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On separability of the functional space with the open-point and bi-point-open topologies

Alexander V. Osipov

Ural Federal University, Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, 16, S.Kovalevskaja street, 620219, Ekaterinburg, Russia

Abstract

In this paper we study the property of separability of functional space $C(X)$ with the open-point and bi-point-open topologies. We show that it is consistent with ZFC that there is a set of reals of cardinality the continuum such that a set $C(X)$ with the open-point topology isn't a separable space. We also show in a set model (the iterated perfect set model) that for every set of reals X a set $C(X)$ with bi-point-open topology is a separable space.

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2000 MSC: 54C40, 54C35, 54D60, 54H11, 46E10

1. Introduction

The space $C(X)$ with the point-open topology (also known as the topology of pointwise convergence) is denoted by $C_p(X)$. It has a subbase consisting of sets of the form

$[x, V]^+ = \{f \in C(X) : f(x) \in V\}$,

where $x \in X$ and V is an open subset of real line $\mathbb{R}$. In paper [2] was introduced two new topologies on $C(X)$ that we call the open-point topology and the bi-point-open topology. The open-point topology on $C(X)$ has a subbase consisting of sets of the form

$[U, r]^- = \{f \in C(X) : f^{-1}(r) \bigcap U \neq \emptyset\}$,

Email address: OAB@list.ru (Alexander V. Osipov)

where U is an open subset of X and $r \in \mathbb{R}$. The open-point topology on $C(X)$ is denoted by h and the space $C(X)$ equipped with the open-point topology h is denoted by $C_h(X)$.

Now the bi-point-open topology on $C(X)$ is the join of the point-open topology p and the open-point topology h. It is the topology having subbase open sets of both kind: $[x, V]^+$ and $[U, r]^-$, where $x \in X$ and V is an open subset of $\mathbb{R}$, while U is an open subset of X and $r \in \mathbb{R}$. The bi-point-open topology on the space $C(X)$ is denoted by ph and the space $C(X)$ equipped with the bi-point-open topology ph is denoted by $C_{ph}(X)$. One can also view the bi-point-open topology on $C(X)$ as the weak topology on $C(X)$ generated by the identity maps $id_1 : C(X) \mapsto C_p(X)$ and $id_2 : C(X) \mapsto C_h(X)$.

In [2] and [1], the separation and countability properties of these two topologies on $C(X)$ have been studied.

In [2] the following statements were proved.

• $C_h(\mathbb{P})$ is separable. (Proposition 5.1.)

• If $C_h(X)$ is separable, then every open subset of X is uncountable. (Theorem 5.2.)

• If X has a countable π-base consisting of nontrivial connected sets, then $C_h(X)$ is separable. (Theorem 5.5.)

• If $C_{ph}(X)$ is separable, then every open subset of X is uncountable. (Theorem 5.8.)

• If X has a countable π-base consisting of nontrivial connected sets and a coarser metrizable topology, then $C_{ph}(X)$ is separable. (Theorem 5.10.)

In the present paper, we will continue to study the separability of spaces $C_h(X)$ and $C_{ph}(X)$.

In this paper we use the following conventions. The symbols $\mathbb{R}$, $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the space of real numbers, irrational numbers, rational numbers and natural numbers, respectively. Recall that a dispersion character $\Delta(X)$ of X is the minimum of cardinalities of its nonempty open subsets.

By *set of reals* we mean a zero-dimensional, separable metrizable space every non-empty open set which has the cardinality the continuum.

2. Main results

Note that if the space $C_h(X)$ is a separable space then $\Delta(X) \geq \mathfrak{c}$. Really, if $A = \{f_i\}$ is a countable dense set of $C_h(X)$ then for each non-empty open set U of X we have $\bigcup f_i(U) = \mathbb{R}$. It follows that $|U| \geq \mathfrak{c}$.

Also note that if the space $C_{ph}(X)$ is a separable space then $C_p(X)$ is a separable space and $C_h(X)$ is a separable. It follows that X is a separable submetrizable (coarser separable metric topology) space and $\Delta(X) = \mathfrak{c}$.

Note also that if the space $C_h(X)$ is a separable space then any point $x \in X$ isn't P-point (point for which the family of neighbourhoods is closed under countable intersections) of X.

Definition 2.1. Let X be a topological space. A set $A \subseteq X$ will be called *$\mathcal{I}$-set* if there is a continuous function $f \in C(X)$ such that $f(A)$ contains an interval $\mathcal{I} = [a, b] \subset \mathbb{R}$.

It is easily seen that in Definition the set $\mathcal{I} = [a, b]$ can be replaced by $\mathbb{C} = 2^\omega$ or $\mathbb{P} = \omega^\omega$.

It is known that there exists a subset $B \subset \mathbb{R}$ such that no uncountable closed set of $\mathbb{R}$ is contained either B or $\mathbb{R} \setminus B$. Such a subset B is called a Bernstein set.

Marcin Kysiak (in personal correspondence) was seen following lemma.

Lemma 2.2. *Let B be a Bernstein set and U be an non-empty open set in B. Then U is $\mathcal{I}$-set.*

Proof. Let $D \subset \mathbb{R} \setminus B$ be a countable dense subset of the real line and let $\{U_n : n \in \omega\}$ be a countable topology base consisting of open intervals with endpoints in D. For every $n \in \omega$ the set $U_n \setminus D$ is homeomorphic to the Baire space ω^ω, and hence it is homeomorphic to its Cartesian square $\omega^\omega \times \omega^\omega$; let $h_n : (U_n \setminus D) \to \omega^\omega \times \omega^\omega$ be a homeomorphism. As every uncountable perfect Polish space is a continuous image of ω^ω, let us fix a continuous mapping F from ω^ω onto $\mathbb{R}$. Let us define $g_n : (U_n \setminus D) \to \mathbb{R}$ as $g_n = F \circ \pi_1 \circ h_n$, where $\pi_1 : \omega^\omega \times \omega^\omega \to \omega^\omega$ is the projection on the first coordinate, i.e. $\pi_1(x, y) = x$ for $x, y \in \omega^\omega$. As for every $x \in \omega^\omega$ the set $h_n^{-1}[\{x\} \times \omega^\omega]$ contains a perfect set, we have $B \bigcap h_n^{-1}[\{x\} \times \omega^\omega] \neq \emptyset$ and consequently $h_n[B] \bigcap (\{x\} \times \omega^\omega) \neq \emptyset$ so $\pi_1 \circ h_n[B] = \omega^\omega$, hence $F \circ \pi_1 \circ h_n[B] = \mathbb{R}$. We have shown that $g_n[B] = \mathbb{R}$ for every $n \in \omega$, where g_n is a continuous function defined on an open interval U_n. As the endpoints of U_n do not belong to B, the function $(g_n) \restriction B$ can be easily extended to a continuous function $f : B \to \mathbb{R}$ which is still onto $\mathbb{R}$ by the property of g_n. Let now $U \subset \mathbb{R}$ be a nonempty open set. As $\{U_n : n \in \omega\}$ was a topology base, there exists $n \in \omega$ such that $U_n \subseteq U$. Then $f[U \bigcap B] = \mathbb{R}$.

□

Theorem 2.3. *Let X be a Tychonoff space and $C_h(X)$ be a separable space. Then X has a π-network consisting of $\mathcal{I}$-sets.*

Proof. Let set $A = \{f_i\}$ be a countable dense subset of $C_h(X)$. Suppose, contrary our claim, that there is non-empty open set U such that for any $f \in C(X)$ the set $f(U)$ don't contains interval of real line. Note that if for every Cantor set $\mathbb{C}$ holds $(\mathbb{R} \setminus f_1(U)) \not\supseteq \mathbb{C}$ then $f_1(U)$ is Bernstein set.

By lemma 2.2, there is continuous function $g \in C(f_1(U))$ such that $g(f_1(U))$ contains interval of real line. This contradicts our assumption. It follow that there is Cantor set $\mathbb{C}_1$ such that $f_1(U) \bigcap \mathbb{C}_1 = \emptyset$. For the set $f_2(U)$ we have that there is Cantor set $\mathbb{C}_2$ such that $\mathbb{C}_2 \subseteq \mathbb{C}_1$ and $f_2(U) \bigcap \mathbb{C}_2 = \emptyset$. We can now proceed analogously to the $f_i(U)$ for each $i > 2$. As a result of the induction, we obtain countable family of Cantor sets $\{\mathbb{C}_i\}_i$ such that $\mathbb{C}_{i+1} \subseteq \mathbb{C}_i$ for each $i \in \mathbb{N}$. Choose $r \in \bigcap_i \mathbb{C}_i$ we have $f_i \notin [U, r]^-$ for each $i \in \mathbb{R}$, which contradicts density of set A.

□

Theorem 2.4. *Let X be a Tychonoff space with countable π-base, then the following are equivalent.*

1. *$C_{ph}(X)$ is a separable space.*
2. *X is separable submetrizable space and it has a countable π-network consisting of $\mathcal{I}$-sets.*

Proof. (1) $\Rightarrow$ (2). The map $id_2 : C_{ph}(X) \mapsto C_h(X)$ is continuous map, hence $C_h(X)$ is separable space. By Theorem 2.3, the space X has a countable π-network consisting of $\mathcal{I}$-sets. The map $id_1 : C_{ph}(X) \mapsto C_p(X)$ is continuous map, hence $C_p(X)$ is separable space. It follow that X is a separable submetrizable space.

(2) $\Rightarrow$ (1). Let $S = \{S_i\}$ be a countable π-network of X consisting of $\mathcal{I}$-sets. By definition of $\mathcal{I}$-sets, for each $S_i \in S$ there is the continuous function $h_i \in C(X)$ such that $h_i(S_i)$ contains an interval $[a_i, b_i]$ of real line. Consider a countable set

$\{h_{i,p,q}(x) = \frac{p-q}{a_i-b_i} * h_i(x) + p - \frac{p-q}{a_i-b_i} * a_i\}$

of continuous functions on X, where $i \in \mathbb{N}$, $p, q \in \mathbb{Q}$. Let $\beta = \{B_j\}$ be countable base of (X, τ_1) where τ_1 is separable metraizable topology on X because of X is separable submetrizable space. For each pair (B_j, B_k) such that $\overline{B_j} \subseteq B_k$ define continuous functions

$$h_{i,p,q,j,k}(x) = \begin{cases} h_{i,p,q}(x) & for\ \ x \in B_j \\ \mathbf{0} & for\ \ x \in X \setminus B_k. \end{cases}$$

and for each $v \in \mathbb{Q}$

$$d_{j,k,v}(x) = \begin{cases} v & for \quad x \in B_j \\ \mathbf{0} & for \quad x \in X \setminus B_k. \end{cases}$$

Let G be the set of finite sum of functions $h_{i,p,q,j,k}$ and $d_{j,k,v}$ where $i, j, k \in \mathbb{N}$ and $p, q, v \in \mathbb{Q}$. We claim that the countable set G is dense set of $C_{ph}(X)$.

By proposition 2.2 in [2], let

$W = [x_1, V_1]^+ \bigcap ... \bigcap [x_m, V_m]^+ \bigcap [U_1, r_1]^- \bigcap ... \bigcap [U_n, r_n]^-$ be a base set of $C_{ph}(X)$ where $n, m \in \mathbb{N}$, $x_i \in X$, V_i is open set of $\mathbb{R}$ for $i \in \overline{1, m}$, U_j is open set of X and $r_j \in \mathbb{R}$ for $j \in \overline{1, n}$ and for $i \neq j$, $x_i \neq x_j$ and $\overline{U_i} \bigcap \overline{U_j} = \emptyset$.

Fix points $y_j \in U_j$ for $j = \overline{1, n}$ and choose $B_{s_l} \in \beta$ for $l = \overline{1, n+m}$ such that $\overline{B_{s_{l_1}}} \bigcap \overline{B_{s_{l_2}}} = \emptyset$ for $l_1 \neq l_2$ and $l_1, l_2 \in \overline{1, n+m}$ and $x_i \in B_{s_l}$ for $l \in \overline{1, m}$ and $y_j \in B_{s_l}$ for $l \in \overline{m+1, n}$. Choose $B_{s'_l} \in \beta$ for $l \in \overline{1, m}$ such that $x_i \in B_{s'_l}$ and $\overline{B_{s'_l}} \subseteq B_{s_l}$ and choose $B_{s'_l} \in \beta$ for $l \in \overline{m+1, n+m}$ such that $y_j \in \overline{B_{s'_l}} \subseteq B_{s_l}$ where $l = j + m$.

Fix points $v_i \in (V_i \bigcap \mathbb{Q})$ for $i \in \overline{1, m}$ and $p_j, q_j \in \mathbb{Q}$ such that $p_j < r_j < q_j$ for $j = \overline{1, n}$.

Consider $g \in G$ such that

$g = d_{s'_1, s_1, v_1} + ... + d_{s'_m, s_m, v_m} + h_{i_1, p_1, q_1, s'_{m+1}, s_{m+1}} + ... + h_{i_n, p_n, q_n, s'_{m+n}, s_{n+m}}$ where $S_{i_k} \subset B_{s'_l} \bigcap U_k$ for $k = \overline{1, n}$ and $l = k + m$.

Note that $g \in W$. This proves theorem.

□

Corollary 2.5. Let X be a Tychonoff space with countable π-base, then the following are equivalent.

1. $C_h(X)$ is a separable space.
2. X has a countable π-network consisting of $\mathcal{I}$-sets.

Corollary 2.6. If X is a separable metrizable space, then the following are equivalent.

1. $C_{ph}(X)$ is a separable space.
2. X has a countable π-network consisting of $\mathcal{I}$-sets.

Theorem 2.7. *If X is a Tychonoff space with network consisting non-trivial connected sets, then the following are equivalent.*

1. $C_{ph}(X)$ *is a separable space.*
2. X *is a separable submetrizable space.*

Proof. (1) $\Rightarrow$ (2). The map $id_1 : C_{ph}(X) \mapsto C_p(X)$ is a continuous map, hence $C_p(X)$ is a separable space. It follow that X is a separable submetrizable space.

(2) $\Rightarrow$ (1). Let X be a separable submetrizable space, i.e. X has coarser separable metric topology τ_1 and γ be network of X consisting non-trivial connected sets. Let $\beta = \{B_i\}$ be a countable base of (X, τ_1). We can assume that β closed under finite union of its elements.

For each finite family $\{B_{s_i}\}_{i=1}^{d} \subset \beta$ such that $\overline{B_{s_i}} \bigcap \overline{B_{s_j}} = \emptyset$ for $i \neq j$ and $i, j \in \overline{1, d}$ and $\{p_i\}_{i=1}^{d} \subset \mathbb{Q}$ we fix $f = f_{s_1,...,s_d,p_1...,p_d} \in C(X)$ such that $f(\overline{B_{s_i}}) = p_i$ for each $i = \overline{1, d}$.

Let G be the set of functions $f_{s_1,...,s_d,p_1...,p_d}$ where $s_i \in \mathbb{N}$ and $p_i \in \mathbb{Q}$ for $i \in \mathbb{N}$. We claim that the countable set G is dense set of $C_{ph}(X)$.

By proposition 2.2 in [2], let

$W = [x_1, V_1]^+ \bigcap ... \bigcap [x_m, V_m]^+ \bigcap [U_1, r_1]^- \bigcap ... \bigcap [U_n, r_n]^-$ be a base set of $C_{ph}(X)$ where $n, m \in \mathbb{N}$, $x_i \in X$, V_i is open set of $\mathbb{R}$ for $i \in \overline{1, m}$, U_j is open set of X and $r_j \in \mathbb{R}$ for $j \in \overline{1, n}$ and for $i \neq j$, $x_i \neq x_j$ and $\overline{U_i} \bigcap \overline{U_j} = \emptyset$.

Choose $B_{s_l} \in \beta$ for $l = \overline{1, n+m}$ such that $\overline{B_{s_{l_1}}} \bigcap \overline{B_{s_{l_2}}} = \emptyset$ for $l_1 \neq l_2$ and $l_1, l_2 \in \overline{1, n+m}$ and $x_i \in B_{s_l}$ for $l \in \overline{1, m}$ and $B_{s_l} \bigcap U_k \neq \emptyset$ for $l \in \overline{m+1, n+m}$ and $k = l - m$. Choose $B_{s'_l} \in \beta$ for $l \in \overline{1, m}$ such that $x_i \in B_{s'_l}$ and $\overline{B_{s'_l}} \subseteq B_{s_l}$ and choose $A_k \in \gamma$ for $k \in \overline{1, m}$ such that $A_k \subseteq (U_l \bigcap B_{s_l})$ where $l = k + m$.

Choose different points $s_k, t_k \in A_k$ for every $k = \overline{1, m}$. Let $S, T \in \beta$ such that $\overline{S} \bigcap \overline{T} = \emptyset$, $\overline{B_l} \bigcap \overline{S} = \emptyset$, $\overline{B_l} \bigcap \overline{T} = \emptyset$ for $l \in \overline{1, m}$ and $s_k \in S$ and $t_k \in T$ for all $k = \overline{1, m}$.

Fix points $v_i \in (V_i \bigcap \mathbb{Q})$ for $i \in \overline{1, m}$.

Choose $p, q \in \mathbb{Q}$ such that $p < \min\{r_i : i = \overline{1, n}\}$ and $q > \max\{r_i : i = \overline{1, n}\}$.

Let

$$f(x) = \begin{cases} p & for \quad x \in \overline{S} \\ q & for \quad x \in \overline{T} \\ v_l & for \quad x \in \overline{B_{s'_l}} \end{cases}$$

where $l \in \overline{1, m}$.

Note that $f \in W$. This proves theorem.

□

Theorem 2.8. *If X is a locally connected space without isolated points, then the following are equivalent.*

1. *$C_{ph}(X)$ is a separable space.*
2. *X is a separable submetrizable space.*

Definition 2.9. Let (X, τ) be a topological space. Define a cardinal function $\xi(X) = min\{|\gamma|$: for every finite family of pairwise disjoint nonempty open subsets $\{V_i\}_{i=1}^{k}$ of X there is family of pairwise disjoint nonempty zero-sets $\gamma' = \{Z_i\}_{i=1}^{k} \subseteq \gamma$ such that $V_i \bigcap Z_i \neq \emptyset$ for $i = \overline{1,k}\}$.

Obviously, that $\xi(X) \leq \pi w(X)$.

Theorem 2.10. *If X is a locally connected space without isolated points, then the following are equivalent.*

1. *$C_h(X)$ is a separable space.*
2. $\xi(X) = \aleph_0$.

Proof. (1) $\Rightarrow$ (2). Let $A = \{f_i\}$ be a countable dense set of $C_h(X)$ and $\beta = \{B_j\}$ be a countable base of $\mathbb{R}$. Define a family $\gamma = \{f_i^{-1}(\overline{B_j}) : i, j \in \mathbb{N}\}$.

Let $\{V_s\}_{s=1}^{k}$ be a finite family of pairwise disjoint nonempty open subsets of X. Consider an open base set $W = [V_1, 1]^- \bigcap ... \bigcap [V_k, k]^-$. Then there are $f_{i'} \in A \bigcap W$ and the family $\{B_{j_s} : s \in B_{j_s}$ for $s \in \overline{1,k}$ and $\overline{B_{j'_s}} \bigcap \overline{B_{j''_s}} = \emptyset$ for $s' \neq s''$ and $s', s'' \in \overline{1,k}\}$ such that $\gamma' = \{f_{i'}^{-1}(\overline{B_{j_s}})\}_{s=1}^{k}$ required the subfamily of γ.

(2) $\Rightarrow$ (1). Let $\gamma = \{F_i\}$ be family of zero-sets from definition of $\xi(X)$ such that $|\gamma| = \xi(X) = \aleph_0$. We can assume that γ closed under finite union of its elements. Consider countable set of continuous functions

$A = \{f_{i,j,p,q} \in C(X) : f_{i,j,p,q}(F_i) = p$ and $f_{i,j,p,q}(F_j) = q$ for $F_i, F_j \in \gamma$ such that $F_i \bigcap F_j = \emptyset$ and $p, q \in \mathbb{Q}\}$.

Let $W = [U_1, r_1]^- \bigcap ... \bigcap [U_n, r_n]^-$ be a base set of $C_h(X)$ where $n \in \mathbb{N}$, U_j is open set of X and $r_j \in \mathbb{R}$ for $j \in \overline{1,n}$ and for $i \neq j$, $\overline{U_i} \bigcap \overline{U_j} = \emptyset$.

Fix connected open sets S_{α_i} such that $S_{\alpha_i} \subset U_i$ for $i = \overline{1,n}$. Since S_{α_i} is not-trivial set there are different points $a_i, b_i \in S_i$ for $i = \overline{1,n}$. Let $\{O_i\}_{i=1}^{n}$ and $\{O^i\}_{i=1}^{n}$ be families of pairwise disjoint nonempty open subsets of X such that $a_i \in O_i$, $b_i \in O^i$ and $(\bigcup_{i=1}^{n} O_i) \bigcap (\bigcup_{i=1}^{n} O^i) = \emptyset$. There are family of pairwise disjoint nonempty zero-set sets $\gamma' = \{F_k\}_{k=1}^{2n} \subset \gamma$ such that $F_k \bigcap O_i \neq \emptyset$ for $k = i$ and $F_k \bigcap O^i \neq \emptyset$ for $k = i + n$. Let $H_1 = \bigcup_{k=1}^{n} F_k$ and $H_2 = \bigcup_{k=n+1}^{2n} F_k$, then consider $f = f_{i',j',p,q} \in A$ such that $f(F'_i) = p$ and $f(F'_j) = q$ where $F'_i = H_1$, $F'_j = H_2$ for some $i', j' \in \mathbb{N}$ and $p, q \in \mathbb{Q}$ such that $p < \min\{r_i : i = \overline{1,n}\}$ and $q > \max\{r_i : i = \overline{1,n}\}$. Note that $f \in W$. This proves theorem.

□

3. Consistent counter examples

Recall that a set of reals X is *null* if for each positive ϵ there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X such that $\sum_n diam(I_n) < \epsilon$. A set of reals X has *strong measure zero* if, for each sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive reals, there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X such that $diam(I_n) < \epsilon_n$ for all n. For example, every Lusin set has strong measure zero.

Example 3.1. (**CH**) *Let X be a set of reals and it has strong measure zero. Consider a space $C_h(X)$. Note that the property has strong measure zero is invariant with respect to continuous mappings [3]. Let $A = \{f_i\}_i \subset C(X)$ be a countable set of continuous functions and $X_i = X$ for each $i \in \mathbb{N}$. Direct sum $Y = \bigoplus_i X_i$ has strong measure zero. Hence a set $F(Y) \subset \mathbb{R}$ has strong measure zero where F is a continuous real-valued function on Y. So if $F \upharpoonright X_i = f_i$ we have that $\bigcup_i f_i(X) \neq \mathbb{R}$. It follows that $C_h(X)$ (a fortiori $C_{ph}(X)$) isn't a separable space.*

In [4] was shown that it is consistent with ZFC that for any set of reals of cardinality the continuum, there is a (uniformly) continuous map from that set onto the closed unit interval. In fact, this holds in the iterated perfect set model.

Theorem 3.2. *(the iterated perfect set model)*
If X is a separable metrizable space, then the following are equivalent.

1. *$C_{ph}(X)$ is a separable space.*
2. $\Delta(X) = \mathfrak{c}$.

Proof. (2) $\Rightarrow$ (1). Note that in the iterated perfect set model every nonempty open set of X is a $\mathcal{I}$-set. Really, suppose that U is a nonempty open set of X, but it isn't a $\mathcal{I}$-set. Then U is a set of reals of cardinality the continuum.

Note that for each point $x \in U$ there exist continuous function $f : X \to \mathcal{I}$ such that $f^{-1}(0) \supseteq X \setminus U$ and $f^{-1}(1) \ni x$. Clearly, that there is $r \in \mathcal{I}$ such that $r \notin f(U)$. It follows that $f^{-1}([r, 1])$ is clopen neighborhood of x and U is a zero-dimensional subspace of X. Let W be an open set such that $\overline{W} \subset U$. Then $\overline{W}$ is a set of reals of cardinality the continuum. By the iterated perfect set model there exist continuous function h from $\overline{W}$ onto the closed unit interval $\mathcal{I}$. Therefore, from Tietze-Urysohn Extension Theorem, there is a continuous function $F : X \to \mathbb{R}$ such that $F \upharpoonright \overline{W} = h$ and $F(U) \supseteq \mathcal{I}$. This contradicts our assumption.

By Theorem 2.6, $C_{ph}(X)$ is a separable space.

□

4. Remarks

Now we state two results of [1] that give for the density of the spaces $C_h(X)$ and $C_{ph}(X)$.

1.([1], Theorem 4.21) If X is a locally connected space with no isolated points, then $d(C_h(X)) = \pi w(X)$.

2.([1], Theorem 4.22) If X is a locally connected space with no isolated points, then $d(C_{ph}(X)) = \pi w(X) \cdot iw(X)$.

We note that these results are false (equality can not be !), but in these results meaning an upper bound for the density of the spaces $C_h(X)$ and $C_{ph}(X)$.

Theorem 4.1. *If X is a locally connected space with no isolated points, then $d(C_h(X)) \leq \pi w(X)$.*

Theorem 4.2. *If X is a locally connected space with no isolated points, then $d(C_{ph}(X)) \leq \pi w(X) \cdot iw(X)$.*

Now we give an example where there is no equality.

Example 4.3. *Let $X = \oplus_{\alpha<\mathfrak{c}}\mathbb{R}_\alpha$ be a direct sum of real lines $\mathbb{R}$. Then X is a separable submetrizable space i.e. $iw(X) = \aleph_0$. Clearly, that $\pi w(X) = \mathfrak{c}$. By Theorem 2.8, $C_{ph}(X)$ is separable, and, hence, $C_h(X)$ is separable.*

Proposition 4.4. *If $C_h(X)$ is a separable space, then $C_h(\beta X)$ is a separable space.*

Proof. Note that $C_h(X)$ is homeomorphic to $C_h(X,(0,1))$. Let $A=\{f_i\}$ be a countable dense set of $C_h(X,(0,1))$. Then set $\{\widetilde{f_i}\}$ is countable dense subset of $C_h(\beta X,(0,1))$ where $\widetilde{f_i}\restriction X=f_i$. Really let $W=[U_1,r_1]^-\bigcap...\bigcap[U_n,r_n]^-$ be a base set of $C_h(\beta X)$ where $n\in\mathbb{N}$, U_j is open set of βX and $r_j\in\mathbb{R}$ for $j\in\overline{1,n}$ and for $i\neq j$, $\overline{U_i}\bigcap\overline{U_j}=\emptyset$. Clearly that $V=[V_1,r_1]^-\bigcap...\bigcap[V_n,r_n]^-$ be a open set of $C_h(X)$ where $n\in\mathbb{N}$, $V_j=X\bigcap U_j$ is open set of X and $r_j\in\mathbb{R}$ for $j\in\overline{1,n}$ and for $i\neq j$, $\overline{V_i}\bigcap\overline{V_j}=\emptyset$. There is $f_i'\in A\bigcap V$ and it follows that $\widetilde{f_i'}\in W$.

□

Example 4.5. *Let $X=\mathbb{R}$. By Theorem 2.6, $C_{ph}(X)$ is a separable space, but $C_{ph}(\beta X)$ is not a separable space because βX is not a separably submetrizable space.*

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