee \overleftarrow{\mathcal{P}}) = 3$.

Proposition 2.13 and Lemma 2.1 easily imply

Corollary 2.16. *If $\mathcal{V}$ is an arbitrary epigroup variety and $\mathcal{X}$ is a completely regular variety then $\deg(\mathcal{V} \vee \mathcal{X}) = \deg(\mathcal{V})$.* $\qquad\square$

2.7. **Some properties of special elements of lattices.** The following claim is well known.

Lemma 2.17 ([3, Lemma II.1.1]). *If an element of a lattice L is distributive and modular in L then it is standard in L.* $\qquad\square$

This fact together with [2, Theorem 255(iii)] imply the following

Lemma 2.18. *An element of a lattice L is neutral in L if and only if it is distributive, codistributive and modular in L.* $\qquad\square$

Let I be a lattice identity of the form $s = t$ where s and t are lattice terms depending on ordering set of variables $x_0, x_1, \ldots, x_n$. An element x of a lattice L is called an *I-element*[2] if $s(x, x_1, \ldots, x_n) = t(x, x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in$

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[2]Probably, it would be more correct to say about (I, σ)-*elements* where $\sigma = (x_0, x_1, \ldots, x_n)$ is a permutation on the set of variables occurring in I.

L. Special elements of all types mentioned above are I-elements for appropriate identities I and ordering of variables occurring in I (this claim is evident for neutral, [co]standard and [co]distributive elements, and follows from the fact that the modular law may be written in the form of identity for modular, upper-modular and lower-modular elements).

Lemma 2.19 ([11, Corollary 2.1]). *Let I be a non-trivial lattice identity, L a lattice with 0, $x \in L$ and a an atom and a neutral element of the lattice L. Then x is an I-element in L if and only if $x \vee a$ has the same property.* $\square$

2.8. $\mathcal{SL}$ and $\mathcal{ZM}$ are atoms. It is evident that atoms of the lattice **EPI** coincide with atoms of the lattice **SEM**. The list of atoms of the latter lattice is generally known (see [17], for instance). In particular, the following is valid.

Lemma 2.20. *The varieties $\mathcal{SL}$ and $\mathcal{ZM}$ are atoms of the lattice* **EPI**. $\square$

3. The varieties $\mathcal{SL}$ and $\mathcal{ZM}$ are neutral

Proposition 3.1. *The variety $\mathcal{SL}$ is a neutral element of the lattice* **EPI**.

Proof. In view of Lemma 2.18, it suffices to verify that the variety $\mathcal{SL}$ is distributive, codistributive and modular. Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary epigroup varieties.

Distributivity. We need to verify the inclusion

$$(\mathcal{SL} \vee \mathcal{X}) \wedge (\mathcal{SL} \vee \mathcal{Y}) \subseteq \mathcal{SL} \vee (\mathcal{X} \wedge \mathcal{Y})$$

because the opposite inclusion is evident. Suppose that the identity $u = v$ holds in $\mathcal{SL} \vee (\mathcal{X} \wedge \mathcal{Y})$. In particular, it holds in $\mathcal{SL}$, whence $c(u) = c(v)$ by Lemma 2.5(i). Let $u \equiv w_0, w_1, \ldots, w_n \equiv v$ be a deduction of this identity from identities of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Further considerations are given by induction on n.

Induction base. If $n = 1$ then the identity $u = v$ holds in one of the varieties $\mathcal{X}$ or $\mathcal{Y}$. Whence, it holds in one of the varieties $\mathcal{SL} \vee \mathcal{X}$ or $\mathcal{SL} \vee \mathcal{Y}$, and therefore $(\mathcal{SL} \vee \mathcal{X}) \wedge (\mathcal{SL} \vee \mathcal{Y})$ satisfies $u = v$.

Induction step. Let now $n > 1$. Consider the words $w'_1, \ldots, w'_{n-1}$ obtained from the words $w_1, \ldots, w_{n-1}$ respectively by equating all the letters that are not occur in u, for some letter of $c(u)$. Clearly, the sequence of words $u, w'_1, \ldots, w'_{n-1}, v$ also is a deduction of the identity $u = v$ from identities of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Thus, we may assume that $c(w_1), \ldots, c(w_{n-1}) \subseteq c(u)$. If $c(w_0) = c(w_1) = \cdots = c(w_n)$ then the sequence $w_0, w_1, \ldots, w_n$ is a deduction of the identity $u = v$ from identities of the varieties $\mathcal{SL} \vee \mathcal{X}$ and $\mathcal{SL} \vee \mathcal{Y}$, and we are done. Suppose now that $c(w_k) \neq c(w_{k+1})$ for some $0 \leq k \leq n - 1$. Let i be the least index with $c(w_i) \neq c(w_{i+1})$ and j be the greatest index with $c(w_j) \neq c(w_{j-1})$. Suppose that $i > 0$. Then $c(w_i) = c(u) = c(v)$ and consequences of words $w_0, w_1, \ldots, w_i$ and $w_i, w_{i+1}, \ldots, w_n$ are deductions of the identities $u = w_i$ and $w_i = v$ respectively from the identities of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Lemma 2.5(i) implies now that the identities $u = w_i$ and $w_i = v$ hold in the variety $\mathcal{SL} \vee (\mathcal{X} \wedge \mathcal{Y})$. By induction assumption these identities hold also in the variety $(\mathcal{SL} \vee \mathcal{X}) \wedge (\mathcal{SL} \vee \mathcal{Y})$. Whence, the last variety satisfies the

identity $u = v$ too. The case when $j < n$ may be considered quite analogously. Thus, we may suppose that $i = 0$ and $j = n$. In other words, $c(u) \neq c(w_1)$ and $c(v) \neq c(w_{n-1})$.

The identity $u = w_1$ holds in one of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Suppose that it holds in $\mathcal{X}$. Since $c(u) \neq c(w_1)$, Lemma 2.5(i) implies that $\mathcal{SL} \nsubseteq \mathcal{X}$. Let S be an epigroup in $\mathcal{X}$ and ζ a homomorphism from F to S. For a word w, we denote by w^ζ the image of w under ζ. It is well known (see [15, 16], for instance) that a variety that does not contain $\mathcal{SL}$ consists of archimedean epigroups. Further, a set of group elements in an archimedean epigroup is an ideal of this epigroup. In particular, this is the case for the epigroup S. Now we are going to check that $u^\zeta \in \mathrm{Gr}\, S$. Since $c(w_1) \subset c(u)$, there is a letter $x \in c(u) \setminus c(w_1)$. Substituting x^ω for x in the identity $u = w_1$, we obtain the identity $u' = w_1$ that holds in $\mathcal{X}$. Therefore, $\mathcal{X}$ satisfies the identity $u = u'$. The word u' contains a subword x^ω. Since $(x^\omega)^\zeta \in \mathrm{Gr}\, S$ and $\mathrm{Gr}\, S$ is an ideal in S, we have that $u^\zeta \in \mathrm{Gr}\, S$. Therefore, $\mathcal{X}$ satisfies the identity $u = uu^\omega$. Similar arguments show that the identity $u = uu^\omega$ holds in $\mathcal{Y}$ whenever $\mathcal{Y}$ satisfies $u = w_1$, and that one of the varieties $\mathcal{X}$ and $\mathcal{Y}$ satisfies the identity $v = vv^\omega$. Therefore, the sequence of words

$$ u, uu^\omega, w_1 u^\omega, \ldots, w_{n-1} u^\omega, vu^\omega, vw_1^\omega, \ldots, vw_{n-1}^\omega, vv^\omega, v $$

is a deduction of the identity $u = v$ from identities of the varieties $\mathcal{SL} \vee \mathcal{X}$ and $\mathcal{SL} \vee \mathcal{Y}$. Hence this identity holds in $(\mathcal{SL} \vee \mathcal{X}) \wedge (\mathcal{SL} \vee \mathcal{Y})$.

Codistributivity. In view of Lemma 2.20, if $\mathcal{W}$ is an arbitrary epigroup variety then either $\mathcal{W} \supseteq \mathcal{SL}$ or $\mathcal{W} \wedge \mathcal{SL} = \mathcal{T}$. We need to verify that

$$ \mathcal{SL} \wedge (\mathcal{X} \vee \mathcal{Y}) = (\mathcal{SL} \wedge \mathcal{X}) \vee (\mathcal{SL} \wedge \mathcal{Y}). $$

Clearly, both the parts of this equality coincides with $\mathcal{SL}$ whenever at least one of the varieties $\mathcal{X}$ or $\mathcal{Y}$ contains $\mathcal{SL}$. It remains to verify that if $\mathcal{X} \nsupseteq \mathcal{SL}$ and $\mathcal{Y} \nsupseteq \mathcal{SL}$ then $\mathcal{X} \vee \mathcal{Y} \nsupseteq \mathcal{SL}$. This claim immediately follows from the fact that there is a non-trivial identity $u = v$ such that an epigroup variety $\mathcal{W}$ does not contain the variety $\mathcal{SL}$ if and only if $\mathcal{W}$ satisfies the identity $u = v$ (in particular, the identity $(x^\omega y^\omega x^\omega)^\omega = x^\omega$ has such a property, see [16, Corollary 3.2], for instance).

Modularity. Let $\mathcal{X} \subseteq \mathcal{Y}$. We need to verify that

$$ (\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y} \subseteq (\mathcal{SL} \wedge \mathcal{Y}) \vee \mathcal{X} $$

because the opposite inclusion is evident. If $\mathcal{SL} \subseteq \mathcal{Y}$ then both the parts of the inclusion evidently coincides with $\mathcal{SL} \vee \mathcal{X}$. Let now $\mathcal{SL} \nsubseteq \mathcal{Y}$. Then Lemma 2.20 implies that $\mathcal{SL} \wedge \mathcal{Y} = \mathcal{T}$, whence $(\mathcal{SL} \wedge \mathcal{Y}) \vee \mathcal{X} = \mathcal{X}$. Suppose that an identity $u = v$ holds in the variety $\mathcal{X}$. It suffices to verify that this identity holds in $(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y}$ too. If $c(u) = c(v)$ then $u = v$ holds in $\mathcal{SL}$ by Lemma 2.5(i). Whence it satisfies in $\mathcal{SL} \vee \mathcal{X}$, and we are done. Let now $c(u) \neq c(v)$. Since $\mathcal{SL} \nsubseteq \mathcal{Y}$, Lemma 2.5(i) implies that $\mathcal{Y}$ satisfies an identity $s = t$ with $c(s) \neq c(t)$. We may assume without any loss that there is a letter $y \in c(t) \setminus c(s)$. Moreover, we may assume that $c(s) = \{x\}$ and $c(t) = \{x, y\}$ (if this is not the case then we equate all letters but y to x in the identity $s = t$ and multiply the identity we obtain on x from the right). Let s_1 [respectively t_1] be the word obtained

from the word s [respectively t] by replacing the letters x and y each to other. Clearly, the identity $s_1 = t_1$ follows from the identity $s = t$.

Consider the case when the word v may be obtained from u by replacing of one letter to another one. We may assume without loss of generality that we substitute the letter y for the letter x. Let u_1, u_2, u_3 and u_4 be the words obtained from u by substitution of the words s, s_1, t and t_1 respectively for the letter x. Since $x \notin c(v)$, the identities $u_1 = v$, $u_2 = v$, $u_3 = v$ and $u_4 = v$ are obtained from $u = v$ by the same substitutions. Therefore, the words u, v, u_1, u_2, u_3 and u_4 are equal each to other in the variety $\mathcal{X}$. Further, $c(u) = c(u_1)$, $c(v) = c(u_2)$ and $c(u_3) = c(u_4)$, whence the identities $u = u_1$, $v = u_2$ and $u_3 = u_4$ hold in $\mathcal{SL} \vee \mathcal{X}$ by Lemma 2.5(i). The identities $u_1 = u_3$ and $u_2 = u_4$ follow from $s = t$ and $s_1 = t_1$ respectively. Hence these identities hold in $\mathcal{Y}$. Thus, the sequence of words u, u_1, u_3, u_4, u_2, v is a deduction of the identity $u = v$ from the identities of the varieties $\mathcal{SL} \vee \mathcal{X}$ and $\mathcal{Y}$.

Finally, consider an arbitrary identity $u = v$ that holds in $\mathcal{X}$. Replacing one by one all the letters from $c(u) \setminus c(v)$ by some letters of $c(v)$, we obtain the sequence $u, w_1, \ldots, w_m$, in which any adjacent words differ by replacing one letter. In the sequence of identities $u = v, w_1 = v, \ldots, w_m = v$, every identity (except the first one) is obtained from the previous one by replacing one letter. Therefore the words $u, w_1, \ldots, w_m$ are equal each to other in $\mathcal{X}$. As we have proved above, this implies that the identities $u = w_1 = \cdots = w_m$ hold in $(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y}$. Analogously, if we replace in the identity $v = w_m$ all letters from $c(v) \setminus c(w_m)$ by an arbitrary letter from $c(w_m)$, then we obtain a sequence of identities $v = w_1' = \cdots = w_n'$ that hold in $(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y}$ as well. In particular, these identities hold in $\mathcal{X}$. Moreover, since $c(w_m) = c(w_n')$, Lemma 2.5(i) implies that the identity $w_m = w_n'$ holds in $\mathcal{SL} \vee \mathcal{X}$. Therefore, the identity $u = v$ holds in $(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y}$. $\square$

Proposition 3.2. *The variety $\mathcal{ZM}$ is a neutral element of the lattice* **EPI**.

Proof. In view of Lemma 2.18, it suffices to check that $\mathcal{ZM}$ is distributive, codistributive and modular. Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary epigroup varieties.

Distributivity. We need to verify that

$$(\mathcal{ZM} \vee \mathcal{X}) \wedge (\mathcal{ZM} \vee \mathcal{Y}) \subseteq \mathcal{ZM} \vee (\mathcal{X} \wedge \mathcal{Y})$$

because the opposite inclusion is evident. Suppose that an identity $u = v$ holds in $\mathcal{ZM} \vee (\mathcal{X} \wedge \mathcal{Y})$. We aim to check that this identity is satisfied by the variety $(\mathcal{ZM} \vee \mathcal{X}) \wedge (\mathcal{ZM} \vee \mathcal{Y})$. The identity $u = v$ holds in $\mathcal{ZM}$ and there is a deduction of this identity from identities of the varieties $\mathcal{X}$ and $\mathcal{Y}$. In other words, there are words $u_0, u_1, \ldots, u_n$ such that $u_0 \equiv u$, $u_n \equiv v$ and, for each $i = 0, 1, \ldots, n - 1$, the identity $u_i = u_{i+1}$ holds in one of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Let $u_0, u_1, \ldots, u_n$ be the shortest sequence of words with such properties. If all the words $u_0, u_1, \ldots, u_n$ are not letters then $u_0 = u_1 = \cdots = u_n$ holds in $\mathcal{ZM}$. This means that the sequence of words $u_0, u_1, \ldots, u_n$ is a deduction of the identity $u = v$ from identities of the varieties $\mathcal{ZM} \vee \mathcal{X}$ and $\mathcal{ZM} \vee \mathcal{Y}$, whence $u = v$ holds in $(\mathcal{ZM} \vee \mathcal{X}) \wedge (\mathcal{ZM} \vee \mathcal{Y})$. Let now i be an index such that $u_i \equiv x$ for some letter x. Clearly, $0 < i < n$ because the variety $\mathcal{ZM}$ satisfies the identity $u_0 = u_n$ but does not satisfy any identity of the kind

$x = w$. The identity $u_{i-1} = x$ holds in one of the varieties $\mathcal{X}$ and $\mathcal{Y}$, say in $\mathcal{X}$. Then $\mathcal{Y}$ satisfies the identity $x = u_{i+1}$. Since both the identities $u_{i-1} = x$ and $x = u_{i+1}$ fail in $\mathcal{ZM}$, we have that $\mathcal{ZM}$ is contained neither in $\mathcal{X}$ nor in $\mathcal{Y}$. Therefore, the varieties $\mathcal{X}$ and $\mathcal{Y}$ are completely regular. By Lemma 2.1 each of the identities $u_0 = \overline{\overline{u_0}}$ and $\overline{\overline{u_n}} = u_n$ holds in one of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Further, for each $i = 0, 1, \ldots, n-1$ one of the varieties $\mathcal{X}$ and $\mathcal{Y}$ satisfies the identity $\overline{\overline{u_i}} = \overline{\overline{u_{i+1}}}$. The words u_0 and u_n are not letters, whence the variety $\mathcal{ZM}$ satisfies the identities $u_0 = \overline{\overline{u_0}} = \overline{\overline{u_1}} = \cdots = \overline{\overline{u_n}} = u_n$. Summarizing all we say, we obtain that the sequence of words $u_0, \overline{\overline{u_0}}, \overline{\overline{u_1}}, \ldots, \overline{\overline{u_n}}, u_n$ is a deduction of the identity $u = v$ from the identities of the varieties $\mathcal{ZM} \vee \mathcal{X}$ and $\mathcal{ZM} \vee \mathcal{Y}$. Therefore, this identity holds in $(\mathcal{ZM} \vee \mathcal{X}) \wedge (\mathcal{ZM} \vee \mathcal{Y})$.

Codistributivity. In view of Lemma 2.20, if $\mathcal{W}$ is an arbitrary epigroup variety then either $\mathcal{W} \supseteq \mathcal{ZM}$ or $\mathcal{W} \wedge \mathcal{ZM} = \mathcal{T}$. We need to verify that

$$\mathcal{ZM} \wedge (\mathcal{X} \vee \mathcal{Y}) = (\mathcal{ZM} \wedge \mathcal{X}) \vee (\mathcal{ZM} \wedge \mathcal{Y}).$$

Clearly, both the parts of this equality equals $\mathcal{ZM}$ whenever at least one of the varieties $\mathcal{X}$ or $\mathcal{Y}$ contains $\mathcal{ZM}$. It remains to verify that if $\mathcal{X} \not\supseteq \mathcal{ZM}$ and $\mathcal{Y} \not\supseteq \mathcal{ZM}$ then $\mathcal{X} \vee \mathcal{Y} \not\supseteq \mathcal{ZM}$. This claim immediately follows from the well-known fact that an epigroup variety $\mathcal{W}$ does not contain the variety $\mathcal{ZM}$ if and only if $\mathcal{W}$ is completely regular.

Modularity. For any epigroup variety $\mathcal{X}$, we put $\mathrm{CR}(\mathcal{X}) = \mathcal{CR} \wedge \mathcal{X}$ where $\mathcal{CR}$ is the variety of all completely regular epigroups. Suppose that $\mathcal{X} \subseteq \mathcal{Y}$. We need to prove that

(3.1) $(\mathcal{ZM} \vee \mathcal{X}) \wedge \mathcal{Y} \subseteq (\mathcal{ZM} \wedge \mathcal{Y}) \vee \mathcal{X}$

because the opposite inclusion is evident. If $\mathcal{ZM} \subseteq \mathcal{X}$ [respectively $\mathcal{ZM} \subseteq \mathcal{Y}$] then both the parts of the inclusion (3.1) coincide with $\mathcal{X}$ [with $\mathcal{ZM} \vee \mathcal{X}$]. Let now $\mathcal{ZM} \not\subseteq \mathcal{X}$ and $\mathcal{ZM} \not\subseteq \mathcal{Y}$. Then the varieties $\mathcal{X}$ and $\mathcal{Y}$ are completely regular. Therefore,

$$(\mathcal{ZM} \vee \mathcal{X}) \wedge \mathcal{Y} \subseteq \mathrm{CR}(\mathcal{ZM} \vee \mathcal{X}).$$

Further, $\mathcal{ZM} \wedge \mathcal{Y} = \mathcal{T}$, whence the right part of the inclusion (3.1) coincides with $\mathcal{X}$. Let $u = v$ be an identity that holds in $\mathcal{X}$. Lemmas 2.1 and 2.3 imply that $\mathcal{ZM} \vee \mathcal{X}$ satisfies the identity $\overline{\overline{u}} = \overline{\overline{v}}$. Therefore, $u = v$ in $\mathrm{CR}(\mathcal{ZM} \vee \mathcal{X})$ by Lemma 2.1. We have proved that $\mathrm{CR}(\mathcal{ZM} \vee \mathcal{X}) \subseteq \mathcal{X}$, whence

$$(\mathcal{ZM} \vee \mathcal{X}) \wedge \mathcal{Y} \subseteq \mathrm{CR}(\mathcal{ZM} \vee \mathcal{X}) \subseteq \mathcal{X} = \mathcal{T} \vee \mathcal{X} = (\mathcal{ZM} \wedge \mathcal{Y}) \vee \mathcal{X}.$$

Proposition is proved. $\square$

For convenience of references, we formulate the following fact that immediately follows from Lemmas 2.19 and 2.20 and Propositions 3.1 and 3.2.

Corollary 3.3. *Let I be a non-trivial lattice identity and $\mathcal{W}$ is one of the varieties $\mathcal{SL}$, $\mathcal{ZM}$ or $\mathcal{SL} \vee \mathcal{ZM}$. An epigroup variety $\mathcal{X}$ is an I-element of the lattice* **EPI** *if and only if the variety $\mathcal{X} \vee \mathcal{W}$ has the same property.* $\square$

4. Upper-modular varieties

Here we verify Theorem 1.5. To do this, we need several auxiliary statements.

Lemma 4.1. *If a strongly permutative epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **EPI** *then $\mathcal{V}$ is commutative.*

Proof. In view of Corollary 2.9, $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an Abelian group variety, $m \geq 0$ and $\mathcal{N}$ is a nilvariety. If $\deg(\mathcal{V}) \leq 2$ then $\mathcal{N} \subseteq \mathcal{ZM}$, and we are done. Let now $\deg(\mathcal{V}) > 2$. By Proposition 2.13 $\mathcal{V}$ contains the variety $\mathcal{X} = \mathrm{var}\{x^2 = xyz = 0,\ xy = yx\}$. Suppose that $\mathcal{V}$ is not commutative. Let $\mathcal{G}'$ be a non-abelian group variety. Since $\mathcal{V}$ is strongly permutative, every group in $\mathcal{V}$ is abelian. Therefore, the variety $(\mathcal{G}' \wedge \mathcal{V}) \vee \mathcal{X}$ is commutative. Since $\mathcal{X} \subseteq \mathcal{V}$ and the variety $\mathcal{V}$ is upper-modular, we have that $(\mathcal{G}' \wedge \mathcal{V}) \vee \mathcal{X} = (\mathcal{G}' \vee \mathcal{X}) \wedge \mathcal{V}$. We see that the variety $(\mathcal{G}' \vee \mathcal{X}) \wedge \mathcal{V}$ is commutative. Hence there is a deduction of the identity $xy = yx$ from the identities of the varieties $\mathcal{G}' \vee \mathcal{X}$ and $\mathcal{V}$. In particular, there is a word v such that $v \not\equiv xy$ and the identity $xy = v$ holds either in $\mathcal{G}' \vee \mathcal{X}$ or in $\mathcal{V}$. The claims (i) and (iii) of Lemma 2.6 imply that a variety with the identity $xy = v$ is either commutative or a variety of degree ≤ 2. The variety $\mathcal{G}' \vee \mathcal{X}$ is neither commutative (because $\mathcal{G}'$ is non-abelian) nor a variety of degree ≤ 2 (because $\deg(\mathcal{X}) > 2$). Since $\deg(\mathcal{V}) > 2$, we have that $\mathcal{V}$ is commutative. $\square$

A semigroup variety is called *proper* if it differs from the variety of all semigroups. It is proved in [25, Theorem 1.1] that if $\mathcal{V}$ is a proper upper-modular in **SEM** variety then, first, $\mathcal{V}$ is periodic[3], and, second, every nilsubvariety of $\mathcal{V}$ is commutative and satisfies the identity (1.1). As we have already mentioned in Subsection 1.2, the epigroup analog of the first claim is not true. Our next step is the following partial epigroup analog of the second claim.

Proposition 4.2. *If a strongly permutative epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **EPI** *then every nil-semigroup in $\mathcal{V}$ satisfies the identity* (1.1).

Proof. According to Lemma 4.1 the variety $\mathcal{V}$ is commutative. If all nil-semigroups in $\mathcal{V}$ are singleton then the desirable conclusion is evident. Suppose now that $\mathcal{V}$ contains a non-singleton nil-semigroup N. Let $\mathcal{N}$ be the variety generated by N. Clearly, this variety is commutative. It is evident that $\mathcal{ZM} \subseteq \mathcal{N}$. We need to verify that $\mathcal{N}$ satisfies the identity (1.1). Put

$$\mathcal{I} = \mathrm{var}\{x^2 y = xy^2,\ xy = yx,\ x^2 yz = 0\}$$

and $\mathcal{N}' = \mathcal{N} \wedge \mathcal{I}$. It is clear that $\mathcal{ZM} \subseteq \mathcal{I}$, whence $\mathcal{ZM} \subseteq \mathcal{N}'$.

A semigroup analog of the proposition we verify is proved in [25] (see the last paragraph of Section 3 in that article). The arguments used there are based on the fact that there is a variety $\mathcal{X}$ such that the following two claims are valid:

    (i) $(\mathcal{X} \vee \mathcal{N}') \wedge \mathcal{V} \subseteq \mathcal{I}$;

---

[3]Note that this is a very special case of the following result obtained in [13, Theorem 1]: if I is a non-trivial lattice identity then a proper semigroup variety is periodic whenever it is an I-element of the lattice **SEM**.

(ii) if $v \in \{x^2y, xyx, yx^2\}$ and $w \in \{xy^2, yxy, y^2x\}$ then the identity $v = w$ fails in $\mathcal{X}$.

In [25] some periodic group variety plays the role of $\mathcal{X}$. Here we should take another $\mathcal{X}$. Namely, put $\mathcal{X} = \mathcal{LZM} \vee \mathcal{RZM}$ where

$$\mathcal{LZM} = \mathrm{var}\{xyz = xy\} \quad \text{and} \quad \mathcal{RZM} = \mathrm{var}\{xyz = yz\}.$$

The variety $\mathcal{X}$ satisfies the identity $xyzxy = xy$. Therefore, Lemma 2.5(i) implies that $\mathcal{SL} \not\subseteq \mathcal{X}$. Further, substituting 1 for x and y in the identity $xyzxy = xy$, we obtain that all groups in $\mathcal{X}$ are singleton. Hence every commutative semigroup in $\mathcal{X}$ is a nil-semigroup. Further, $\mathcal{X}$ satisfies the identity $xy = (xy)^2$, whence all nil-semigroups in $\mathcal{X}$ lie in $\mathcal{ZM}$ by Lemma 2.6(ii). Since the variety $\mathcal{X} \wedge \mathcal{V}$ is commutative, $\mathcal{X} \wedge \mathcal{V} \subseteq \mathcal{ZM}$. The variety $\mathcal{V}$ is upper-modular and $\mathcal{N}' \subseteq \mathcal{V}$. Therefore,

$$(\mathcal{X} \vee \mathcal{N}') \wedge \mathcal{V} = (\mathcal{X} \wedge \mathcal{V}) \vee \mathcal{N}' \subseteq \mathcal{ZM} \vee \mathcal{N}' = \mathcal{N}' \subseteq \mathcal{I}.$$

We have proved the claim (i). To verify the claim (ii), we note that if $\mathcal{X}$ satisfies a semigroup identity $v = w$ then the words v and w have the same prefix of length 2 and the same suffix of length 2. Clearly, this is not the case whenever $v \in \{x^2y, xyx, yx^2\}$ and $w \in \{xy^2, yxy, y^2x\}$. Now we can complete the proof by the same arguments as in the last paragraph of [25, Section 3]. □

The proof of the following statement repeats almost literally the 'only if' part of the proof of Theorem 2 in [33].

Proposition 4.3. *If a nilvariety of epigroups $\mathcal{X}$ satisfies the identities* (1.1) *and $xy = yx$ then $\mathcal{X}$ is an upper-modular element of the lattice* **EPI**.

Proof. It is easy to prove (see [33, Lemma 2.7], for instance) that $\mathcal{X}$ satisfies the identity $x^2yz = 0$. Thus, $\mathcal{X} \subseteq \mathcal{I}$. Put

$$U = \{x^2, x^3, x^2y, x_1x_2 \cdots x_n \mid n \in \mathbb{N}\}.$$

It is evident that any subvariety of $\mathcal{I}$ may be given in $\mathcal{I}$ only by identities of the type $u = v$ or $u = 0$ where $u, v \in U$. Lemma 2.6 implies that if $u, v \in U$ and $u \not\equiv v$ then $u = v$ implies in $\mathcal{I}$ the identity $u = 0$. Now it is very easy to check that the lattice $L(\mathcal{I})$ has the form shown on Fig. 2 where

$\mathcal{I}_n = \mathrm{var}\{x^2yz = x_1x_2 \cdots x_n = 0, \ x^2y = xy^2, \ xy = yx\}$ where $n \geq 4$,

$\mathcal{J} = \mathrm{var}\{x^2yz = x^3 = 0, \ x^2y = xy^2, \ xy = yx\}$,

$\mathcal{J}_n = \mathrm{var}\{x^2yz = x^3 = x_1x_2 \cdots x_n = 0, \ x^2y = xy^2, \ xy = yx\}$ where $n \geq 4$,

$\mathcal{K} = \mathrm{var}\{x^2y = 0, \ xy = yx\}$,

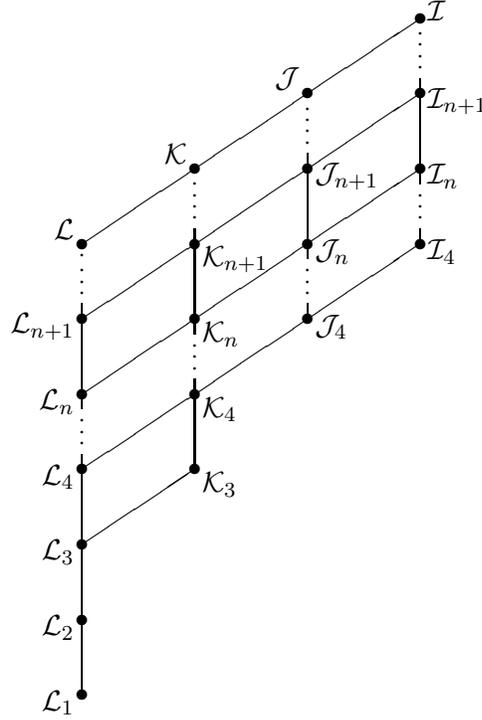$\mathcal{K}_n = \mathrm{var}\{x^2y = x_1x_2 \cdots x_n = 0, \ xy = yx\}$ where $n \geq 3$,

$\mathcal{L} = \mathrm{var}\{x^2 = 0, \ xy = yx\}$,

$\mathcal{L}_n = \mathrm{var}\{x^2 = x_1x_2 \cdots x_n = 0, \ xy = yx\}$ where $n \in \mathbb{N}$.

Note that $\mathcal{L}_1 = \mathcal{T}$ and $\mathcal{L}_2 = \mathcal{ZM}$.

Let $\mathcal{X} \subseteq \mathcal{I}$. We have to check that if $\mathcal{Y} \subseteq \mathcal{X}$ and $\mathcal{Z}$ is an arbitrary epigroup variety then $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X} = (\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$. For a variety $\mathcal{M}$ with $\mathcal{M} \subseteq \mathcal{I}$, we denote by $\mathcal{M}^*$ the least of the varieties $\mathcal{I}, \mathcal{J}, \mathcal{K}$ and $\mathcal{L}$ that contains $\mathcal{M}$. Fig. 2

FIGURE 2. The lattice $L(\mathcal{I})$

shows that if $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{I}$ then $\mathcal{M}_1 = \mathcal{M}_2$ if and only if $\deg(\mathcal{M}_1) = \deg(\mathcal{M}_2)$ and $\mathcal{M}_1^* = \mathcal{M}_2^*$. Therefore, we have to verify the following two equalities:

$$(4.1) \qquad \deg\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}\big) = \deg\big((\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}\big),$$

$$(4.2) \qquad \big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}\big)^* = \big((\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}\big)^*.$$

The equality (4.1). Put $\deg(\mathcal{X}) = k$, $\deg(\mathcal{Y}) = \ell$ and $\deg(\mathcal{Z}) = m$. Clearly, $\ell \le k$ because $\mathcal{Y} \subseteq \mathcal{X}$. The set of all natural numbers with the operations min and max is a distributive lattice. Now Corollaries 2.14 and 2.15 apply and we conclude that

$$\deg\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}\big) = \min\{\max\{m, \ell\}, k\} = \max\{\min\{m, k\}, \min\{\ell, k\}\}$$
$$= \max\{\min\{m, k\}, \ell\} = \deg\big((\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}\big).$$

The equality (4.1) is proved.

The equality (4.2). Clearly, this equality is equivalent to the following claim: if u is one of the words x^3, x^2y and x^2 then the variety $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$ satisfies the identity $u = 0$ if and only if the variety $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$ does so. It suffices to verify that $u = 0$ holds in $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$ whenever it is so in $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$ because the opposite claim immediately follows from the evident inclusion $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y} \subseteq (\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$. Further considerations are divided into two cases.

Case 1: $u \equiv x^n$ where $n \in \{2, 3\}$. Then $x^n = 0$ in $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$. This means that $x^n = 0$ in $\mathcal{Y}$ and there is a deduction of the identity $x^n = 0$ from the identities of the varieties $\mathcal{Z}$ and $\mathcal{X}$. In particular, there is a word v such that $v \not\equiv x^n$ and $x^n = v$ holds in either $\mathcal{Z}$ or $\mathcal{X}$. If $x^n = v$ in $\mathcal{X}$ then the fact that $\mathcal{X}$ is a nil-variety together with the claims (i) and (ii) of Lemma 2.6 imply that $x^n = 0$ in $\mathcal{X}$, and moreover in $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$. Let now $x^n = v$ in $\mathcal{Z}$. The case when v is a semigroup word may be considered by the same arguments as in the Case 1 in the 'only if' part of the proof of Theorem 2 in [33]. Let now v is a non-semigroup word. Then, in view of Lemma 2.3, the identity $v = 0$ holds in the variety $\mathcal{X}$, and therefore in $\mathcal{Y}$. Recall that $x^n = 0$ in $\mathcal{Y}$. Therefore, the identity $x^n = v$ holds in $\mathcal{Y}$. Since this identity holds in $\mathcal{Z}$ as well, we obtain that it holds in $\mathcal{Y} \vee \mathcal{Z}$. We see that the sequence x^n, v, 0 is a deduction of the identity $x^n = 0$ from the identities of the varieties $\mathcal{Y} \vee \mathcal{Z}$ and $\mathcal{X}$, whence $x^n = 0$ holds in $(\mathcal{Y} \vee \mathcal{Z}) \wedge \mathcal{X}$.

Case 2: $u \equiv x^2 y$. We have to check that if the variety $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$ satisfies the identity (1.2) then the variety $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$ also satisfies this identity. Put

$$W = \{x^2 y, xyx, yx^2, y^2 x, yxy, xy^2\}.$$

The variety $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$ is commutative. Therefore, it suffices to verify that this variety satisfies an identity $w = 0$ for some word $w \in W$. By the hypothesis the variety $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$ satisfies the identity (1.2). This means that this identity holds in $\mathcal{Y}$ and there is a deduction of the identity from identities of the varieties $\mathcal{X}$ and $\mathcal{Z}$. Let $x^2 y \equiv u_0, u_1, \ldots, u_n, 0$ be an arbitrary such deduction. The case when $u_n \in W$ may be considered by the same way as in the 'only if' part of the proof of Theorem 2 in [33].

Let now $u_n \notin W$. Since $u_0 \in W$, there is an index $i > 0$ such that $u_i \notin W$ while $u_{i-1} \in W$. The identity $u_{i-1} = u_i$ holds in one of the varieties $\mathcal{Z}$ and $\mathcal{X}$. If $u_{i-1} = u_i$ holds in $\mathcal{X}$ then $\mathcal{X}$ satisfies the identity $u_{i-1} = 0$ (this follows from [33, Lemma 2.5] whenever u_i is a semigroup word and from Lemma 2.3 otherwise). Therefore, $u_{i-1} = 0$ holds in $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$. Since $u_{i-1} \in W$, we are done.

Finally, suppose that $u_{i-1} = u_i$ holds in $\mathcal{Z}$. If u_i is a semigroup word then we may complete the proof by the same arguments as in the 'only if' part of the proof of Theorem 2 in [33]. Suppose now that u_i is not a semigroup word. Lemma 2.3 implies then that the variety $\mathcal{Y}$ satisfies the identity $u_i = 0$. Hence the identity $u_{i-1} = u_i$ holds in $\mathcal{Y}$, and therefore the variety $\mathcal{Y} \vee \mathcal{Z}$ satisfies this identity. Applying Lemma 2.3 again, we conclude that $u_i = 0$ holds in $\mathcal{X}$. Whence, the sequence u_{i-1}, u_i, 0 is a deduction of the identity $u_{i-1} = 0$ from the identities of the varieties $\mathcal{Y} \vee \mathcal{Z}$ and $\mathcal{X}$. Thus $u_{i-1} = 0$ holds in $(\mathcal{Y} \vee \mathcal{Z}) \wedge \mathcal{X}$, and we are done.

The equality (4.2) is proved. Thus, we have proved Proposition 4.3.   □

Proof of Theorem 1.5. *Necessity.* Let $\mathcal{V}$ be a strongly permutative upper-modular epigroup variety. In view of Corollary 2.9, $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an abelian group variety, $m \geq 0$ and $\mathcal{N}$ is a nilvariety. Lemma 4.1 and Proposition 4.2 imply respectively that $\mathcal{N}$ is commutative and satisfies the identity (1.1). The variety $\mathcal{C}_m$ contains a nilsubvariety $\mathrm{var}\{x^m = 0, \ xy = yx\}$.

Clearly, this variety does not satisfy the identity (1.1) whenever $m \geq 3$. Now Proposition 4.2 applies again and we conclude that $m \leq 2$. If the variety $\mathcal{N}$ satisfies the identity (1.2) then the claim (ii) of Theorem 1.5 holds. Suppose now that the identity (1.2) fails in $\mathcal{N}$. By [35, Lemma 7] this implies that $\mathcal{N}$ contains the variety $\mathcal{J}$. We need to verify that $\mathcal{G} = \mathcal{T}$ and $m \leq 1$. Arguing by contradiction, suppose that either $\mathcal{G} \neq \mathcal{T}$ or $m \geq 2$. Then $\mathcal{V}$ contains a variety of the form $\mathcal{X} \vee \mathcal{J}$ where $\mathcal{X}$ is either a non-trivial abelian group variety or the variety $\mathcal{C}_2$. It is well known and may be easily checked that the variety $\mathcal{C}_2$ is generated by the epigroup

$$C_2 = \langle e, a \mid e^2 = e, \ ea = ae = a, \ a^2 = 0 \rangle = \{e, a, 0\}$$

and e is the unit of C_2. Thus, $\mathcal{M}$ is generated by an epigroup with unit in any case. Suppose that $\mathcal{X}$ satisfies the identity (1.1). Substituting 1 for y in this identity, we have that $x^2 = x$ holds in $\mathcal{X}$. But this identity is false both in a non-trivial group variety and in the variety $\mathcal{C}_2$. As it is verified in the proof of [35, Lemma 8], this implies that (1.1) is false in any nil-semigroup in $\mathcal{X} \vee \mathcal{J}$. But this contradicts Proposition 4.2.

Sufficiency. If $\mathcal{V}$ satisfies the claim (i) of Theorem 1.5 then $\mathcal{V}$ is upper-modular by Proposition 4.3 and Corollary 3.3. Suppose now that $\mathcal{V}$ satisfies the claim (ii). In other words, $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an abelian group variety, $0 \leq m \leq 2$, and $\mathcal{N}$ satisfies the identities $xy = yx$ and (1.2). Let $\mathcal{Y} \subseteq \mathcal{V}$ and $\mathcal{Z}$ an arbitrary epigroup variety. We aim to verify that

$$(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V} = (\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}.$$

As we have already mentioned in the proof of Lemma 2.8, the variety $\mathcal{C}_m$ is generated by the $(m+1)$-element combinatorial cyclic monoid C_m and the set $X = \{m \in \mathbb{N} \mid C_m \in \mathcal{X}\}$ has the greatest element. For any $m \geq 0$, let c_m be a generator of C_m. Put $C = \prod_{m \in X} C_m$. Then the semigroup C is not an epigroup because no power of the element $(\ldots, c_m, \ldots)_{m \in X}$ belongs to a subgroup of C. Thus, the set X has the greatest element. We denote this element by m and put $C(\mathcal{X}) = \mathcal{C}_m$. It is clear that the varieties $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}$ and $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}$ are commutative. In view of Corollary 2.9, it suffices to verify the following three equalities:

$$(4.3) \qquad \mathrm{Gr}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big) = \mathrm{Gr}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big),$$

$$(4.4) \qquad C\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big) = C\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big),$$

$$(4.5) \qquad \mathrm{Nil}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big) = \mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big).$$

The equality (4.3). If $\mathcal{G}$ is a periodic group variety then we denote by $\exp(G)$ the *exponent* of $\mathcal{G}$, that is the least number n such that $\mathcal{G}$ satisfies the identity $x = x^{n+1}$. For a non-periodic group variety $\mathcal{G}$, we put $\exp(\mathcal{G}) = \infty$. Put $\mathcal{G}_1 = \mathrm{Gr}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big)$ and $\mathcal{G}_2 = \mathrm{Gr}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big)$. Since $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{V}$, we have that $\mathcal{G}_1$ and $\mathcal{G}_2$ are abelian group varieties. To prove the equality (4.3), it suffices to verify that $\exp(\mathcal{G}_1) = \exp(\mathcal{G}_2)$. This claim is verified by the same arguments as the analogous claim in the proof of Theorem 1.2 in [25].

The equality (4.4). Here and below we need the following easy remark. It is evident that if $m \geq 3$ then the identity (1.2) fails in the variety $\mathrm{Nil}(\mathcal{C}_m) = \mathrm{var}\{x^m = 0,\ xy = yx\}$. This means that each part of the equality (4.4) coincides with the variety $\mathcal{C}_m$ for some $0 \leq m \leq 2$. Then we may complete the proof of equality (4.4) by the same arguments as in the proof of the equality (4.2) in [25].

The equality (4.5). The varieties $\mathcal{G}$, $\mathcal{C}_2$ and $\mathcal{N}$ satisfy the identity $x^2 y = \overline{\overline{x}}^{\,2} y$. By Lemma 2.3 the variety $\mathrm{Nil}(\mathcal{V})$ satisfies the identity (1.2). Fig. 2 shows that it suffices to check the following two claims: first, the varieties $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}$ and $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}$ have the same degree, and second, the variety $\mathrm{Nil}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big)$ satisfies the identity

$$(4.6) \qquad\qquad\qquad x^2 = 0$$

if and only if the variety $\mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y})\big)$ satisfies this identity. The former claim may be verified by the same way as in the proof of the equality (4.3) in [25] with references to Proposition 2.13, Corollary 2.15 and Corollary 2.16 of the present article rather than Proposition 2.11, Lemma 2.13 and Lemma 2.12 of [25] respectively.

It remains to verify that the variety $\mathrm{Nil}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big)$ satisfies the identity (4.6) whenever $\mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big)$ does so (because the opposite claim is evident). Suppose that the identity (4.6) holds in $\mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big)$. Corollary 2.9 imply that the variety $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}$ is the join of some group variety, the variety $\mathcal{C}_m$ for some $m \geq 0$ and the variety $\mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big)$. Here $m \leq 2$ because $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y} \subseteq \mathcal{V}$. This implies that the variety $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}$ satisfies the identity $x^2 = \overline{\overline{x^2}}$. In particular, this identity holds in both the varieties $\mathcal{Y}$ and $\mathcal{Z} \wedge \mathcal{V}$. Therefore, there is a sequence of words $u_0, u_1, \ldots, u_k$ such that $u_0 \equiv x^2$, $u_k \equiv \overline{\overline{x^2}}$ and, for each $i = 0, 1, \ldots, k-1$, the identity $u_i = u_{i+1}$ holds in one of the varieties $\mathcal{Z}$ or $\mathcal{V}$. We may assume that $u_i \not\equiv u_{i+1}$ for each $i = 0, 1, \ldots, k-1$. Arguments from the proof of the equality (4.3) in [25] show that it suffices to check that the identity (4.6) holds in one of the varieties $\mathrm{Nil}(\mathcal{Z})$ or $\mathrm{Nil}(\mathcal{V})$. This fact follows from Lemma 2.6 whenever u_1 is a semigroup word and from Lemma 2.3 otherwise.

We complete the proof of Theorem 1.5. $\qquad\qquad\qquad\qquad\qquad\square$

Corollary 4.4. *A periodic strongly permutative semigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **SEM** *if and only if $\mathcal{V}$ is an upper-modular element of the lattice* **EPI**.

Proof. Necessity. Let $\mathcal{V}$ be a periodic strongly permutative semigroup variety and $\mathcal{V}$ is an upper-modular element of **SEM**. It follows from results of [5] and the proof of Proposition 1 in [35] that $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ for some Abelian periodic group variety $\mathcal{V}$, some $m \geq 0$ and some nilvariety $\mathcal{N}$. By [25, Theorem 1.1], if $\mathcal{X}$ is a proper semigroup variety that is an upper-modular element of **SEM** and $\mathcal{Y}$ is a nilsubvariety of $\mathcal{X}$ then $\mathcal{Y}$ is commutative. Thus $\mathcal{N}$ is commutative. This implies that the variety $\mathcal{V}$ is commutative too. Now we may apply [25, Theorem 1.2] and conclude that $\mathcal{V}$ satisfies one of the claims (i) or (ii) of Theorem 1.5. Therefore $\mathcal{V}$ is an upper-modular variety.

Sufficiency. Let $\mathcal{V}$ be a periodic strongly permutative semigroup variety and $\mathcal{V}$ is an upper-modular variety. By Theorem 1.5, either $\mathcal{V}$ satisfies the claim (i) of this theorem or $\mathcal{V}$ satisfies the claim (ii) of this theorem and the variety $\mathcal{G}$ from this claim is periodic. In both these cases $\mathcal{V}$ is an upper-modular element of **SEM** by [25, Theorem 1.2]. $\square$

Theorem 1.5 readily implies the following

Corollary 4.5. *If a strongly permutative epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **EPI** *then the lattice $L(\mathcal{V})$ is distributive.*

Proof. If $\mathcal{V}$ satisfies the claim (i) of Theorem 1.5 then it is periodic, whence it may be considered as a semigroup variety. In this case, it suffices to take into account a description of commutative semigroup varieties with distributive subvariety lattice obtained in [36]. Suppose now that $\mathcal{V}$ satisfies the claim (ii) of Theorem 1.5. Then $\mathcal{V} \subseteq \mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{N}$ where $\mathcal{N}$ satisfies the commutative law and the identity (1.2). In view of Proposition 2.10, $L(\mathcal{V})$ is embeddable into the direct product of the lattices $L(\mathcal{AG})$ and $L(\mathcal{C}_2 \vee \mathcal{N})$. The former lattice is generally known to be distributive. Finally, the variety $\mathcal{C}_2 \vee \mathcal{N}$ is periodic, whence it may be considered as a semigroup variety. To complete the proof, it remains to note that the lattice $L(\mathcal{C}_2 \vee \mathcal{N})$ is distributive by the mentioned result of [36]. $\square$

5. Codistributive varieties

Here we verify Theorem 1.4.

Necessity. Here and in Section 9 we need the following statement that may be verified by repeating literally arguments from the first paragraph of the proof of Theorem 1.1 in [27].

Lemma 5.1. *Let an epigroup variety $\mathcal{V}$ be a codistributive element of the lattice* **EPI***. If $\mathcal{V}$ does not contain the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ then $\mathcal{V}$ is a variety of degree ≤ 2.* $\square$

Let now $\mathcal{V}$ be a strongly permutative codistributive variety. It is evident that $\mathcal{V}$ is upper-modular. Theorem 1.5 implies that the variety $\mathcal{V}$ is commutative. Hence $\mathcal{P}, \overleftarrow{\mathcal{P}} \nsubseteq \mathcal{V}$. Lemma 5.1 implies that $\mathcal{V}$ is a variety of degree ≤ 2. It remains to refer to Corollary 2.9 and the observation that the variety $\mathcal{C}_m$ has a degree > 2 whenever $m \geq 2$.

Sufficiency. In view of Corollary 3.3, it suffices to verify that an abelian group variety $\mathcal{G}$ is codistributive. Let $\mathcal{Y}$ and $\mathcal{Z}$ be arbitrary epigroup varieties. Every epigroup variety is either periodic or contains the variety $\mathcal{AG}$. Suppose that at least one of the varieties $\mathcal{Y}$ and $\mathcal{Z}$, say $\mathcal{Y}$, contains $\mathcal{AG}$. Then $\mathcal{Y} \vee \mathcal{Z} \supseteq \mathcal{Y} \supseteq \mathcal{AG} \supseteq \mathcal{G}$ and therefore,

$$\mathcal{G} \wedge (\mathcal{Y} \vee \mathcal{Z}) = \mathcal{G} = \mathcal{G} \vee (\mathcal{G} \wedge \mathcal{Z}) = (\mathcal{G} \wedge \mathcal{Y}) \vee (\mathcal{G} \wedge \mathcal{Z}).$$

Hence we may assume that the varieties $\mathcal{Y}$ and $\mathcal{Z}$ are periodic. Now we may complete the proof by the same arguments as in the proof of the implication c) $\longrightarrow$ a) of Theorem 1.2 in [27]. $\square$

Comparing Theorem 1.2 in [27] with Theorem 1.4, we obtain the following

Corollary 5.2. *A periodic strongly permutative semigroup variety $\mathcal{V}$ is a codistributive element of the lattice* **SEM** *if and only if $\mathcal{V}$ is a codistributive element of the lattice* **EPI**. $\qquad\square$

6. Modular varieties

Here we are going to prove Theorems 1.6–1.8. We need several auxiliary facts.

Proposition 6.1. *If an epigroup variety $\mathcal{V}$ is a modular element of the lattice* **EPI** *then $\mathcal{V}$ is periodic.*

Proof. Let $\mathcal{V}$ be a modular epigroup variety. Suppose that $\mathcal{V}$ is non-periodic. Being an epigroup variety, $\mathcal{V}$ satisfies the identity $x^n = x^n x^\omega$ for some natural n. Consider varieties

$$\mathcal{N}_1 = \mathrm{var}\{x_1 x_2 \ldots x_{n+3} = 0\} \text{ and } \mathcal{N}_2 = \mathrm{var}\{x_1 x_2 \ldots x_{n+3} = 0,\ x^{n+1}y = x^n y^2\}.$$

To prove that $\mathcal{V}$ is non-modular, we are going to check that the varieties $\mathcal{V}$, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{V} \vee \mathcal{N}_1$ and $\mathcal{V} \wedge \mathcal{N}_1$ form the 5-element non-modular sublattice N_5. Note that $\mathcal{N}_2 \subseteq \mathcal{N}_1$. Whence, to achieve our aim, it suffices to verify the equalities $\mathcal{V} \vee \mathcal{N}_1 = \mathcal{V} \vee \mathcal{N}_2$ and $\mathcal{V} \wedge \mathcal{N}_1 = \mathcal{V} \wedge \mathcal{N}_2$.

The inclusion $\mathcal{V} \vee \mathcal{N}_2 \subseteq \mathcal{V} \vee \mathcal{N}_1$ is evident. It is evident also that a non-trivial identity $u = v$ holds in $\mathcal{N}_2$ if and only if either $\ell(u), \ell(v) \ge n+3$ or $u = v$ coincides with the identity $x^{n+1}y = x^n y^2$. Let $u = v$ be a non-trivial identity that is satisfied by the variety $\mathcal{V} \vee \mathcal{N}_2$. Substituting y^2 for y in the identity $x^{n+1}y = x^n y^2$, we have $x^{n+1}y^2 = x^n y^4$ that implies $x^{n+3} = x^{n+4}$. Therefore, a variety satisfying $x^{n+1}y = x^n y^2$ is periodic. Since the identity $u = v$ holds in a non-periodic variety $\mathcal{V}$, it differs from the identity $x^{n+1}y = x^n y^2$. Therefore, $\ell(u), \ell(v) \ge n+3$. This implies that $u = v$ holds in $\mathcal{N}_1$ and therefore, in $\mathcal{V} \vee \mathcal{N}_1$. Thus, $\mathcal{V} \vee \mathcal{N}_1 \subseteq \mathcal{V} \vee \mathcal{N}_2$. The equality $\mathcal{V} \vee \mathcal{N}_1 = \mathcal{V} \vee \mathcal{N}_2$ is proved.

The inclusion $\mathcal{V} \wedge \mathcal{N}_2 \subseteq \mathcal{V} \wedge \mathcal{N}_1$ is evident. The variety $\mathcal{V} \wedge \mathcal{N}_1$ is a nilvariety and is contained in $\mathcal{V}$. Since $\mathcal{V}$ satisfies $x^n = x^n x^\omega$, Lemma 2.6(ii) implies that $x^n = 0$ holds in $\mathcal{V} \wedge \mathcal{N}_1$. Therefore, $x^{n+1}y = 0 = x^n y^2$ in $\mathcal{V} \wedge \mathcal{N}_1$. We see that $\mathcal{V} \wedge \mathcal{N}_1 \subseteq \mathcal{N}_2$, whence $\mathcal{V} \wedge \mathcal{N}_1 \subseteq \mathcal{V} \wedge \mathcal{N}_2$. The equality $\mathcal{V} \wedge \mathcal{N}_1 = \mathcal{V} \wedge \mathcal{N}_2$ is proved as well. $\qquad\square$

We denote by **PER** the lattice of all periodic semigroup varieties.

Lemma 6.2. *Let $\mathcal{V}$ be a nilvariety that is a modular element of the lattice* **EPI**. *If $\mathcal{V}$ satisfies a non-substitutive identity $u = v$ then it satisfies also the identity $u = 0$.*

Proof. If the identity $u = v$ is not a semigroup one then Lemma 2.3 is applied with the desirable conclusion. So, we may assume that $u = v$ is a semigroup identity. Note that the variety $\mathcal{V}$ is periodic, whence it may be considered as a semigroup variety. Clearly, $\mathcal{V}$ is a modular element of the lattice **PER**. It is verified in [22, Proposition 2.2] that if a semigroup variety is modular in the lattice **SEM** then it has the property we verify. All varieties that appear in the proof of this claim are periodic. Therefore, the desirable conclusion is true for modular elements of the lattice **PER**, and we are done. $\qquad\square$

The formulation of the following statement and its proof are closely related with the formulation and proof of Lemma 3.1 of the article [12]. But we need slightly modify some terminology from this article. Lemma 3.1 of [12] deals with the notions of equivalent and non-stable pairs of (semigroup) words defined in [12]. Here we need some modification of the first notion and do not require the second one at all. So, we call semigroup words u and v *equivalent* if $u \equiv \xi(v)$ for some automorphism ξ on F. Clearly, if words u and v are equivalent semigroup words and $c(u) = c(v)$ then $u = v$ is a substitutive identity.

Lemma 6.3. *Let $\mathcal{V}$ be an epigroup variety that is a modular element of the lattice* **EPI** *and let u, v, s and t be pairwise non-equivalent words of the same length depending on the same letters. If the variety $\mathcal{V}$ satisfies the identities $u = v$ and $s = t$ then it satisfies also the identity $u = s$.*

Proof. In view of Proposition 6.1, the variety $\mathcal{V}$ is periodic. Whence, it may be considered as a semigroup variety. Clearly, $\mathcal{V}$ is a modular element of the lattice **PER**. The proof of [12, Lemma 3.1] readily implies that if u, v, s and t are pairwise non-equivalent words of the same length depending on the same letters, $\mathcal{V}$ satisfies the identities $u = v$ and $s = t$ and $\mathcal{V}$ does not satisfy the identity $u = s$ then there are periodic varieties (in actual fact, even nilvarieties) $\mathcal{U}$ and $\mathcal{W}$ such that $\mathcal{U} \subseteq \mathcal{W}$ but $(\mathcal{V} \vee \mathcal{U}) \wedge \mathcal{W} \neq (\mathcal{V} \wedge \mathcal{W}) \vee \mathcal{U}$. This contradicts the claim that $\mathcal{V}$ is a modular element of the lattice **PER**. $\square$

Proof of Theorem 1.6. Let $\mathcal{V}$ be a modular epigroup variety. According to Proposition 6.1, the variety $\mathcal{V}$ is periodic. It follows immediately from [12, Proposition 3.3] that if a periodic semigroup variety is a modular element of the lattice **SEM** then it is the join of one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and a nil-variety. Repeating literally arguments from the proof of this statement with references to Proposition 2.7 and Lemma 2.8 of the present work rather than Lemma 2.6 of the article [12] and to Lemma 6.3 of the present work rather than Lemma 3.1 of the article [12], we obtain that the variety $\mathcal{V}$ has the same property. Thus, $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety. It remains to verify that if $\mathcal{N}$ satisfies a non-substitutive identity $u = v$ then $\mathcal{N}$ satisfies also the identity $u = 0$. If the identity $u = v$ is not a semigroup one then Lemma 2.3 is applied with the conclusion that $\mathcal{N}$ satisfies the identity $u = 0$. So, we may assume that $u = v$ is a semigroup identity. Note that the variety $\mathcal{N}$ is periodic, whence it may be considered as a semigroup variety. In this situation the desirable conclusion directly follows from [22, Proposition 2.2]. Theorem 1.6 is proved. $\square$

Let **USEM** denotes the lattice of all varieties of unary semigroups. The following lemma will be helpful here and in Section 7.

Proposition 6.4. *An arbitrary 0-reduced epigroup variety is a modular and lower-modular element of the lattice* **USEM**.

Proof. Let $\mathcal{N}$ be a 0-reduced epigroup variety, while ν a fully invariant congruence on F corresponding to $\mathcal{N}$. Then ν has exactly one non-singleton class (this class includes words that are equal to 0 in $\mathcal{N}$ and only them). The lattice **USEM** is antiisomorphic to the lattice of all fully invariant congruences on

F. Further, the latter lattice is embeddable in the lattice $\mathrm{Eq}(F)$ of all equivalence relations on F. If an equivalence relation ρ on a set X has exactly one non-singleton class then ρ is both modular and upper-modular element in the lattice $\mathrm{Eq}(X)$ (this claim is verified in [6, Proposition 2.2] for modular elements and in [32, Proposition 3] for upper-modular ones). Thus, ν is a modular and upper-modular element in the lattice $\mathrm{Eq}(F)$, and moreover in the lattice of all fully invariant congruences on F. Since the notion of a modular element of a lattice is self-dual, we are done. $\square$

The lattice **EPI** is a sublattice in **USEM**. Whence, Proposition 6.4 immediately implies Theorem 1.7. $\square$

Proof of Theorem 1.8. *Necessity.* Let $\mathcal{V}$ be a commutative modular epigroup variety. By Theorem 1.6 $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety. Corollary 3.3 implies that the variety $\mathcal{N}$ is modular. Since every commutative variety satisfies the identity $x^2y = yx^2$, Lemma 6.2 implies that the identity (1.2) holds in $\mathcal{N}$.

Sufficiency. In view of Corollary 3.3, it suffices to verify that a commutative epigroup variety satisfying the identity (1.2) is modular. This fact may be verified by the same arguments as in the proof of the 'if' part of Theorem 1 in [33]. Theorem 1.8 is proved. $\square$

Theorems 1.8 and 1.5 evidently imply

Corollary 6.5. *If a commutative epigroup variety is a modular element of the lattice* **EPI** *then it is an upper-modular element of this lattice.* $\square$

Theorem 3.1 in [22] and Theorem 1.8 show that the following is true.

Corollary 6.6. *A periodic commutative semigroup variety is a modular element of the lattice* **SEM** *if and only if $\mathcal{V}$ is a modular element of the lattice* **EPI**. $\square$

7. Lower-modular varieties

Here we prove Theorem 1.3. To achieve this aim, we need the following

Proposition 7.1. *If an epigroup variety $\mathcal{V}$ is a lower-modular element of the lattice* **EPI** *then $\mathcal{V}$ is periodic.*

Proof. If S is an epigroup and $x \in S$ then $\overline{\overline{x}} = \overline{\overline{x}}\,x^{\omega} = \overline{\overline{x}} \cdot \overline{x}\,x$. Thus, every epigroup satisfies the identity

$$(7.1) \qquad\qquad \overline{\overline{x}} = \overline{\overline{x}} \cdot \overline{x}\,x.$$

It is known (see [15] or [16], for instance) that there is a natural number n such that the nth power of any element x in an arbitrary member S of $\mathcal{V}$ is a group element. Therefore, $\mathcal{V}$ satisfies the identity $x^n = \overline{\overline{x^n}}$. Let n be the least number with such a property. Put

$$\mathcal{Y} = \mathrm{var}\{x^n y^3 x = \overline{\overline{x^n}}\,y^3 x\} \quad \text{and} \quad \mathcal{Z} = \mathrm{var}\{x^n y^2 x = x^n y^3 x\}.$$

Clearly, $\mathcal{V} \subseteq \mathcal{Y}$. Since $\mathcal{V}$ is lower-modular, the equality

$$(\mathcal{Z} \vee \mathcal{V}) \wedge \mathcal{Y} = (\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V}$$

holds. Note that

$$x^n y^2 x = x^n y^3 x \qquad \text{in the variety } \mathcal{Z}$$
$$= \overline{\overline{x^n}} \, y^3 x \qquad \text{in the variety } \mathcal{Y}$$
$$= \overline{\overline{x^n}} \cdot \overline{x^n} \, x^n y^3 x \qquad \text{in the variety } \mathcal{Z} \wedge \mathcal{Y} \text{ by } (7.1)$$
$$= \overline{\overline{x^n}} \cdot \overline{x^n} \, x^n y^2 x \qquad \text{in the variety } \mathcal{Z}$$
$$= \overline{\overline{x^n}} \, y^2 x \qquad \text{in the variety } \mathcal{Z} \wedge \mathcal{Y} \text{ by } (7.1).$$

We see that $\mathcal{Z} \wedge \mathcal{Y}$ satisfies the identity $x^n y^2 x = \overline{\overline{x^n}} \, y^2 x$. Therefore, the identity $x^n y^2 x = \overline{\overline{x^n}} \, y^2 x$ holds in $(\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V}$ and moreover, in $(\mathcal{Z} \vee \mathcal{V}) \wedge \mathcal{Y}$. This means that there is a deduction of this identity from identities of the varieties $\mathcal{Z} \vee \mathcal{V}$ and $\mathcal{Y}$. In particular, one of the varieties $\mathcal{Z} \vee \mathcal{V}$ or $\mathcal{Y}$ satisfies a non-trivial identity of the form $x^n y^2 x = w$ for some word w. It is evident that any identity of such a kind is false in $\mathcal{Y}$. Whence, it holds in $\mathcal{Z} \vee \mathcal{V}$. It is obvious that the words $x^n y^2 x$ and $x^n y^3 x$ do not contain images of each other relatively to endomorphisms of the unary semigroup F. Hence the identity $x^n y^2 x = x^n y^3 x$ does not imply any identity of the form $x^n y^2 x = w$ where w differs from the words $x^n y^2 x$ and $x^n y^3 x$. In particular, all identities of this form are false in $\mathcal{Z}$. Therefore, the variety $\mathcal{Z} \vee \mathcal{V}$ satisfies the identity $x^n y^2 x = x^n y^3 x$. In particular, this identity holds in $\mathcal{V}$. Thus, $\mathcal{V}$ satisfies the identity $x^{n+3} = x^{n+4}$, whence it is periodic. $\qquad \square$

Repeating literally arguments from the proof of Lemma 6.3 but referring to Proposition 7.1 rather than Proposition 6.1, we obtain the following

Lemma 7.2. *Let $\mathcal{V}$ be an epigroup variety that is a lower-modular element of the lattice* **EPI** *and let u, v, s and t be pairwise non-equivalent words of the same length depending on the same letters. If the variety $\mathcal{V}$ satisfies the identities $u = v$ and $s = t$ then it satisfies also the identity $u = s$.* $\qquad \square$

Proof of Theorem 1.3. *Sufficiency.* Proposition 6.4 immediately implies that a 0-reduced epigroup variety is lower-modular. It remains to refer to Corollary 3.3.

Necessity. In view of Proposition 7.1, the variety $\mathcal{V}$ is periodic. Now we may complete the proof repeating literally arguments from the proof of Proposition 3.3 in [12] but referring to Proposition 2.7 rather than Lemma 2.6 in [12], and to Lemma 7.2 rather than Lemma 3.1 in [12]. $\qquad \square$

Comparing Theorems 1.7 and 1.3, we obtain the following

Corollary 7.3. *If an epigroup variety is a lower-modular element of the lattice* **EPI** *then it is a modular element of this lattice.* $\qquad \square$

By the way, we note that neither of the five other possible interrelations between properties of being a modular variety, a lower-modular variety or an upper-modular variety holds. For instance:

- the variety $\text{var}\{x^2 = 0, \, xy = yx\}$ is modular by Theorem 1.8 but not lower-modular by Theorem 1.3;

- the variety var$\{xyz = 0\}$ is modular and lower-modular by Theorems 1.7 and 1.3 respectively but not upper-modular by Theorem 1.5;
- an arbitrary Abelian periodic group variety is upper-modular by Theorem 1.5 but neither modular nor lower-modular by Theorems 1.6 and 1.3 respectively.

Theorem 1.1 in [14] and Theorem 1.3 show that the following is true.

Corollary 7.4. *A periodic semigroup variety $\mathcal{V}$ is a lower-modular element of the lattice* **SEM** *if and only if $\mathcal{V}$ is a lower-modular element of the lattice* **EPI**. $\qquad\square$

8. An application to definable varieties

Here we discuss an interesting application of Theorems 1.1 and 1.3. A subset A of a lattice $\langle L; \vee, \wedge \rangle$ is called *definable in L* if there exists a first-order formula $\Phi(x)$ with one free variable x in the language of lattice operations $\vee$ and $\wedge$ which *defines A in L*. This means that, for an element $a \in L$, the sentence $\Phi(a)$ is true if and only if $a \in A$. A set X of semigroup [epigroup] varieties is said to be *definable* if it is definable in **SEM** [respectively in **EPI**]. In this situation we will say that the corresponding first-order formula *defines* the set X.

A number of deep results about definable varieties and sets of varieties of semigroups have been obtained in [7] by Ježek and McKenzie. In particular, it was proved there that the set of all 0-reduced varieties is definable. But the article [7] contains no explicit first-order formula that define this set of varieties. The simple first-order formula that define the class of all 0-reduced semigroup varieties was found in [28, Theorem 3.3] or [29, Subsection 3.5]. Here we are going to verify that the same formula defines the set of 0-reduced varieties in the lattice **EPI**.

Theorem 1.3 shows that an epigroup variety is 0-reduced if and only if it is lower-modular and does not contain the variety $\mathcal{SL}$. Obviously, the set of all lower-modular varieties is definable. It remains to define the variety $\mathcal{SL}$. Evidently, the lattices **SEM** and **EPI** have the same set of atoms. Theorem 1.1 together with the well-known description of atoms of the lattice **SEM** (see [17, Section 1], for instance) imply that the lattice **EPI** contains exactly two neutral atoms, namely the varieties $\mathcal{SL}$ and $\mathcal{ZM}$. Recall that a semigroup variety $\mathcal{V}$ is called a *chain* if the lattice $L(\mathcal{V})$ is a chain. Clearly, every chain variety is periodic, whence it may be considered as epigroup variety. It is well known that the variety $\mathcal{ZM}$ is properly contained in some chain variety, while the variety $\mathcal{SL}$ is not [19]. Combining the mentioned observations, we see that the class of all 0-reduced varieties may be defined as the class **K** of epigroup varieties with the following properties:

(i) every member of **K** is a lower-modular variety;
(ii) if $\mathcal{V} \in$ **K** and $\mathcal{V}$ contains some neutral atom $\mathcal{A}$ then $\mathcal{A}$ is properly contained in some chain variety.

It is evident that properties (i) and (ii) may be written by simple first-order formulas with one free variable. We prove the following

Proposition 8.1. *The class of all 0-reduced epigroup varieties is definable in the lattice* **EPI***.* $\square$

9. NEUTRAL AND COSTANDARD VARIETIES

Here we prove Theorem 1.1. The proof will be given by the following scheme:

$$
\begin{array}{ccc}
 & \text{b)} & \\
\nearrow & & \searrow \\
\text{a)} & \longleftarrow & \text{d)}. \\
\searrow & & \nearrow \\
 & \text{c)} &
\end{array}
$$
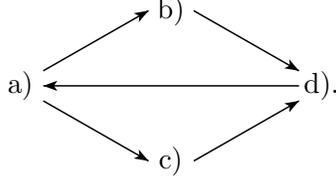
The implications a) $\longrightarrow$ b) and a) $\longrightarrow$ c) are evident, while the implication d) $\longrightarrow$ a) immediately follows from Propositions 3.1 and 3.2, and the well-known fact that the set of all neutral elements of a lattice L forms a sublattice in L (see [2, Theorem 259]). It remains to verify the implications b) $\longrightarrow$ d) and c) $\longrightarrow$ d).

Suppose that $\mathcal{V}$ is either costandard or simultaneously lower-modular and upper-modular. We need to check that $\mathcal{V}$ coincides with one of the varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$ or $\mathcal{SL} \vee \mathcal{ZM}$. In any case, $\mathcal{V}$ is modular (this is evident whenever $\mathcal{V}$ is costandard, and follows from Corollary 7.3 whenever $\mathcal{V}$ is lower-modular and upper-modular). Then we may apply Theorem 1.6 and conclude that $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety. It remains to verify that $\mathcal{N}$ is one of the varieties $\mathcal{T}$ or $\mathcal{ZM}$. By Lemma 2.20 we have to check that $\mathcal{N} \subseteq \mathcal{ZM}$. In other words, we need to verify that $\mathcal{N}$ is a variety of degree ≤ 2. In view of Corollary 3.3, $\mathcal{N}$ is either costandard or simultaneously lower-modular and upper-modular.

Suppose at first that $\mathcal{N}$ is costandard. Then it is codistributive. Now Lemma 5.1 applies with the desirable conclusion. We have proved the implication b) $\longrightarrow$ d).

It remains to consider the case when $\mathcal{N}$ is lower-modular and upper-modular. The variety $\mathcal{N}$ is periodic, whence it may be considered as a semigroup variety. Therefore, $\mathcal{N}$ is a lower-modular and upper-modular element of the lattice **PER**.

It follows from [23, Theorem 1] that if a nilvariety of semigroups is lower-modular in **SEM** then it is 0-reduced. All varieties that appear in the proof of this fact are periodic. So, a nilvariety is 0-reduced whenever it is a lower-modular element in **PER**. In particular, the variety $\mathcal{N}$ is 0-reduced.

Suppose that $\mathcal{N}$ is a variety of degree > 2. It follows from [26, Theorem 1] that if a proper semigroup variety of degree > 2 is upper-modular in **SEM** then it is commutative. All varieties that appear in the proof of this fact are periodic again. So, a variety of degree > 2 is commutative whenever it is an upper-modular element in **PER**. In particular, the variety $\mathcal{N}$ is commutative.

Thus, $\mathcal{N}$ is a 0-reduced and commutative nilvariety. Therefore, $\mathcal{N} \subseteq \mathcal{ZM}$. This completes the proof of the implication c) $\longrightarrow$ d), and hence the proof of Theorem 1.1 as a whole. $\square$

Theorem 1.1 immediately implies the following

Corollary 9.1. *If an epigroup variety is a costandard element of the lattice* **EPI** *then it is a standard element of this lattice.*   □

Comparing Theorem 1.3 in [27] with Theorem 1.1, we obtain the following

Corollary 9.2. *For a periodic semigroup variety $\mathcal{V}$, the following are equivalent:*

    a) *$\mathcal{V}$ is a costandard element of the lattice* **EPI***;*
    b) *$\mathcal{V}$ is a neutral element of the lattice* **EPI***;*
    c) *$\mathcal{V}$ is a costandard element of the lattice* **SEM***;*
    d) *$\mathcal{V}$ is a neutral element of the lattice* **SEM***.*   □

10. Distributive and standard varieties

Here we prove Theorem 1.2. The implication b)$\longrightarrow$a) of this theorem is evident, whence it suffices to verify the implications a)$\longrightarrow$c), c)$\longrightarrow$a) and a)$\longrightarrow$b).

a)$\longrightarrow$b) Suppose that an epigroup variety $\mathcal{V}$ is distributive. Hence it is lower-modular. Corollary 7.3 implies that $\mathcal{V}$ is modular. Now Lemma 2.17 implies with the conclusion that $\mathcal{V}$ is standard.

a)$\longrightarrow$c) We denote by F_m the free unary semigroup over the alphabet $\{x_1, x_2, \ldots, x_m\}$ and by S_m the symmetric group on the set $\{1, 2, \ldots, m\}$. If $\sigma \in S_m$ and $u \in F_m$ then $\sigma(u)$ denotes the image of the word u under the automorphism of the unary semigroup F_m induced by the action of the permutation σ on indexes of letters. The following statement easily follows from the proof of Lemma 5.1 of the paper [30].

Lemma 10.1. *Let u and v be semigroup words from F_m such that the identity $u = v$ is non-balanced and none of the words u and v contains the image of other one over some endomorphism of the free unary semigroup, and let σ be a permutation from S_m. If the variety $\mathcal{X} = \mathrm{var}\{u = v\}$ satisfies a non-trivial identity of the form $\sigma(u) = w$ where w is a semigroup word then $w \equiv \sigma(v)$.*   □

Note that if the identity $u = v$ is balanced then the class $K_{\{u=v\}}$ is not a variety by Lemma 2.4. Thus, the restriction to the identity $u = v$ to be non-balanced in Lemma 10.1 is natural.

The following assertion is an 'epigroup analog' of Lemma 5.2 of the article [30].

Lemma 10.2. *Let $\mathcal{V}$ be an epigroup variety that is a distributive element of the lattice* **EPI** *and let u, v be semigroup words such that $c(u) = c(v)$ and one of the following holds:*

    (i) *none of the words u and v contains the image of other one over some endomorphism of the free unary semigroup;*
    (ii) *$\ell(u) = \ell(v)$.*

If the variety $\mathcal{V}$ satisfies the identity $u = 0$ then it satisfies the identity $v = 0$.

Proof. We may assume that $u, v \in F_m$ for some natural m. Suppose that $\mathcal{V}$ satisfies the identity $u = 0$.

(i) If $m = 1$ then $u \equiv x^q$ and $v \equiv x^r$ for some q and r. Then one of the words u and v contains an image of another one that contradicts the hypothesis. Therefore, $m > 1$, whence the group S_m contains a non-trivial permutation α. Put $w \equiv \alpha(u)$. Then $\mathcal{V}$ satisfies the identity $w = 0$ that implies $u = w$. If the identity $u = v$ is non-balanced then we may complete the proof repeating literally arguments from the corresponding part of the proof of item (i) in Lemma 5.2 of the paper [30]. Suppose now that the identity $u = v$ is balanced. Let x be a letter with $x \notin c(u)$. Then the identities $xu = v$ and $v = xw$ are non-balanced. According to Lemma 2.4, we may consider the varieties $\mathcal{Y} = \mathrm{var}\{xu = v\}$ and $\mathcal{Z} = \mathrm{var}\{v = xw\}$. Since the variety $\mathcal{V}$ is distributive, $\mathcal{V} \vee (\mathcal{Y} \wedge \mathcal{Z}) = (\mathcal{V} \vee \mathcal{Y}) \wedge (\mathcal{V} \vee \mathcal{Z})$. Clearly, the variety $\mathcal{V} \vee (\mathcal{Y} \wedge \mathcal{Z})$ satisfies the identity $xu = xw$. Therefore, this identity holds in the variety $(\mathcal{V} \vee \mathcal{Y}) \wedge (\mathcal{V} \vee \mathcal{Z})$. Then there exists a deduction of the identity $xu = xw$ from identities of the varieties $\mathcal{V} \vee \mathcal{Y}$ and $\mathcal{V} \vee \mathcal{Z}$. In particular, one of these varieties satisfies a non-trivial identity of the form $xu = w_1$.

Suppose at first that $xu = w_1$ holds in $\mathcal{V} \vee \mathcal{Y}$. Then it holds in $\mathcal{Y}$. Now Lemma 10.1 with the trivial permutation σ from S_m applies, and we conclude that $w_1 \equiv v$. Since $xu = w_1$ in $\mathcal{V}$, we have that $\mathcal{V}$ satisfies the identity $v = xu$ that implies $v = 0$. It remains to consider the case when the identity $xu = w_1$ holds in $\mathcal{V} \vee \mathcal{Z}$. In particular, it holds in $\mathcal{Z}$. Let σ be a permutation of the alphabet such that the restriction of σ on the set $c(u)$ coincides with α^{-1} and $\sigma(x) \equiv x$. Now we apply Lemma 10.1 with such permutation σ. Then we obtain that $w_1 \equiv \sigma(v)$. But the identity $xu = w_1$ holds in $\mathcal{V}$. Hence $\mathcal{V}$ satisfies the identity $xu = \sigma(v)$ that implies $\sigma(v) = 0$ and $v = 0$.

(ii) Here we may repeat literally arguments from the proof of item (ii) in [30, 5.2]. $\qquad\square$

Now we are well prepared to complete the proof of the implication a) $\longrightarrow$ c) of Theorem 1.2. Let $\mathcal{V}$ be a distributive epigroup variety. Then $\mathcal{V}$ is a lower-modular variety. In view of Theorem 1.3, $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a 0-reduced variety. Corollary 3.3 implies that the variety $\mathcal{N}$ is distributive. Being 0-reduced, this variety satisfies the identity $u = 0$ for some word u. We may assume that this identity fails in the class of all nil-semigroups. Lemma 2.3 implies that u is a semigroup word. We may assume that $c(u) = \{x, y\}$. Indeed, if u depends on one letter then we may substitute xy to this letter, and if u depends on two or more letters then we may substitute x to one of these letters and y to all other of them. Now we may repeat literally the corresponding part of the proof of Proposition 3.2 of the article [30] with referring to Lemmas 10.1 and 10.2 of the present article rather than Lemmas 5.1 and 5.2 of the paper [30] respectively. As a result, we obtain that $\mathcal{N}$ satisfies all identities of the form $v = 0$ with $c(v) = c(u)$ and $\ell(v) \geq 3$. In particular, $\mathcal{N}$ satisfies the identities $x^2 y = xyx = yx^2 = 0$. We see that $\mathcal{N}$ is a 0-reduced subvariety of the variety $\mathcal{Q}$. If $\mathcal{N} = \mathcal{Q}$ then we are done. Let now $\mathcal{N} \subset \mathcal{Q}$. Then $\mathcal{N}$ is given in $\mathcal{Q}$ by some set of 0-reduced identities. By Lemma 2.12, $\mathcal{N}$ is given within $\mathcal{Q}$ either by the identity $x^2 = 0$ or by the identity $x_1 x_2 \cdots x_n = 0$ for some n or by these two identities together. Thus $\mathcal{N}$ is one of the varieties $\mathcal{Q}$, $\mathcal{Q}_n$, $\mathcal{R}$ or $\mathcal{R}_n$. The implication a) $\longrightarrow$ c) is proved.

c) $\longrightarrow$ a) Let $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is one of the varieties $\mathcal{Q}$, $\mathcal{Q}_n$, $\mathcal{R}$ or $\mathcal{R}_n$. In view of Corollary 3.3, it suffices to prove that the variety $\mathcal{N}$ is standard. In other words, we may assume that $\mathcal{V}$ is one of the varieties $\mathcal{Q}$, $\mathcal{Q}_n$, $\mathcal{R}$ or $\mathcal{R}_n$. In particular, $\mathcal{V}$ is periodic, whence it may be considered as a semigroup variety.

Let $\mathcal{Y}$ and $\mathcal{Z}$ be arbitrary epigroup varieties. We need to verify that

$$\mathcal{V} \vee (\mathcal{Y} \wedge \mathcal{Z}) = (\mathcal{V} \vee \mathcal{Y}) \wedge (\mathcal{V} \vee \mathcal{Z}).$$

It suffices to check that $(\mathcal{V} \vee \mathcal{Y}) \wedge (\mathcal{V} \vee \mathcal{Z}) \subseteq \mathcal{V} \vee (\mathcal{Y} \wedge \mathcal{Z})$ because the contrary inclusion is evident. In other words, we have to prove that an arbitrary identity holds in $(\mathcal{V} \vee \mathcal{Y}) \wedge (\mathcal{V} \vee \mathcal{Z})$ whenever it holds in $\mathcal{V} \vee (\mathcal{Y} \wedge \mathcal{Z})$. In view of [30, Lemma 2.2] we may suppose that $\mathcal{Y}, \mathcal{Z} \supseteq \mathcal{SL}$.

Let $u = v$ be an identity that holds in $\mathcal{V} \vee (\mathcal{Y} \wedge \mathcal{Z})$. Then it holds in $\mathcal{V}$ and there exists a deduction of this identity from identities of the varieties $\mathcal{Y}$ and $\mathcal{Z}$. Let

(10.1) $$u \equiv w_0 \longrightarrow w_1 \longrightarrow \cdots \longrightarrow w_k \equiv v$$

be the shortest such deduction. Since $\mathcal{Y}, \mathcal{Z} \supseteq \mathcal{SL}$, Lemma 2.5(i) implies that $c(w_0) = c(w_1) = \cdots = c(w_k)$.

We use an induction by k. The induction base is evident because if $k = 1$ then the identity $u = v$ holds in one of the varieties $\mathcal{V} \vee \mathcal{Y}$ or $\mathcal{V} \vee \mathcal{Z}$, whence it holds in the intersection of these varieties. Let now $k > 1$. The identity $u = v$ holds in the variety $\mathcal{V}$. Since $\mathcal{V}$ is 0-reduced, $u = v = 0$ holds in $\mathcal{V}$. If $w_i = 0$ in $\mathcal{V}$ for some $0 < i < k$ then our considerations may be completed by repeating literally arguments from the corresponding part of the proof of Proposition 3.3 of the paper [30]. In view of Lemma 2.12, we may assume that each of the words $w_1, w_2, \ldots, w_{k-1}$ either is a linear word or coincides with the word x^2.

Suppose that $w_i \equiv x^2$ for some $0 < i < k$. The choice of the deduction (10.1) guarantees that the words $w_0, w_1, \ldots, w_k$ are pairwise different. Since $c(w_0) = c(w_1) = \cdots = c(w_k) = \{x\}$, we have that each of the varieties $\mathcal{Y}$ and $\mathcal{Z}$ satisfies a non-trivial identity of the form $x^m = x^n$. Therefore, the varieties $\mathcal{Y}$ and $\mathcal{Z}$ are periodic. Hence they may be considered as semigroup varieties. This permits to complete our considerations by repeating literally arguments from the corresponding part of the proof of Proposition 3.3 of the paper [30].

It remains to consider the case when the words $w_1, w_2, \ldots, w_{k-1}$ are linear. We may assume that the words u and v are non-linear (as usual, this claim may be verified by the same arguments as in the proof of [30, Proposition 3.3]). Besides that, we will assume without loss of generality that the identity $u = w_1$ holds in $\mathcal{Y}$ and therefore, $w_1 = w_2$ holds in $\mathcal{Z}$. Further considerations are divided into three cases.

Case 1: $k = 2$. Here the deduction (10.1) has the form $u \to w_1 \to v$, $u = w_1$ in $\mathcal{Y}$ and $w_1 = v$ in $\mathcal{Z}$. Since the words u and v are non-linear, the varieties $\mathcal{Y}$ and $\mathcal{Z}$ are periodic. Hence they may be considered as semigroup varieties. This permits to complete a consideration of this case by repeating literally arguments from the Case 1 in the proof of Proposition 3.3 of the paper [30].

Case 2: $k = 3$. Here the deduction (10.1) has the form $u \to w_1 \to w_2 \to v$ and the identities $u = w_1$ and $w_2 = v$ hold in $\mathcal{Y}$. Since the words u and

v are non-linear, the variety $\mathcal{Y}$ is periodic. Hence it may be considered as a semigroup variety. This variety satisfies the identity $x_1 x_2 \cdots x_m = u$ and $\ell(u) > m$. Therefore, $\mathcal{Y}$ is a variety of degree $\leq m$ (see [10, Lemma 1]). According to [25, Proposition 2.11], $\mathcal{Y}$ satisfies an identity of the form

$$x_1 x_2 \cdots x_m = x_1 x_2 \cdots x_{i-1}(x_i \cdots x_j)^t x_{j+1} \cdots x_m$$

for some $t > 1$ and $0 \leq i \leq j \leq m$. This permits to complete a consideration of this case by repeating literally arguments from the Case 2 in the proof of Proposition 3.3 of the paper [30].

Case 3: $k > 3$. This case may be considered by repeating literally arguments from the Case 3 in the proof of Proposition 3.3 of the paper [30].

We complete the proof of the implication c) $\longrightarrow$ a) and of Theorem 1.2 as a whole. $\qquad\square$

Comparing Theorem 1.1 in [30] with Theorem 1.2, we obtain the following

Corollary 10.3. *For a periodic semigroup variety $\mathcal{V}$, the following are equivalent:*

 a) *$\mathcal{V}$ is a distributive element of the lattice* **EPI***;*
 b) *$\mathcal{V}$ is a standard element of the lattice* **EPI***;*
 c) *$\mathcal{V}$ is a distributive element of the lattice* **SEM***;*
 d) *$\mathcal{V}$ is a standard element of the lattice* **SEM***.* $\qquad\square$

11. Open questions

It is verified in [27, Theorem 1.1] that if a proper semigroup variety $\mathcal{V}$ is codistributive in **SEM** then the square of any member of $\mathcal{V}$ is completely regular. We do not know, whether the epigroup analog of this fact is true.

Question 11.1. *Is it true that if an epigroup variety $\mathcal{V}$ is a codistributive element of the lattice* **EPI** *then the square of any member of $\mathcal{V}$ is completely regular?*

As we have already mentioned in Section 4, it is proved in [25, Theorem 1.1] that if $\mathcal{V}$ is a proper upper-modular in **SEM** semigroup variety then every nilsubvariety of $\mathcal{V}$ is commutative and satisfies the identity (1.1). Proposition 4.2 gives a partial epigroup analog of this assertion. We do not know, whether the full analog is true.

Question 11.2. *Suppose that an epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **EPI** *and let $\mathcal{N}$ be a nilsubvariety of $\mathcal{V}$. Is it true that the variety $\mathcal{N}$*

 a) *is commutative;*
 b) *satisfies the identity* (1.1)?

Proposition 4.2 shows that the affirmative answer to Question 11.2a) would immediately implies the same answer to Question 11.2b).

Further, it is verified in [26, Theorem 1] that every proper upper-modular in **SEM** variety is either commutative or has a degree ≤ 2. We do not know, whether the epigroup analog of this alternative is valid. We formulate the corresponding question together with its weaker version.

Question 11.3. *Suppose that an epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **EPI**. *Is it true that the variety $\mathcal{V}$*

    a) *either is commutative or has a degree ≤ 2;*
    b) *either is permutative or has a finite degree?*

The affirmative answer to Question 11.3a) together with Theorem 1.5 would immediately imply a complete description of upper-modular epigroup varieties of degree > 2.

If $\mathcal{V}$ is a proper upper-modular in **SEM** variety then the lattice $L(\mathcal{V})$ is modular [26, Corolary 2] and every subvariety of $\mathcal{V}$ is upper-modular in **SEM** [26, Corolary 3]. This makes a natural the following

Question 11.4. *Suppose that an epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **EPI**. *Is it true that*

    a) *the lattice $L(\mathcal{V})$ is modular;*
    b) *every subvariety of $\mathcal{V}$ is an upper-modular element of the lattice* **EPI**?

We note that the affirmative answer to Question 11.4a) would immediately implies the same answer to Question 11.4b) (see [25, Lemma 2.1]).

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