Special elements of the lattice of monoid varieties

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Abstract. We completely classify all neutral and costandard elements in the lattice $\mathbb{MON}$ of all monoid varieties. Further, we prove that an arbitrary upper-modular element of $\mathbb{MON}$ except the variety of all monoids is either a completely regular or a commutative variety. Finally, we verify that all commutative varieties of monoids are codistributive elements of $\mathbb{MON}$. Thus, the problems of describing codistributive or upper-modular elements of $\mathbb{MON}$ are completely reduced to the completely regular case.

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1. Introduction and summary

The lattice of all semigroup varieties denoted hereinafter by $\mathbb{SEM}$ has been the subject of an intensive research over the last five decades. An extensive and quite diverse material has been accumulated in this direction. It is systematically presented in the survey [14]. In sharp contrast, the lattice $\mathbb{MON}$ of all monoid varieties has received much less attention over the years (when referring to monoid varieties, we consider monoids as algebras with an associative binary operation and the nullary operation that fixes the unit element). Up to the recent time, the latest lattice have been considered in the articles [6, 12, 21] only. However, recently interest in the lattice $\mathbb{MON}$ has grown. This is confirmed by the fact that several papers devoted principally to an examination of identities of monoids contain some non-trivial results about lattices of varieties (see [7, 8, 10, 11], for instance).

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Many questions about the lattice $\mathbb{SEM}$ are formulated in terms of identities. It is proved in the early 1970's in [1, 2] that the lattice $\mathbb{SEM}$ does not satisfy any non-trivial identity. A similar fact for the lattice $\mathbb{MON}$ was established quite recently in [4]. In view of this result, it seems natural to study varieties of monoids with different identities in subvariety lattice. The most important lattice identities are the distributive and modular laws. The problem of describing monoid varieties with distributive subvariety lattice seems to be quite difficult. Indeed, it turns out that even the examination of a much more stronger restriction "to be a chain" to lattices of monoid varieties required very considerable efforts (see [5]).

In this paper, we study several restrictions on the monoid varieties related to the distributive and modular laws. More exactly, we consider special elements of several types in the lattice $\mathbb{MON}$. Let us recall definitions of those of them which will be used below. An element x of a lattice L is called

neutral if	$\forall\, y,z \in L\colon$	$(x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x);$
costandard if	$\forall\, y,z \in L\colon$	$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z);$
codistributive if	$\forall\, y,z \in L\colon$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z);$
modular if	$\forall\, y,z \in L\colon$	$y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y;$
upper-modular if	$\forall\, y,z \in L\colon$	$y \le x \longrightarrow x \wedge (y \vee z) = y \vee (x \wedge z).$

Lower-modular elements are defined dually to upper-modular once. It is well known that an element $x \in L$ is neutral if and only if, for any $y, z \in L$, the elements x, y and z generate a distributive sublattice of L (see [3, Theorem 254], for instance). Note that neutral elements play an important role in the general theory of lattices. In particular, it is well known that if a is a neutral element in a lattice L then L is decomposable into a subdirect product of the principal ideal and the principal filter of L generated by a (see [3, Theorem 254], for instance). Thus, the knowledge of which elements of a lattice are neutral gives essential information on the structure of the lattice as a whole. It is evident that a neutral element is both lower-modular and costandard; a costandard element is modular; a codistributive element is upper-modular. It is well known also that a costandard element is codistributive (see [3, Theorem 253], for instance). Some information about special elements in arbitrary lattices can be found in [3, Section III.2] and [13, Chapter 1].

There are many interesting and deep results about special elements of the mentioned above types in the lattice $\mathbb{SEM}$ (see the surveys [14, Section 14] and [18]). In particular, neutral elements of the lattice $\mathbb{SEM}$ have been completely described in [20, Proposition 4.1], while in [17, Theorem 1.3] it is proved that a semigroup variety is a costandard element of the lattice $\mathbb{SEM}$ if and only if it is a neutral element of this lattice. Codistributive elements of $\mathbb{SEM}$ were examined in [17], while upper-modular elements of $\mathbb{SEM}$ were considered in [15, 16].

Special elements in the lattice $\mathbb{MON}$ were not studied so far. The main results of this work give a complete descriptions of neutral and costandard elements of the lattice $\mathbb{MON}$. Besides that, we obtain a valuable information about codistributive and upper-modular elements of $\mathbb{MON}$.

In order to formulate the first main result of the article, we fix notation for a few concrete varieties. The trivial variety of monoids is denoted by $\mathbf{T}$, while $\mathbf{MON}$ denotes the variety of all monoids. We denote by $\mathbf{SL}$ the variety of all semilattice monoids. Our first main result is the following theorem.

Theorem 1.1. *For a monoid variety* $\mathbf{V}$*, the following are equivalent:*

(i) $\mathbf{V}$ *is a modular, lower-modular and upper-modular element of the lattice* $\mathbb{MON}$*;*
(ii) $\mathbf{V}$ *is a neutral element of the lattice* $\mathbb{MON}$*;*
(iii) $\mathbf{V}$ *is one of the varieties* $\mathbf{T}$*,* $\mathbf{SL}$ *or* $\mathbf{MON}$*.*

In order to formulate the second main result of the article, we need some notation. We denote by F^1 the free monoid over a countably infinite alphabet. The words (i.e., elements of F^1) unlike letters are written in bold. Two sides of identities we connect by the symbol $\approx$, while the symbol $=$ denotes the equality relation on F^1. For an identity system Σ, we denote by var Σ the variety of monoids given by Σ. Put

$$\mathbf{C}_n = \mathrm{var}\{x^n \approx x^{n+1}, xy \approx yx\}$$

where $n \geq 2$. Our second main result is the following theorem.

Theorem 1.2. *For a monoid variety* $\mathbf{V}$*, the following are equivalent:*

(i) $\mathbf{V}$ *is a modular and upper-modular element of the lattice* $\mathbb{MON}$*;*
(ii) $\mathbf{V}$ *is a costandard element of the lattice* $\mathbb{MON}$*;*
(iii) $\mathbf{V}$ *is one of the varieties* $\mathbf{T}$*,* $\mathbf{SL}$*,* $\mathbf{C}_2$ *or* $\mathbf{MON}$*.*

As we have already mentioned above, an element of $\mathbb{SEM}$ is neutral if and only if it is costandard. Theorems 1.1 and 1.2 show that in the lattice $\mathbb{MON}$ these properties are not equivalent.

The following assertion gives a valuable information about upper-modular elements of the lattice $\mathbb{MON}$. A variety of monoids is called *proper* if it differs from the variety of all monoids. A variety of monoids is said to be *completely regular* if it consists of *completely regular monoids* (i.e., unions of groups).

Proposition 1.3. *If a proper monoid variety* $\mathbf{V}$ *is an upper-modular element of the lattice* $\mathbb{MON}$ *then* $\mathbf{V}$ *is either commutative or completely regular.*

This proposition immediately implies that any proper monoid variety that is a codistributive element of $\mathbb{MON}$ also is either commutative or completely regular. To determine codistributive elements in the completely regular case, we need to consider, in particular, periodic group varieties. The lattice of periodic group varieties is modular but not distributive. Therefore, it contains the 5-element modular non-distributive sublattice. It is evident that all three pairwise non-comparable elements of this sublattice are non-codistributive elements of $\mathbb{MON}$. We see that the problem of describing codistributive elements of $\mathbb{MON}$ in the completely regular case is closely related to the problem of describing periodic group varieties with distributive subvariety lattice. The latter problem seems to be extremely difficult (see [14, Subsection 11.2] for more detailed comments), whence the former problem is extremely difficult too. Fortunately, out of the completely regular case, the problem of describing codistributive and even upper-modular elements of $\mathbb{MON}$ possesses the complete decision. Proposition 1.3 shows that, to achieve this goal, it suffices to consider commutative varieties. It tuns out that the following assertion is true.

Proposition 1.4. *Every commutative monoid variety is a codistributive element of the lattice* $\mathbb{MON}$.

We note that Propositions 1.3 and 1.4 play an important role in the proof of Theorems 1.1 and 1.2.

The article consists of four sections. Section 2 contains definitions, notation and auxiliary results. In Section 3 several auxiliary results about modular or lower-modular elements of the lattice $\mathbb{MON}$ are collected. Finally, Section 4 is devoted to the proofs of Theorems 1.1 and 1.2 and Propositions 1.3 and 1.4.

2. Preliminaries

We start with the fact that is a part of the semigroup folklore (it is noted in [8, Section 1.1], for instance).

Proposition 2.1. *Let M be a monoid. We denote by $\mathbf{V}$ the monoid variety generated by M and by $\mathbf{W}$ the semigroup variety generated by this monoid. Then the map $\mathbf{V} \mapsto \mathbf{W}$ is an injective homomorphism of the lattice* $\mathbb{MON}$ *into the lattice* $\mathbb{SEM}$.

Obviously, the varieties $\mathbf{T}$ and $\mathbf{MON}$ are neutral elements of the lattice $\mathbb{MON}$. It is proved in [20, Proposition 2.4] that $\mathbf{SL}$ is a neutral element of the lattice $\mathbb{SEM}$. Using Proposition 2.1 we have that the following statement is true.

Lemma 2.2. *The varieties $\mathbf{T}$, $\mathbf{SL}$ and $\mathbf{MON}$ are neutral elements of the lattice* $\mathbb{MON}$.

The variety of all Abelian groups whose exponent divides n is denoted by $\mathbf{A}_n$. We note that $\mathbf{A}_1 = \mathbf{T}$. We need the following result obtained in [6].

Lemma 2.3. *If $\mathbf{V}$ is a periodic commutative monoid variety then $\mathbf{V} = \mathbf{A}_n \vee \mathbf{M}$ where n is some natural number and $\mathbf{M}$ is one of the varieties $\mathbf{T}$, $\mathbf{SL}$ or $\mathbf{C}_m$ for some $m \geq 2$.*

The *content* of a word $\mathbf{w}$, i.e., the set of all letters occurring in $\mathbf{w}$ is denoted by $\mathrm{con}(\mathbf{w})$. The following statement is well known and can be easily verified (see [5, Lemma 2.1], for instance).

Lemma 2.4. *For a monoid variety* $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is a group variety;*
b) $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$ *with* $\mathrm{con}(\mathbf{u}) \neq \mathrm{con}(\mathbf{v})$*;*
c) $\mathbf{SL} \nsubseteq \mathbf{V}$.

A letter is called *simple in a word* $\mathbf{w}$ if it occurs in $\mathbf{w}$ only once. The following fact is well known and may be easily verified.

Lemma 2.5. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{C}_2$ *if and only if* $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ *and the set of all simple in* $\mathbf{u}$ *letters coincides with the set of all simple in* $\mathbf{v}$ *letters.*

Put $\mathbf{D} = \mathrm{var}\{x^2 \approx x^3, x^2y \approx xyx \approx yx^2\}$.

Lemma 2.6 ([5, Lemma 2.14])**.** *If a variety of monoids* $\mathbf{V}$ *is non-completely regular and non-commutative then* $\mathbf{D} \subseteq \mathbf{V}$.

To avoid a confusion, we note that, in [5], the variety $\mathbf{D}$ is denoted by $\mathbf{D}_1$, while $\mathbf{D}$ denotes another variety.

Lemma 2.7 ([15, Lemma 2.6])**.** *If* $\mathbf{V}$ *is a semigroup variety that satisfies the identity* $x^n \approx x^{n+1}$ *for some* n *and* $\mathbf{G}$ *is a variety of periodic groups then* $\mathbf{G}$ *is the largest group subvariety of* $\mathbf{G} \vee \mathbf{X}$.

3. On modular and lower-modular elements in the lattice $\mathbb{MON}$

The assertions provided in this section will be used in the proof of Theorems 1.1 and 1.2 and Proposition 1.3. The following assertion was communicated to the author by M.V. Volkov. But the proof given above is found by the author.

Lemma 3.1. *Let* $\mathbf{V}$ *be a non-commutative completely regular monoid variety. Then*

$$\mathbf{C}_2 \vee (\mathbf{D} \wedge \mathbf{V}) = \mathbf{C}_2 \subset \mathbf{D} = \mathbf{D} \wedge (\mathbf{C}_2 \vee \mathbf{V}).$$

In particular, $\mathbf{V}$ *is not a modular element of the lattice* $\mathbb{MON}$ *and* $\mathbf{C}_2$ *is not a lower-modular element of this lattice.*

Proof. It easily follows from Lemmas 4.4 and 4.5(ii) of [7], for instance, that the subvariety lattice of the variety $\mathbf{D}$ is the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}$. This fact implies that the variety $\mathbf{SL}$ is a maximal completely regular subvariety of the variety $\mathbf{D}$. The variety $\mathbf{V} \wedge \mathbf{D}$ is completely regular. Hence $\mathbf{D} \wedge \mathbf{V} \subseteq \mathbf{SL}$. Thus, $\mathbf{C}_2 \vee (\mathbf{D} \wedge \mathbf{V}) = \mathbf{C}_2$. On the other hand, the variety $\mathbf{C}_2 \vee \mathbf{V}$ is non-completely regular and non-commutative because $\mathbf{V}$ is non-commutative and $\mathbf{C}_2$ is non-completely regular. Then Lemma 2.6 implies that $\mathbf{D} \subseteq \mathbf{V} \vee \mathbf{C}_2$, whence $\mathbf{D} = \mathbf{D} \wedge (\mathbf{C}_2 \vee \mathbf{V})$. □

A word $\mathbf{w}$ is called an *isoterm* for a class of monoids if each monoid in the class does not satisfy any non-trivial identity of the form $\mathbf{w} \approx \mathbf{w}'$. The empty word (i.e., the unit element of the monoid F^1) is denoted by λ. Let us fix notation for the following two words:

$$\mathbf{s} = yxyzxz \quad \text{and} \quad \mathbf{t} = yxzxyxz.$$

Put $\mathbf{B}_{2,3} = \operatorname{var}\{x^2 \approx x^3\}$ and $\mathbf{Q} = \operatorname{var}\{\mathbf{s} \approx \mathbf{t}\}$. Clearly, $\mathbf{Q} \subset \mathbf{B}_{2,3}$.

Lemma 3.2. *If $\mathbf{V}$ is a commutative monoid variety containing a non-trivial group then*

$$\mathbf{Q} \vee (\mathbf{B}_{2,3} \wedge \mathbf{V}) \subset \mathbf{B}_{2,3} \wedge (\mathbf{Q} \vee \mathbf{V}).$$

In particular, $\mathbf{V}$ is not a modular element of the lattice $\mathbb{MON}$ and $\mathbf{Q}$ is not a lower-modular element of this lattice.

Proof. In view of Lemma 2.5, $\mathbf{C}_2 \subseteq \mathbf{Q}$. Since is $\mathbf{V}$ is commutative, we have that $\mathbf{B}_{2,3} \wedge \mathbf{V} \subseteq \mathbf{C}_2$. Therefore,

$$\mathbf{Q} \vee (\mathbf{B}_{2,3} \wedge \mathbf{V}) = \mathbf{Q} \subseteq \mathbf{B}_{2,3} \wedge (\mathbf{Q} \vee \mathbf{V}).$$

We are going to verify that this inclusion is strict. It suffices to establish that $\mathbf{B}_{2,3} \wedge (\mathbf{Q} \vee \mathbf{V})$ violates $\mathbf{s} \approx \mathbf{t}$. If, otherwise, $\mathbf{B}_{2,3} \wedge (\mathbf{Q} \vee \mathbf{V})$ satisfies $\mathbf{s} \approx \mathbf{t}$ then there is a sequence of pairwise distinct words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_k$ such that $\mathbf{w}_0 = \mathbf{s}$, $\mathbf{w}_k = \mathbf{t}$ and the identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds either in $\mathbf{B}_{2,3}$ or in $\mathbf{Q} \vee \mathbf{V}$ for every $0 \le i < k$. We note that the identity $\mathbf{s} \approx \mathbf{w}_1$ does not hold in the variety $\mathbf{B}_{2,3}$ because $\mathbf{s}$ is an isoterm for $\mathbf{B}_{2,3}$. Thus, this identity holds in $\mathbf{Q} \vee \mathbf{V}$. In particular, it holds in $\mathbf{Q}$, whence there exists a *deduction* of the identity $\mathbf{s} \approx \mathbf{w}_1$ from the identity $\mathbf{s} \approx \mathbf{t}$, i.e., a sequence of pairwise distinct words

$$\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m \tag{3.1}$$

such that $\mathbf{v}_0 = \mathbf{s}$, $\mathbf{v}_m = \mathbf{w}_1$ and, for any $0 \le i < m$, there exist words $\mathbf{a}_i$, $\mathbf{b}_i$ and endomorphism ξ_i of F^1 such that ether $\mathbf{v}_i = \mathbf{a}_i \xi_i(\mathbf{s}) \mathbf{b}_i$ and $\mathbf{v}_{i+1} = \mathbf{a}_i \xi_i(\mathbf{t}) \mathbf{b}_i$ or $\mathbf{v}_i = \mathbf{a}_i \xi_i(\mathbf{t}) \mathbf{b}_i$ and $\mathbf{v}_{i+1} = \mathbf{a}_i \xi_i(\mathbf{s}) \mathbf{b}_i$. We can assume without loss of generality that the sequence (3.1) is the shortest deduction of the identity $\mathbf{s} \approx \mathbf{w}_1$ from $\mathbf{s} \approx \mathbf{t}$.

Let η be an arbitrary endomorphism of the monoid F^1. If η maps at least one of the letters x, y or z into the empty word then the words $\eta(\mathbf{s})$ and $\eta(\mathbf{t})$ have one of the forms given in Table 1. We see that in all cases $\eta(\mathbf{s})$ and $\eta(\mathbf{t})$ contain a subword of the form $\mathbf{w}^2$. We note that the words $\mathbf{s}$ and $\mathbf{t}$ are square-free. This fact and the information collected in Table 1 imply that if the equality $\mathbf{a} = \mathbf{c}\eta(\mathbf{b})\mathbf{d}$ holds where $\mathbf{a}, \mathbf{b} \in \{\mathbf{s}, \mathbf{t}\}$, $\mathbf{c}, \mathbf{d} \in F^1$ and η is an endomorphism of F^1 that maps at least one of the letters x, y and z into the empty word then $\eta(\mathbf{b}) = \lambda$. This fact will be repeatedly used below to obtain a contradiction.

Suppose first that $\mathbf{s} = \mathbf{v}_0 = \mathbf{a}_0 \xi_0(\mathbf{s}) \mathbf{b}_0$ and $\mathbf{v}_1 = \mathbf{a}_0 \xi_0(\mathbf{t}) \mathbf{b}_0$. If the words $\mathbf{a}_0$ and $\mathbf{b}_0$ are empty then $\xi_0(a) = a$ for each $a \in \{x, y, z\}$. Then $\mathbf{v}_1 = \mathbf{t}$. Suppose now that at least one of the words $\mathbf{a}_0$ and $\mathbf{b}_0$ is non-empty. Then the endomorphism ξ_0 maps one of the letters x, y and z into the empty word. We have verified in the previous paragraph that $\xi_0(\mathbf{s}) = \xi_0(\mathbf{t}) = \lambda$, whence

TABLE 1. The forms of the words $\eta(\mathbf{s})$ and $\eta(\mathbf{t})$ depending on the endomorphism η

The word	The form of the word if		
	$\eta(x)=\lambda$	$\eta(y)=\lambda$	$\eta(z)=\lambda$
$\eta(\mathbf{s})$	$\mathbf{p}^2\mathbf{q}^2$	$(\mathbf{pq})^2$	$(\mathbf{pq})^2$
$\eta(\mathbf{t})$	$(\mathbf{pq})^2$	$\mathbf{pqp}^2\mathbf{q}$	$\mathbf{pq}^2\mathbf{pq}$

$\mathbf{v}_0=\mathbf{v}_1=\mathbf{a}_0\mathbf{b}_0$ in this case. We obtain a contradiction with the fact that the words $\mathbf{v}_0$ and $\mathbf{v}_1$ are distinct.

Suppose now that $\mathbf{s}=\mathbf{v}_0=\mathbf{a}_0\xi_0(\mathbf{t})\mathbf{b}_0$ and $\mathbf{v}_1=\mathbf{a}_0\xi_0(\mathbf{s})\mathbf{b}_0$. Since the length of $\mathbf{s}$ is less than the length of $\mathbf{t}$, the endomorphism ξ_0 maps one of the letters x, y and z into the empty word. As we have seen above, $\xi_0(\mathbf{t})=\lambda$ in this case. This contradicts with the inequality $\mathbf{v}_0\ne\mathbf{v}_1$. Thus, we have verified that $\mathbf{v}_1=\mathbf{t}$.

Suppose that $\mathbf{t}=\mathbf{v}_1=\mathbf{a}_1\xi_1(\mathbf{s})\mathbf{b}_1$ and $\mathbf{v}_2=\mathbf{a}_1\xi_1(\mathbf{t})\mathbf{b}_1$. Note that the number of occurrences of the letter x in $\xi_1(\mathbf{s})$ is not equal to 3, whence $x\in\operatorname{con}(\mathbf{a}_1\mathbf{b}_1)$. Since neither the first nor the last letter of the word $\mathbf{t}$ does not coincide with x, the length of the word $\mathbf{a}_1\mathbf{b}_1$ is more than 1. Then the endomorphism ξ_1 maps one of the letters x, y and z into the empty word. Then $\xi_1(\mathbf{s})=\lambda$, whence $\mathbf{v}_1=\mathbf{v}_2$, a contradiction.

Suppose now that $\mathbf{t}=\mathbf{v}_1=\mathbf{a}_1\xi_1(\mathbf{t})\mathbf{b}_1$ and $\mathbf{v}_2=\mathbf{a}_1\xi_1(\mathbf{s})\mathbf{b}_1$. If the words $\mathbf{a}_1$ and $\mathbf{b}_1$ are empty then $\xi_1(a)=a$ for each $a\in\{x,y,z\}$. Then $\mathbf{v}_2=\mathbf{s}$. But this is impossible because the sequence (3.1) is the shortest deduction of the identity $\mathbf{s}\approx\mathbf{w}_1$ from the identity $\mathbf{s}\approx\mathbf{t}$. So, at least one of the words $\mathbf{a}_1$ and $\mathbf{b}_1$ is non-empty. Then the endomorphism ξ_1 maps one of the letters x, y and z into the empty word. Then $\xi_1(\mathbf{t})=\lambda$, whence $\mathbf{v}_1=\mathbf{v}_2$. We obtain a contradiction with the fact that the words $\mathbf{v}_2$ and $\mathbf{v}_1$ are distinct. Thus, we have proved that $m=1$ and $\mathbf{v}_1=\mathbf{w}_1=\mathbf{t}$. Therefore, the identity $\mathbf{s}\approx\mathbf{t}$ holds in the variety $\mathbf{Q}\vee\mathbf{V}$. Then this variety satisfies the identity $x^2\approx x^3$. But this is impossible because $\mathbf{V}$ contains a non-trivial group. □

Put $\mathbf{F}=\operatorname{var}\{xyx\approx xyx^2, x^2y^2\approx y^2x^2, x^2y\approx x^2yx, xytxy\approx yxtxy\}$. Clearly, $\mathbf{F}\subset\mathbf{B}_{2,3}$.

Lemma 3.3. *If $n>2$ then*

$$(\mathbf{C}_n\wedge\mathbf{B}_{2,3})\vee\mathbf{F}\subset(\mathbf{C}_n\vee\mathbf{F})\wedge\mathbf{B}_{2,3}.$$

In particular, $\mathbf{C}_n$ with $n>2$ is not a modular element of the lattice $\mathbb{MON}$ and $\mathbf{F}$ is not a lower-modular element of this lattice.

Proof. Evidently, $(\mathbf{C}_n\wedge\mathbf{B}_{2,3})\vee\mathbf{F}\subseteq(\mathbf{C}_n\vee\mathbf{F})\wedge\mathbf{B}_{2,3}$. We are going to verify that this inclusion is strict. Lemma 2.5 implies that $\mathbf{C}_2\subseteq\mathbf{F}$. Then

$$(\mathbf{C}_n\wedge\mathbf{B}_{2,3})\vee\mathbf{F}=\mathbf{C}_2\vee\mathbf{F}=\mathbf{F}.$$

Thus, we need to verify that $\mathbf{F}\subset(\mathbf{C}_n\vee\mathbf{F})\wedge\mathbf{B}_{2,3}$. It suffices to establish that $(\mathbf{C}_n\vee\mathbf{F})\wedge\mathbf{B}_{2,3}$ violates $xyx\approx xyx^2$. If, otherwise, $(\mathbf{C}_n\vee\mathbf{F})\wedge\mathbf{B}_{2,3}$ satisfies

$xyx \approx xyx^2$ then there is a sequence of pairwise distinct words (3.1) such that $\mathbf{v}_0 = xyx$, $\mathbf{v}_m = xyx^2$ and, for any $0 \le i < m$, the identity $\mathbf{v}_i \approx \mathbf{v}_{i+1}$ holds either in $\mathbf{B}_{2,3}$ or in $\mathbf{C}_n \vee \mathbf{F}$. We note that the identity $xyx \approx \mathbf{v}_1$ does not hold in the variety $\mathbf{B}_{2,3}$ because xyx is an isoterm for $\mathbf{B}_{2,3}$. On the other hand, the identity $xyx \approx \mathbf{v}_1$ holds in $\mathbf{C}_n$ if and only if this identity follows from commutativity. Then $\mathbf{v}_1 \in \{x^2y, yx^2\}$. Put

$$\mathbf{E} = \mathrm{var}\{x^2 \approx x^3, x^2y \approx xyx, x^2y^2 \approx y^2x^2\}.$$

It is evident that $\mathbf{E} \subseteq \mathbf{F}$. Comparison of Propositions 4.2 and 6.9(i) of the article [5] shows that this inclusion is strict. Thus,

$$\mathbf{E} \subset \mathbf{F} \tag{3.2}$$

(we note that in [5] the variety $\mathbf{F}$ is denoted by $\mathbf{F}_1$). If $\mathbf{v}_1 = x^2y$ then $\mathbf{F}$ satisfies the identity $x^2y \approx xyx$. We obtain a contradiction with (3.2). If $\mathbf{v}_1 = yx^2$ then $\mathbf{F}$ satisfies the identities

$$xyx \approx \mathbf{v}_1 = yx^2 \approx yx^3 \approx xyx^2 \approx x^2yx \approx x^2y,$$

and we have a contradiction with (3.2) again. So, $(\mathbf{C}_n \vee \mathbf{F}) \wedge \mathbf{B}_{2,3}$ violates $xyx \approx xyx^2$, whence $\mathbf{F} \subset (\mathbf{C}_n \vee \mathbf{F}) \wedge \mathbf{B}_{2,3}$. □

4. Proofs of the main results

Proof of Proposition 1.3. Let $\mathbf{V}$ be a proper non-commutative non-completely regular monoid variety that is an upper-modular element of the lattice $\mathbb{MON}$. Then $\mathbf{D} \subseteq \mathbf{V}$ by Lemma 2.6. It is proved in [15, Lemma 2.16] that the variety of all semigroups is generated by all minimal non-Abelian varieties of groups. This fact and Proposition 2.1 imply that there exists a minimal non-Abelian group variety $\mathbf{G}$ such that $\mathbf{G} \nsubseteq \mathbf{V}$. Then $\mathbf{V} \wedge \mathbf{G} = \mathbf{A}_n$ for some positive integer n, whence $\mathbf{C}_2 \vee (\mathbf{V} \wedge \mathbf{G}) = \mathbf{C}_2 \vee \mathbf{A}_n$. On the other hand, $\mathbf{D} \subseteq \mathbf{C}_2 \vee \mathbf{G}$ by Lemma 3.1. Taking into account the fact that $\mathbf{D} \subseteq \mathbf{V}$, we have that $\mathbf{D} \subseteq \mathbf{V} \wedge (\mathbf{C}_2 \vee \mathbf{G})$. The variety $\mathbf{A}_n \vee \mathbf{C}_2$ is commutative, while $\mathbf{D}$ is non-commutative, whence

$$\mathbf{C}_2 \vee (\mathbf{V} \wedge \mathbf{G}) \neq \mathbf{V} \wedge (\mathbf{C}_2 \vee \mathbf{G}).$$

Since $\mathbf{C}_2 \subset \mathbf{D} \subseteq \mathbf{V}$, we obtain a contradiction with the fact that the variety $\mathbf{V}$ is an upper-modular element of the lattice $\mathbb{MON}$. □

Proof of Proposition 1.4. Let $\mathbf{V}$ be a commutative monoid variety and $\mathbf{Y}$, $\mathbf{Z}$ be arbitrary monoid varieties. Put $\mathbf{X} = \mathbf{V} \wedge (\mathbf{Y} \vee \mathbf{Z})$ and $\mathbf{W} = (\mathbf{V} \wedge \mathbf{Y}) \vee (\mathbf{V} \wedge \mathbf{Z})$. Evidently, $\mathbf{W} \subseteq \mathbf{X}$. We need to verify that $\mathbf{X} \subseteq \mathbf{W}$. If $\mathbf{V} \subseteq \mathbf{Y}$ then

$$\mathbf{X} = \mathbf{V} \wedge (\mathbf{Y} \vee \mathbf{Z}) = \mathbf{V} = \mathbf{V} \vee (\mathbf{V} \wedge \mathbf{Z}) = (\mathbf{V} \wedge \mathbf{Y}) \vee (\mathbf{V} \wedge \mathbf{Z}) = \mathbf{W},$$

and we are done. Therefore, we may assume that $\mathbf{V} \nsubseteq \mathbf{Y}$. By symmetry, $\mathbf{V} \nsubseteq \mathbf{Z}$. If $\mathbf{V}$ is periodic then $\mathbf{X}$ is periodic too. If $\mathbf{V}$ is non-periodic then $\mathbf{V}$ is the variety of all commutative monoids. Since $\mathbf{V} \nsubseteq \mathbf{Y}$ and $\mathbf{V} \nsubseteq \mathbf{Z}$, the varieties $\mathbf{Y}$ and $\mathbf{Z}$ are periodic, whence $\mathbf{Y} \vee \mathbf{Z}$ is periodic too. Thus, $\mathbf{X}$ is a periodic commutative variety. Then Lemma 2.3 imply that $\mathbf{X} = \mathbf{M} \vee \mathbf{A}_s$ for some s where $\mathbf{M}$ is one of the varieties $\mathbf{T}$, $\mathbf{SL}$ or $\mathbf{C}_n$ with $n \geq 2$. Evidently, $\mathbf{M} \subseteq \mathbf{V}$ and $\mathbf{M} \subseteq \mathbf{Y} \vee \mathbf{Z}$. Now we are going to prove that either $\mathbf{M} \subseteq \mathbf{Y}$ or $\mathbf{M} \subseteq \mathbf{Z}$. If $\mathbf{M} = \mathbf{T}$ then we are done. If $\mathbf{M} = \mathbf{SL}$ then the required fact follows from Lemma 2.4. Let now $\mathbf{M} = \mathbf{C}_n$ with $n \geq 2$. Suppose that $\mathbf{M} \nsubseteq \mathbf{Y}$ and $\mathbf{M} \nsubseteq \mathbf{Z}$. It is proved in [5, Lemma 2.5] that if a monoid variety does not contain $\mathbf{C}_n$ then this variety satisfies the identity $x^{n-1} \approx x^{n-1+\ell}$ for some natural ℓ. This fact implies that there are natural numbers i and j such that $x^{n-1} \approx x^{n-1+i}$ holds in $\mathbf{Y}$ and $x^{n-1} \approx x^{n-1+j}$ holds in $\mathbf{Z}$. Then the variety $\mathbf{Y} \vee \mathbf{Z}$ satisfies the identity $x^{n-1} \approx x^{n-1+ij}$. We obtain a contradiction with the fact that $\mathbf{M} \subseteq \mathbf{Y} \vee \mathbf{Z}$. Thus, we have proved that either $\mathbf{M} \subseteq \mathbf{Y}$ or $\mathbf{M} \subseteq \mathbf{Z}$. Since $\mathbf{M} \subseteq \mathbf{V}$, we get that either $\mathbf{M} \subseteq \mathbf{V} \wedge \mathbf{Y}$ or $\mathbf{M} \subseteq \mathbf{V} \wedge \mathbf{Z}$. Therefore, $\mathbf{M} \subseteq \mathbf{W}$.

Now we are going to verify that $\mathbf{A}_s \subseteq \mathbf{W}$. We note also that $\mathbf{W}$ is a periodic commutative variety. Then Lemma 2.3 applies with the conclusion that $\mathbf{W} = \mathbf{M}' \vee \mathbf{A}_r$ for some r where $\mathbf{M}'$ is one of the varieties $\mathbf{T}$, $\mathbf{SL}$ or $\mathbf{C}_n$ with $n \geq 2$. Suppose that s does not divide r. Then there exist a prime number p and a positive integer k such that p^k divides s but does not divide r. Put $q = p^k$. Then $\mathbf{A}_q \subseteq \mathbf{A}_s \subseteq \mathbf{X}$ but $\mathbf{A}_q \nsubseteq \mathbf{A}_r$. It is easy to see that any group subvariety of the variety $\mathbf{W} = \mathbf{M}' \vee \mathbf{A}_r$ is contained in $\mathbf{A}_r$ (this follows from Proposition 2.1 and Lemma 2.7, for instance). Therefore, $\mathbf{A}_q \nsubseteq \mathbf{W}$. Since $\mathbf{A}_q \subseteq \mathbf{X}$, we have that $\mathbf{A}_q \subseteq \mathbf{Y} \vee \mathbf{Z}$. It is proved in [17, Theorem 1.2] that every variety of periodic Abelian groups is a codistributive element of the lattice $\mathbb{SEM}$. This fact and Proposition 2.1 imply that

$$\mathbf{A}_q = \mathbf{A}_q \wedge (\mathbf{Y} \vee \mathbf{Z}) = (\mathbf{A}_q \wedge \mathbf{Y}) \vee (\mathbf{A}_q \wedge \mathbf{Z}).$$

Since the subvariety lattice of the variety $\mathbf{A}_q$ is a chain, we have that $\mathbf{A}_q$ coincides with one of the varieties $\mathbf{A}_q \wedge \mathbf{Y}$ or $\mathbf{A}_q \wedge \mathbf{Z}$, whence either $\mathbf{A}_q \subseteq \mathbf{Y}$ or $\mathbf{A}_q \subseteq \mathbf{Z}$. Taking into account that $\mathbf{A}_q \subseteq \mathbf{X} \subseteq \mathbf{V}$, we obtain a contradiction with the fact that $\mathbf{A}_q \nsubseteq \mathbf{W}$. Thus, s divides r. Then $\mathbf{A}_s \subseteq \mathbf{A}_r \subseteq \mathbf{W}$. Therefore, $\mathbf{X} = \mathbf{M} \vee \mathbf{A}_s \subseteq \mathbf{W}$. Proposition 1.4 is proved. □

Propositions 1.3 and 1.4 completely reduce the problems of describing codistributive or upper-modular elements in $\mathbb{MON}$ to consideration of completely regular varieties. As we have noted in Section 1, there are non-codistributive elements of $\mathbb{MON}$ among periodic group varieties. But the following question is open.

Question 4.1. Is every completely regular monoid variety an upper-modular element of $\mathbb{MON}$?

The lattice $\mathbb{SEM}$ contains upper modular but not codistributive elements (see [18, Subsection 3.9]). On the other hand, in the lattice of all commutative semigroup varieties the properties to be upper-modular and codistributive elements are equivalent [19, Theorem 1.1]. The following question is open so far.

Question 4.2. Does there exist an upper-modular but not codistributive element of the lattice $\mathbb{MON}$?

Clearly, the negative answer to Question 4.2 immediately implies the negative answer to Question 4.1.

Proof of Theorem 1.2. The implication (ii) $\Rightarrow$ (i) is obvious. It remains to prove the implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (iii). Let $\mathbf{V}$ be a proper monoid variety that is a modular and upper-modular element of the lattice $\mathbb{MON}$. Then Proposition 1.3 implies that $\mathbf{V}$ is either completely regular or commutative. The case when $\mathbf{V}$ is completely regular and non-commutative is impossible by Lemma 3.1. Therefore, $\mathbf{V}$ is commutative. Further, Lemma 3.2 implies that all groups of $\mathbf{V}$ are trivial. Then $\mathbf{V}$ satisfies the identity $x^n \approx x^{n+1}$ for some n, whence $\mathbf{V} \subseteq \mathbf{C}_n$. By Lemma 2.3 (see also [5, Proposition 5.1], for instance), $\mathbf{V}$ coincides with one of the varieties $\mathbf{T}$, $\mathbf{SL}$ or $\mathbf{C}_k$ for some $k \le n$. Finally, Lemma 3.3 implies that the case $\mathbf{V} = \mathbf{C}_k$ with $k \ge 3$ is impossible.

(iii) $\Rightarrow$ (ii). In view of Lemma 2.2, the varieties $\mathbf{T}$, $\mathbf{SL}$ and $\mathbf{MON}$ are neutral elements of the lattice $\mathbb{MON}$. Then these varieties are costandard elements of $\mathbb{MON}$ as well. It remains to prove that the variety $\mathbf{C}_2$ is a costandard element of this lattice too.

It is easy to verify that an element of a lattice is costandard whenever it is modular and codistibutive. This claim readily follows from [3, Theorem 253] or [13, Proposition 1.7], for instance (to avoid a confusion, we note that, in [13], modular elements are called *s-modular* once, while “a modular element” means the same as “a lower-modular element” in our terminology). In view of the mentioned fact and Proposition 1.4, it suffices to prove that $\mathbf{C}_2$ is a modular element of the lattice $\mathbb{MON}$. Assume the contrary. Then [9, Proposition 2.1] implies that there exist the varieties $\mathbf{U}$ and $\mathbf{W}$ such that $\mathbf{U} \subset \mathbf{W}$, $\mathbf{U} \wedge \mathbf{C}_2 = \mathbf{W} \wedge \mathbf{C}_2$ and $\mathbf{U} \vee \mathbf{C}_2 = \mathbf{W} \vee \mathbf{C}_2$. If $\mathbf{C}_2 \subseteq \mathbf{U}$ then $\mathbf{W} \wedge \mathbf{C}_2 = \mathbf{U} \wedge \mathbf{C}_2 = \mathbf{C}_2$, whence $\mathbf{C}_2 \subseteq \mathbf{W}$. But then $\mathbf{U} = \mathbf{C}_2 \vee \mathbf{U} = \mathbf{C}_2 \vee \mathbf{W} = \mathbf{W}$. This contradicts the choice of $\mathbf{U}$ and $\mathbf{W}$. Thus, $\mathbf{C}_2 \nsubseteq \mathbf{U}$. Analogously, $\mathbf{C}_2 \nsubseteq \mathbf{W}$. It is proved in [5, Corollary 2.6] that a monoid variety $\mathbf{X}$ is completely regular if and only if $\mathbf{C}_2 \nsubseteq \mathbf{X}$. Therefore, the varieties $\mathbf{U}$ and $\mathbf{W}$ are completely regular.

Suppose that $\mathbf{U}$ is a group variety. Then $\mathbf{SL} \nsubseteq \mathbf{U}$. If $\mathbf{W}$ is a non-group variety then $\mathbf{SL} \subseteq \mathbf{W}$ by Lemma 2.4. Then $\mathbf{U} \wedge \mathbf{C}_2 = \mathbf{T}$ but $\mathbf{SL} \subseteq \mathbf{W} \wedge \mathbf{C}_2$. We obtain a contradiction with the equality $\mathbf{U} \wedge \mathbf{C}_2 = \mathbf{W} \wedge \mathbf{C}_2$. Therefore, $\mathbf{W}$ is a group variety. Proposition 2.1 and Lemma 2.7 imply that $\mathbf{U}$ is the largest group subvariety of the variety $\mathbf{U} \vee \mathbf{C}_2$. But this is impossible because $\mathbf{W}$ is a group variety and

$$\mathbf{U} \subset \mathbf{W} \subset \mathbf{W} \vee \mathbf{C}_2 = \mathbf{U} \vee \mathbf{C}_2.$$

We see that $\mathbf{U}$ is a non-group variety. Then $\mathbf{SL} \subseteq \mathbf{U}$ by Lemma 2.4. In this case $\mathbf{SL} \subseteq \mathbf{U} \wedge \mathbf{C}_2 = \mathbf{W} \wedge \mathbf{C}_2 \subseteq \mathbf{W}$. Therefore, $\mathbf{W}$ is also a non-group variety. Since $\mathbf{U}$ is completely regular, it satisfies $x \approx x^{n+1}$ for some positive integer n. Let n be the least number with such a property, while Σ be an identity basis of the variety $\mathbf{U}$. We denote by ζ the endomorphism of the monoid F^1

which maps each letter x into the word x^{n+1}. Put

$$\Sigma^* = \{\zeta(\mathbf{u}) \approx \zeta(\mathbf{v}) \mid \mathbf{u} \approx \mathbf{v} \in \Sigma\}.$$

Obviously, $\mathbf{U} = \mathrm{var}\{x \approx x^{n+1}, \Sigma^*\}$. If $\mathbf{p} \approx \mathbf{q} \in \Sigma^*$ then $\mathrm{con}(\mathbf{p}) = \mathrm{con}(\mathbf{q})$ by Lemma 2.4. According to Lemma 2.5, the variety $\mathbf{C}_2$ satisfies the identity system Σ^*. Taking into account that $\mathbf{U} \vee \mathbf{C}_2 = \mathbf{W} \vee \mathbf{C}_2$, we obtain that $\mathbf{W}$ satisfies the identity system Σ^* too. Since the identity $x^2 \approx x^3$ holds in $\mathbf{C}_2$ and the identity $x \approx x^{n+1}$ holds in $\mathbf{U}$, the variety $\mathbf{U} \vee \mathbf{C}_2 = \mathbf{W} \vee \mathbf{C}_2$ satisfies $x^2 \approx x^{n+2}$. Taking into account that $\mathbf{W}$ is completely regular, we get that $x \approx x^{n+1}$ holds in $\mathbf{W}$. Then $\mathbf{W} \subseteq \mathbf{U}$. We obtain a contradiction with the choice of the varieties $\mathbf{U}$ and $\mathbf{W}$. Thus, we have proved that $\mathbf{C}_2$ is a modular, and therefore, a costandard element of the lattice $\mathbb{MON}$. □

Proof of Theorem 1.1. The implication (iii) ⇒ (ii) follows from Lemma 2.2, while the implication (ii ⇒ (i) is obvious. It remains to prove the implication (i) ⇒ (iii). Let $\mathbf{V}$ be a modular, lower-modular and upper-modular element of the lattice $\mathbb{MON}$. Theorem 1.2 implies that $\mathbf{V}$ coincides with one of the varieties $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{C}_2$ or $\mathbf{MON}$. Lemma 3.1 implies that $\mathbf{V} \neq \mathbf{C}_2$, and we are done. □

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