Classification of selectors for sequences of dense sets of $C_p(X)$ 

Alexander V. Osipov  
Krasovskii Institute of Mathematics and Mechanics, Ural Federal University,  
Ural State University of Economics, Yekaterinburg, Russia

Abstract

For a Tychonoff space $X$, we denote by $C_p(X)$ the space of all real-valued continuous functions on $X$ with the topology of pointwise convergence. In this paper we investigate different selectors for sequences of dense sets of $C_p(X)$. We give the characteristics of selection principles $S_1(P, Q)$, $S_{fin}(P, Q)$ and $U_{fin}(P, Q)$ for $P, Q \in \{D, S, A\}$, where

- $D$ — the family of a dense subsets of $C_p(X)$;
- $S$ — the family of a sequentially dense subsets of $C_p(X)$;
- $A$ — the family of a 1-dense subsets of $C_p(X)$, through the selection principles of a space $X$.

Keywords:
$S_1(S, S)$, $U_{fin}(S, S)$, $S_1(D, S)$, $S_1(S, D)$, $S_{fin}(S, D)$, $S_1(D, D)$, $S_{fin}(D, D)$, $S_1(A, A)$, $U_{fin}(S, D)$, $S_1(S, A)$, $S_{fin}(A, A)$, function spaces, selection principles, $C_p$ theory, Scheepers Diagram

2000 MSC: 37F20, 26A03, 03E75, 54C35

1. Introduction

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by $\omega$. Let $\mathbb{R}$ be the real line, we put $I = [0, 1] \subset \mathbb{R}$, and $\mathbb{Q}$ be the rational numbers. For a space $X$, we denote by $C_p(X)$ the space of all real-valued continuous functions on $X$ with the topology of pointwise convergence. The symbol $0$ stands for the constant function to 0.

Email address: OAB@list.ru (Alexander V. Osipov)
Basic open sets of $C_p(X)$ are of the form
$$[x_1, ..., x_k, U_1, ..., U_k] = \{ f \in C(X) : f(x_i) \in U_i, \ i = 1, ..., k \},$$
where each $x_i \in X$ and each $U_i$ is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of the point $f$ as $< f, A, \epsilon >$ where $< f, A, \epsilon > := \{ g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in A \}, A$ is a finite subset of $X$ and $\epsilon > 0$.

In this paper, by cover we mean a nontrivial one, that is, $U$ is a cover of $X$ if $X = \bigcup U$ and $X \notin U$.

An open cover $U$ of a space $X$ is:

- an $\omega$-cover if $X$ does not belong to $U$ and every finite subset of $X$ is contained in a member of $U$.
- a $\gamma$-cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $U$.

For a topological space $X$ we denote:

- $\mathcal{O}$ — the family of open covers of $X$;
- $\Gamma$ — the family of open $\gamma$-covers of $X$;
- $\Gamma_{cl}$ — the family of clopen $\gamma$-covers of $X$;
- $\Omega$ — the family of open $\omega$-covers of $X$;
- $\mathcal{D}$ — the family of a dense subsets of $X$;
- $\mathcal{S}$ — the family of a sequentially dense subsets of $X$.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set $X$. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \omega \}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n \}_{n \in \omega}$ such that for each $n$, $b_n \in A_n$, and $\{b_n : n \in \omega \}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \omega \}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n \}_{n \in \omega}$ of finite sets such that for each $n$, $B_n \subseteq A_n$, and $\bigcup_{n \in \omega} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $U_1, U_2, ..., \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq U_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega \} \in \mathcal{B}$.

The following prototype of many classical properties is called ”$\mathcal{A}$ choose $\mathcal{B}$” in [39].

$(\mathcal{A})^\mathcal{B}$: For each $\mathcal{U} \in \mathcal{A}$ there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$.

Then $S_{fin}(\mathcal{A}, \mathcal{B})$ implies $(\mathcal{A})^\mathcal{B}$.  

2
Many equivalence hold among these properties, and the surviving ones appear in the following Diagram (where an arrow denote implication), to which no arrow can be added except perhaps from $U_{\text{fin}}(\Gamma, \Gamma)$ or $U_{\text{fin}}(\Gamma, \Omega)$ to $S_{\text{fin}}(\Gamma, \Omega)$ [16].

Fig. 1. The Scheepers Diagram.

The papers [16, 17, 32, 37, 41] have initiated the simultaneous consideration of these properties in the case where $\mathcal{A}$ and $\mathcal{B}$ are important families of open covers of a topological space $X$.

2. Main definitions and notation

Let $X$ be a topological space, and $x \in X$. A subset $A$ of $X$ converges to $x$, $x = \lim A$, if $A$ is infinite, $x \notin A$, and for each neighborhood $U$ of $x$, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{ A \subseteq X : x \in \overline{A} \setminus A \}$;
- $\Gamma_x = \{ A \subseteq X : x = \lim A \}$.

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ without specifying $x$, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

- A space $X$ has countable fan tightness (Arhangel’skii’s countable fan tightness), if $X \models S_{\text{fin}}(\Omega_x, \Omega_x)$ [2].
• A space $X$ has countable strong fan tightness (Sakai’s countable strong fan tightness), if $X \models S_1(\Omega_x, \Gamma_x)$ \cite{28}.

• A space $X$ has countable selectively sequentially fan tightness (Arhangel’skii’s property $\alpha_4$), if $X \models S_{\text{fin}}(\Gamma_x, \Gamma_x)$ \cite{1}.

• A space $X$ has countable strong selectively sequentially fan tightness (Arhangel’skii’s property $\alpha_2$), if $X \models S_1(\Gamma_x, \Gamma_x)$ \cite{1}.

• A space $X$ has strictly Fréchet-Urysohn at $x$, if $X \models S_1(\Omega_x, \Gamma_x)$ \cite{30}.

• A space $X$ has almost strictly Fréchet-Urysohn at $x$, if $X \models S_{\text{fin}}(\Omega_x, \Gamma_x)$.

• A space $X$ has the weak sequence selection property, if $X \models S_1(\Gamma_x, \Omega_x)$ \cite{33}.

• A space $X$ has the sequence selection property, if $X \models S_{\text{fin}}(\Gamma_x, \Omega_x)$.

The following implications hold

\[
S_1(\Gamma_x, \Gamma_x) \Rightarrow S_{\text{fin}}(\Gamma_x, \Gamma_x) \Rightarrow S_1(\Gamma_x, \Omega_x) \Rightarrow S_{\text{fin}}(\Gamma_x, \Omega_x) \\
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
S_1(\Omega_x, \Gamma_x) \Rightarrow S_{\text{fin}}(\Omega_x, \Gamma_x) \Rightarrow S_1(\Omega_x, \Omega_x) \Rightarrow S_{\text{fin}}(\Omega_x, \Omega_x)
\]

We write $\Pi(\mathcal{A}, \mathcal{B}_x)$ without specifying $x$, we mean $(\forall x)\Pi(\mathcal{A}, \mathcal{B}_x)$.

• A space $X$ has countable fan tightness with respect to dense subspaces, if $X \models S_{\text{fin}}(\mathcal{D}, \Omega_x)$ \cite{6}.

• A space $X$ has countable strong fan tightness with respect to dense subspaces, if $X \models S_1(\mathcal{D}, \Omega_x)$ \cite{6}.

• A space $X$ has almost strictly Fréchet-Urysohn at $x$ with respect to dense subspaces, if $X \models S_{\text{fin}}(\mathcal{D}, \Gamma_x)$.

• A space $X$ has strictly Fréchet-Urysohn at $x$ with respect to dense subspaces, if $X \models S_1(\mathcal{D}, \Gamma_x)$.

• A space $X$ has countable selectively sequentially fan tightness with respect to dense subspaces, if $X \models S_{\text{fin}}(\mathcal{S}, \Gamma_x)$.

• A space $X$ has countable strong selectively sequentially fan tightness with respect to dense subspaces, if $X \models S_1(\mathcal{S}, \Gamma_x)$.

• A space $X$ has the sequence selection property with respect to dense subspaces, if $X \models S_{\text{fin}}(\mathcal{S}, \Omega_x)$.

• A space $X$ has the weak sequence selection property with respect to dense subspaces, if $X \models S_1(\mathcal{S}, \Omega_x)$.
The following implications hold

\[ S_1(S, \Gamma_x) \Rightarrow S_{\text{fin}}(S, \Gamma_x) \Rightarrow S_1(S, \Omega_x) \Rightarrow S_{\text{fin}}(S, \Omega_x) \]

\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]

\[ S_1(D, \Gamma_x) \Rightarrow S_{\text{fin}}(D, \Gamma_x) \Rightarrow S_1(D, \Omega_x) \Rightarrow S_{\text{fin}}(D, \Omega_x) \]

- A space \( X \) is \( R \)-separable, if \( X \models S_1(D, D) \) (Def. 47, [6]).
- A space \( X \) is \( M \)-separable (selective separability), if \( X \models S_{\text{fin}}(D, D) \).
- A space \( X \) is selectively sequentially separable, if \( X \models S_{\text{fin}}(S, S) \) (Def. 1.2, [7]).

The following implications hold

\[ S_1(S, S) \Rightarrow S_{\text{fin}}(S, S) \Rightarrow S_1(S, D) \Rightarrow S_{\text{fin}}(S, D) \]

\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]

\[ S_1(D, S) \Rightarrow S_{\text{fin}}(D, S) \Rightarrow S_1(D, D) \Rightarrow S_{\text{fin}}(D, D) \]

If \( X \) is a space and \( A \subseteq X \), then the sequential closure of \( A \), denoted by \([A]_{\text{seq}}\), is the set of all limits of sequences from \( A \). A set \( D \subseteq X \) is said to be sequentially dense if \( X = [D]_{\text{seq}} \). If \( D \) is a countable sequentially dense subset of \( X \) then \( X \) call sequentially separable space.

Call \( X \) strongly sequentially dense in itself, if every dense subset of \( X \) is sequentially dense, and, \( X \) strongly sequentially separable, if \( X \) is separable and every countable dense subset of \( X \) is sequentially dense. Clearly, every strongly sequentially separable space is sequentially separable, and every sequentially separable space is separable.

We recall that a subset of \( X \) that is the complete preimage of zero for a certain function from \( C(X) \) is called a zero-set. A subset \( O \subseteq X \) is called a cozero-set (or functionally open) of \( X \) if \( X \setminus O \) is a zero-set.

Recall that the \( i \)-weight \( iw(X) \) of a space \( X \) is the smallest infinite cardinal number \( \tau \) such that \( X \) can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than \( \tau \).

**Theorem 2.1.** (Noble [21]) Let \( X \) be a space. A space \( C_p(X) \) is separable if and only if \( iw(X) = \aleph_0 \).
Definition 2.2. A space $X$ has $V$-property ($X \models V$), if there exist a condensation (one-to-one continuous mapping) $f : X \mapsto Y$ from the space $X$ on a separable metric space $Y$, such that $f(U) = F_\sigma$-set of $Y$ for any cozero-set $U$ of $X$.

Theorem 2.3. (Velichko [12]). Let $X$ be a Tychonoff space. A space $C_p(X)$ is sequentially separable if and only if $X \models V$.

Recall that the cardinal $p$ is the smallest cardinal so that there is a collection of $p$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega_1 \leq p \leq c$.

For $f, g \in \mathbb{N}^\mathbb{N}$, let $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n$. $b$ is the minimal cardinality of a $\leq^*$-unbounded subset of $\mathbb{N}^\mathbb{N}$. A set $B \subset [\mathbb{N}]^\infty$ is unbounded if the set of all increasing enumerations of elements of $B$ is unbounded in $\mathbb{N}^\mathbb{N}$, with respect to $\leq^*$. It follows that $|B| \geq b$. A subset $S$ of the real line is called a $Q$-set if each one of its subsets is a $G_\delta$. The cardinal $q$ is the smallest cardinal so that for any $\kappa < q$ there is a $Q$-set of size $\kappa$. (See [10] for more on small cardinals including $p$).

3. $S_1(\mathcal{D}, \mathcal{D})$ — $R$-separable

In [28] (Lemma, Theorem 1), M. Sakai proved:

Theorem 3.1. (Sakai) For each space $X$ the following are equivalent.

1. $C_p(X) \models S_1(\Omega_x, \Omega_x)$.
2. $X^n \models S_1(\mathcal{O}, \mathcal{O})$ ($X^n$ has Rothberger’s property $C''$) for each $n \in \omega$.
3. $X \models S_1(\Omega, \Omega)$.

In [35], Theorem 13) M. Scheeper was proved the following result

Theorem 3.2. (Scheeper) For each separable metric space $X$, the following are equivalent:

1. $C_p(X) \models S_1(\mathcal{D}, \mathcal{D})$;
2. $X \models S_1(\Omega, \Omega)$.

By Theorem 57 in [6], [28] and Theorem 2.1 we have

Theorem 3.3. For a space $X$, the following are equivalent:
1. \( C_p(X) \models S_1(D, D); \)
2. \( C_p(X) \models S_1(\Omega_x, \Omega_x), \text{ and is separable}; \)
3. \( C_p(X) \models S_1(D, \Omega_x), \text{ and is separable}; \)
4. \( X \models S_1(\Omega, \Omega), \text{ and } iw(X) = \aleph_0; \)
5. \( X^n \models S_1(\mathcal{O}, \mathcal{O}) \text{ for each } n \in \omega, \text{ and } iw(X) = \aleph_0. \)

**Corollary 3.4.** For a separable metrizable space \( X \), the following are equivalent:

1. \( C_p(X) \models S_1(D, D) \text{ [}R\text{-separable}]; \)
2. \( C_p(X) \models S_1(\Omega_x, \Omega_x) \text{ [}countable strong fan tightness]; \)
3. \( C_p(X) \models S_1(D, \Omega_x) \text{ [}countable strong fan tightness with respect to dense subspaces]; \)
4. \( X \models S_1(\Omega, \Omega); \)
5. \( X^n \models S_1(\mathcal{O}, \mathcal{O}) \text{ for each } n \in \omega \text{ [}X^n \text{ is Rothberger}]. \)

**4. \( S_{fin}(D, D) \text{ — } M\text{-separable} \)**

In (2), Theorem 2.2.2 in (4) A.V. Arhangel’skii was proved the following result.

**Theorem 4.1. (Arhangel’skii) For a space \( X \), the following are equivalent:**

1. \( C_p(X) \models S_{fin}(\Omega_x, \Omega_x); \)
2. \( (\forall n \in \omega) X^n \models S_{fin}(\mathcal{O}, \mathcal{O}). \)

It is known (see [16]) that \( X \models S_{fin}(\Omega, \Omega) \text{ iff } (\forall n \in \omega) X^n \models S_{fin}(\mathcal{O}, \mathcal{O}). \)

By Theorem 21 in [6] and Theorem 3.9 in [16], we have a next result.

**Theorem 4.2. For a space \( X \), the following are equivalent:**

1. \( C_p(X) \models S_{fin}(D, D); \)
2. \( X \models S_{fin}(\Omega, \Omega) \text{ and } iw(X) = \aleph_0; \)
3. \( (\forall n \in \omega) X^n \in S_{fin}(\mathcal{O}, \mathcal{O}) \text{ and } iw(X) = \aleph_0; \)
4. \( C_p(X) \models S_{fin}(\Omega_x, \Omega_x) \text{ and is separable}; \)
5. \( C_p(X) \models S_{fin}(D, \Omega_x) \text{ and is separable}. \)

**Corollary 4.3.** For a separable metrizable space \( X \), the following are equivalent:
1. \( C_p(X) \models S_{\text{fin}}(\mathcal{D}, \mathcal{D}) \) [M-separable];
2. \( C_p(X) \models S_{\text{fin}}(\Omega_x, \Omega_x) \) [countable fan tightness];
3. \( C_p(X) \models S_{\text{fin}}(\mathcal{D}, \Omega_x) \) [countable strong fan tightness with respect to dense subspaces];
4. \( X \models S_{\text{fin}}(\Omega, \Omega) \);
5. \( X^n \models S_{\text{fin}}(\mathcal{O}, \mathcal{O}) \) for each \( n \in \omega \) [\( X^n \) is Menger].

5. \( S_1(\mathcal{D}, \mathcal{S}) \)

Pytkeev [26] and independently Gerlits [14], see also [4] and [20], proved

**Theorem 5.1.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \) is Fréchet-Urysohn;
2. \( C_p(X) \) is sequential;
3. \( C_p(X) \) is a \( k \)-space.

Gerlits and Nagy [13] proved

**Theorem 5.2.** (Gerlits, Nagy) For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\Omega_x, \Gamma_x) \);
2. \( C_p(X) \) is Fréchet-Urysohn;
3. \( X \models S_1(\Omega, \Gamma) \);
4. \( X \models (\Omega, \Gamma) \).

**Theorem 5.3.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \) is strongly sequentially dense in itself;
2. \( X \models S_1(\Omega, \Gamma) \).

*Proof.* (1) \( \Rightarrow \) (2). By Theorem 5.2, \( S_1(\Omega, \Gamma) = (\Omega, \Gamma) \). Let \( U \in \Omega \) and \( P \) be a dense subset of \( C_p(X) \). A set \( \mathcal{D} := \{ f \in C(X) : f \upharpoonright K = h \) for \( h \in P \), and \( f \upharpoonright (X \setminus U) = 1 \) for a finite subset \( K \subset U \) where \( U \in \mathcal{U} \).

Since \( \mathcal{U} \) is a \( \omega \)-cover of \( X \) and \( P \) is a dense subset of \( C_p(X) \), we claim that \( \mathcal{D} \) is a dense subset of \( C_p(X) \).

Fix \( g \in C(X) \). Let \( K \) be a finite subset of \( X \), \( \epsilon > 0 \) and \( W = \langle g, K, \epsilon \rangle \) be a base neighborhood of \( g \), then there is \( U \in \mathcal{U} \) such that \( K \subset U \) and \( h \in W \) for some \( h \in P \). Since \( f \upharpoonright K = h \upharpoonright K \) for some \( f \in \mathcal{D} \), then
\( f \in W \). By (1), \( D \) is a sequentially dense subset of \( C_p(X) \). Then there exists a sequence \( \{f_i\}_{i \in \omega} \) such that for each \( i \), \( f_i \in D \), and \( \{f_i\}_{i \in \omega} \) converge to 0.

By definition of \( f_i \), \( f_i \rvert K_i = h_i \) for some finite set \( K_i \) and \( h_i \in P \), and \( f_i \rvert (X \setminus U_i) = 1 \) for some \( U_i \in \mathcal{U} \).

Consider a sequence \( \{U_i\}_{i \in \omega} \).

(a). \( U_i \in \mathcal{U} \).

(b). \( \{U_i : i \in \omega\} \) is a \( \gamma \)-cover of \( X \).

Let \( K \) be a finite subset of \( X \) and \( W = [K, (-1, 1)] \) be a base neighborhood of 0, then there is \( i' \in \omega \) such that \( f_i \in W \) for each \( i > i' \). It follows that \( K \subset U_i \) for each \( i > i' \). We thus get that \( X \models \Gamma \), and, hence, \( X \models S_1(\Omega, \Gamma) \).

(2) \( \Rightarrow \) (1). By Theorem 5.2, \( C_p(X) \) is Fréchet-Urysohn, and, hence, \( C_p(X) \) is strongly sequentially dense in itself.

\[ \square \]

**Corollary 5.4.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\Omega_x, \Gamma_x) \);
2. \( C_p(X) \) is Fréchet-Urysohn;
3. \( X \models S_1(\Omega, \Gamma) \);
4. \( X \models (a) \);,
5. \( C_p(X) \) is sequential;
6. \( C_p(X) \) is a k-space;
7. \( C_p(X) \) is strongly sequentially dense in itself.

Note that \( S_1(\Omega, \Gamma) = S_{\text{fin}}(\Omega, \Gamma) \) (see [16]).

**Theorem 5.5.** For a space \( X \), the following statements are equivalent:

1. \( X \models S_{\text{fin}}(\Omega, \Gamma) \);
2. \( C_p(X) \models S_{\text{fin}}(\Omega_x, \Gamma_x) \).

**Proof.** By Theorem 5.2 it suffices to prove (2) \( \Rightarrow \) (1).

(2) \( \Rightarrow \) (1). Let \( \{\mathcal{U}_n\}_{n \in \omega} \) be a sequence of open \( \omega \)-covers of \( X \). We set \( A_n = \{f \in C(X) : f \rvert (X \setminus U) = 0 \text{ for } U \in \mathcal{U}_n\} \). It is not difficult to see that each \( A_n \) is dense in \( C(X) \) since each \( \mathcal{U}_n \) is an \( \omega \)-cover of \( X \) and \( X \) is Tychonoff. Let \( f \) be the constant function to 1. By the assumption there exist \( \{f^1_n : i = 1, \ldots, k(n)\} \subset A_n \) such that \( \bigcup_{n \in \omega} \{f^i_n\}_{i=1}^{k(n)} \) converge to \( f \). Consider subsequence \( \{f^1_n\}_{n \in \omega} \subset \bigcup_{n \in \omega} \{f^i_n\}_{i=1}^{k(n)} \). Note that \( \{f^1_n\}_{n \in \omega} \) also converge to \( f \).

9
For each $f^i_n$ we take $U_n \in \mathcal{U}_n$ such that $f^i_n \upharpoonright (X \setminus U_n) = 0$.

Set $\mathcal{U} = \{U_n : n \in \omega\}$. For each finite subset $\{x_1, ..., x_k\}$ of $X$ we consider the basic open neighborhood of $f [x_1, ..., x_k; W_1, ..., W]$, where $W = (0, 2)$.

Note that there is $n' \in \omega$ such that $[x_1, ..., x_k; W_1, ..., W]$ contains $f^i_n$ for $n > n'$. This means $\{x_1, ..., x_k\} \subset U_n$ for $n > n'$. Consequently $\mathcal{U}$ is an $\gamma$-cover of $X$.

\begin{proof}
\quad
\end{proof}

**Theorem 5.6.** For a space $X$, the following statements are equivalent:

1. $C_p(X) \models S_1(D, S)$;
2. $C_p(X)$ is strongly sequentially dense in itself and is separable;
3. $X \models S_1(\Omega, \Gamma)$ and $iw(X) = \aleph_0$;
4. $C_p(X) \models S_1(\Omega_x, \Gamma_x)$ and is separable;
5. $C_p(X) \models S_1(D, \Gamma_x)$ and is separable;
6. $C_p(X) \models S_{fin}(D, S)$;
7. $X \models S_{fin}(\Omega, \Gamma)$ and $iw(X) = \aleph_0$;
8. $C_p(X) \models S_{fin}(\Omega_x, \Gamma_x)$ and is separable;
9. $C_p(X) \models S_{fin}(D, \Gamma_x)$ and is separable.

\begin{proof}
(1) $\Rightarrow$ (2). Let $D$ be a dense subset of $C_p(X)$. By $S_1(D, S)$, for sequence $\{D_i : D_i = D \text{ and } i \in \omega\}$ there is a sequence $(d_i : i \in \omega)$ such that for each $i$, $d_i \in D_i$, and $\{d_i : i \in \omega\}$ is a countable sequentially dense subset of $C_p(X)$. It follows that $D$ is a sequentially dense subset of $C_p(X)$.

(2) $\Rightarrow$ (1). Let $\{D_i\}_{i \in \omega}$ be a sequence of dense subsets of $C_p(X)$. Since $X \models S_1(\Omega, \Gamma)$, then $X \models S_1(\Omega, \Omega)$ and, by Theorem 5.3, $C_p(X) \models S_1(D, D)$. Then there is a sequence $(d_i)_{i \in \omega}$ such that for each $i$, $d_i \in D_i$, and $\{d_i : i \in \omega\}$ is a countable dense subset of $C_p(X)$. By (2), $\{d_i : i \in \omega\}$ is a countable sequentially dense subset of $C_p(X)$, i.e. $\{d_i : i \in \omega\} \in S$.

(2) $\Rightarrow$ (3). By Theorem 5.3 and Theorem 2.1.

(3) $\Leftrightarrow$ (4). By Theorem 5.2.

(4) $\Rightarrow$ (5) is immediate.

(5) $\Rightarrow$ (2). Let $D \in D$, $f \in C(X)$ and $\{D_n\}_{n \in \omega}$ such that $D_n = D$ for each $n \in \omega$. By (5), there is a sequence $\{f_n\}_{n \in \omega}$ such that for each $n$, $f_n \in D_n$, and $\{f_n\}_{n \in \omega}$ converge to $f$. It follows that $D$ is a sequentially dense subset of $C_p(X)$.

(7) $\Leftrightarrow$ (8). By Theorem 5.5 and Theorem 2.1.

(8) $\Rightarrow$ (9) is immediate.
(1) \Rightarrow (6) is immediate.
(3) \Rightarrow (7) is immediate.
(9) \Rightarrow (2) is proved similarly the implication (5) \Rightarrow (2).
(6) \Rightarrow (2) is proved similarly the implication (1) \Rightarrow (2).

\[ \]

Corollary 5.7. For a separable metrizable space \( X \), the following are equivalent:

1. \( C_p(X) \models S_1(\mathcal{D}, \mathcal{S}) \);
2. \( C_p(X) \) is strongly sequentially dense in itself;
3. \( C_p(X) \) is strongly sequentially separable;
4. \( C_p(X) \models S_1(\Omega_x, \Gamma_x) \);
5. \( C_p(X) \models S_1(\mathcal{D}, \Gamma_x) \);
6. \( X \models S_1(\Omega, \Gamma) \);
7. \( C_p(X) \models S_{\text{fin}}(\mathcal{D}, \mathcal{S}) \);
8. \( C_p(X) \models S_{\text{fin}}(\Omega_x, \Gamma_x) \);
9. \( C_p(X) \models S_{\text{fin}}(\mathcal{D}, \Gamma_x) \);
10. \( X \models S_{\text{fin}}(\Omega, \Gamma) \).

6. \( S_1(\mathcal{S}, \mathcal{D}) \)

In [29] (Theorem 2.3), M. Sakai proved:

Theorem 6.1. (Sakai) For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\Gamma_x, \Omega_x) \) (the weak sequence selection property);
2. \( X \models S_1(\Gamma_{cl}, \Omega_{cl}) \) and is strongly zero-dimensional.

Definition 6.2. (Sakai) An \( \gamma \)-cover \( \mathcal{U} \) of co-zero sets of \( X \) is \( \gamma_F \)-shrinkable if there exists a \( \gamma \)-cover \( \{ F(U) : U \in \mathcal{U} \} \) of zero-sets of \( X \) with \( F(U) \subset U \) for every \( U \in \mathcal{U} \).

For a topological space \( X \) we denote:
- \( \Gamma_F \) — the family of \( \gamma_F \)-shrinkable \( \gamma \)-covers of \( X \).

Proposition 6.3. For a strongly zero-dimensional space \( X \), the following statements are equivalent:
1. \[ X \models S_1(\Gamma_F, \Omega); \]
2. \[ X \models S_1(\Gamma_{cl}, \Omega_{cl}). \]

**Proposition 6.4.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\Gamma_x, \Omega_x); \)
2. \( X \models S_1(\Gamma_F, \Omega). \)

**Proof.** (1) \( \Rightarrow \) (2). By Theorem 6.1 and Proposition 6.3.

(2) \( \Rightarrow \) (1). Let \( X \models S_1(\Gamma_F, \Omega) \) and \( \{A_i\}_{i \in \omega} \) such that \( A_i \in \Gamma_0 \) for each \( i \in \omega \). Consider \( \mathcal{U}_i = \{f^{-1}(\{-\frac{1}{i}, \frac{1}{i}\}) : f \in A_i\} \) for each \( i \in \omega \). Without loss of generality we can assume that there is \( i' \) such that \( \mathcal{U} \neq X \) for any \( i > i' \) and \( U \in \mathcal{U}_i \). Otherwise there is sequence \( \{f_{i_k}\}_{k \in \omega} \) such that \( \{f_{i_k}\}_{k \in \omega} \) uniform converge to \( 0 \) and \( \{f_{i_k} : k \in \omega\} \in \Omega_0 \).

Note that \( \mathcal{F}_i = \{f^{-1}(\{-\frac{1}{i+1}, \frac{1}{i+1}\}) : f \in A_i\} \) is \( \gamma \)-cover of zero-sets of \( X \). It follows that \( U_i \in \Gamma_F \) for each \( i \in \omega \). By \( X \models S_1(\Gamma_F, \Omega) \), there is a set \( \mathcal{U}_i : i \in \omega \) such that for each \( i \), \( U_i \in \mathcal{U}_i \), and \( \mathcal{U}_i : i \in \omega \) is an element of \( \Omega \).

We claim that \( 0 \in \{f_i : i \in \omega\} \). Let \( W = < 0, K, \epsilon > \) be a base neighborhood of \( 0 \) where \( \epsilon > 0 \) and \( K \) is a finite subset of \( X \), then there are \( i_0 \in \omega \) such that \( \frac{1}{i_0} < \epsilon \) and \( U_{i_0} \supset K \). It follows that \( f_{i_0} \in W \) and, hence, \( 0 \in \{f_i : i \in \omega\} \) and \( C_p(X) \models S_1(\Gamma_x, \Omega_x) \).

By Theorem 6.1 we have that \( X \) is strongly zero-dimensional.

\[ \square \]

**Theorem 6.5.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\mathcal{S}, \mathcal{D}) \) and is sequentially separable;
2. \( X \models S_1(\Gamma_F, \Omega), X \models V; \)
3. \( X \models S_1(\Gamma_{cl}, \Omega_{cl}), X \models V \) and is strongly zero-dimensional;
4. \( C_p(X) \models S_1(\Gamma_x, \Omega_x) \) and is sequentially separable;
5. \( C_p(X) \models S_1(\mathcal{S}, \mathcal{O}_x) \) and is sequentially separable.

**Proof.** (1) \( \Rightarrow \) (2). Let \( \{V_i\} \subset \Gamma_F \) and \( \mathcal{S} = \{h_m : m \in \omega\} \) be a countable sequentially dense subset of \( C_p(X) \). For each \( i \in \omega \) we consider a countable sequentially dense subset \( \mathcal{S}_i \) of \( C_p(X) \) and \( \mathcal{U}_i = \{U_i^m\}_{m \in \omega} \) where \( \mathcal{U}_i \subset V_i \) and \( \mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \omega\} \).

Note that \( \mathcal{U}_i \in \Gamma_F \) for each \( i \in \omega \).
Since $\mathcal{F}_i = \{F(U^m_i) : m \in \omega\}$ is infinite, it is a $\gamma$-cover of zero subsets of $X$. Since $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a sequence $\{h_{m_s} : s \in \omega\} \subset \mathcal{S}$ such that $\{h_{m_s}\}_{s \in \omega}$ converge to $h$. Let $K$ be a finite subset of $X$, $\epsilon > 0$ and $W = \langle h, K, \epsilon \rangle$ be a base neighborhood of $h$, then there is a number $m_0$ such that $K \subset F(U^m_i)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f^{m_s}_i \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f^{m_s}_i \in W$ for each $m_s > m_0$. It follows that a sequence $\{f^{m_s}_i\}_{s \in \omega}$ converge to $h$.

By $C(X) \in S_1(\mathcal{S}, \mathcal{D})$, there is a sequence $\{f^{m(i)}_i : i \in \omega\}$ such that for each $i$, $f^{m(i)}_i \in \mathcal{S}_i$, and $\{f^{m(i)}_i : i \in \omega\}$ is an element of $\mathcal{D}$.

Consider a set $\{U^{m(i)}_i : i \in \omega\}$.

(a). $U^{m(i)}_i \in \mathcal{U}_i$.

(b). $\{U^{m(i)}_i : i \in \omega\}$ is a $\omega$-cover of $X$.

Let $K$ be a finite subset of $X$ and $U = \langle 0, K, (-1, 1) \rangle$ be a base neighborhood of $0$, then there is $f^{m(i)}_{j_0} \in U$ for some $j_0 \in \omega$. It follows that $K \subset U^{m(i)j_0}_i$. We thus get $X \models S_1(\Gamma_F, \Omega)$.

(2) $\Rightarrow$ (4). By Proposition 6.4 and Theorem 2.3.

(2) $\Leftrightarrow$ (3). Clearly, that $\Gamma_{cl} \subset \Gamma_F$. For a strongly zero-dimensional $X$, if $\mathcal{U} \in \Gamma_F$, then there is $\mathcal{W} \in \Gamma_{cl}$ such that $F(U) \subset W \subset U$ for every $U \in \mathcal{U}$ and $W \in \mathcal{W}$.

(3) $\Leftrightarrow$ (4). By Theorem 2.3 and Theorem 6.1.

(4) $\Rightarrow$ (5) is immediate.

(5) $\Rightarrow$ (1). Suppose that $C_p(X)$ is sequentially separable and $C_p(X) \models S_1(\mathcal{S}, \Omega_\omega)$.

Let $D = \{d_n : n \in \omega\}$ be a dense subspace of $C_p(X)$. Given a sequence of sequentially dense subspace of $C_p(X)$, enumerate it as $\{S_{n,m} : n, m \in \omega\}$. For each $n \in \omega$, pick $d_{n,m} \in S_{n,m}$ so that $d_n \in \{d_{n,m} : m \in \omega\}$. Then $\{d_{n,m} : m, n \in \omega\}$ is dense in $C_p(X)$.

\[\square\]

**Corollary 6.6.** For a separable metrizable space $X$, the following statements are equivalent:

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{D})$;
2. $X \models S_1(\Gamma_F, \Omega)$;
3. $X \models S_1(\Gamma_{cl}, \Omega_{cl})$, and is strongly zero-dimensional;
4. \( C_p(X) \models S_1(\Gamma_x, \Omega_x) \);
5. \( C_p(X) \models S_1(S, \Omega_x) \).

7. \( S_{\text{fin}}(S, \mathcal{D}) \)

**Theorem 7.1.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_{\text{fin}}(\Gamma_x, \Omega_x) \);
2. \( X \models S_{\text{fin}}(\Gamma_F, \Omega) \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( \{\mathcal{V}_n\}_{n \in \omega} \) be a sequence \( \gamma_F \)-shrinkable \( \gamma \)-covers of \( X \). Let \( \mathcal{U}_n = \{U_{n,m} : m \in \omega\} \subset \mathcal{V}_n \) for each \( n \in \omega \). Note that \( \mathcal{U}_n \in \Gamma_F \) for each \( n \in \omega \).

For \( n, m \in \omega \), let \( f_{n,m} : X \mapsto [0, 1] \) be the continuous function satisfying \( F(U_{n,m}) = f_{n,m}^{-1}(0) \) and \( X \setminus U_{n,m} = f_{n,m}^{-1}(1) \). For each \( n \in \omega \), \( \{F(U_{n,m}) : m \in \omega\} \) is a \( \gamma \)-cover of \( X \), it follows that \( \{f_{n,m}\}_{m \in \omega} \) is a sequence converging pointwise to 0. By \( C_p(X) \models S_{\text{fin}}(\Gamma_x, \Omega_x) \), there is a sequence \( \{F_n = \{f_{n,m_1}, f_{n,m_2}, \ldots, f_{n,m_{k_n}}\}\}_{n \in \omega} \) such that \( F_n \subset \{f_{n,m}\}_{m \in \omega} \) for each \( n \in \omega \) and \( \bigcup_{n \in \omega} F_n \in \Omega_0 \). Then \( \bigcup_{n \in \omega} \{U_{n,m_1}, U_{n,m_2}, \ldots, U_{n,m_{k_n}}\} \) is an \( \omega \)-cover of \( X \).

(2) \( \Rightarrow \) (1). For each \( n \in \omega \), let \( A_n \in \Gamma_0 \).

For \( n \in \omega \) and \( f \in A_n \), let \( Z_{n,f} = \{x \in X : |f(x)| \leq \frac{1}{2^n+1}\} \), \( U_{n,f} = \{x \in X : |f(x)| < \frac{1}{2^n}\} \). For each \( n \in \omega \), we put \( \mathcal{U}_n = \{U_{n,f} : f \in A_n\} \). If the set \( \{n \in \omega : X \in \mathcal{U}_n\} \) is infinite, \( X = U_{n_1,f_1} = U_{n_2,f_2} = \ldots \) for some sequences \( \{n_j\}_{j \in \omega} \) and \( f_i \in A_{n_i} \), where \( \{n_j\}_{j \in \omega} \) is strictly increasing. This means that \( \{f_i\}_{i \in \omega} \) is a sequence converging uniformly to 0. If the set \( \{n \in \omega : X \in \mathcal{U}_n\} \) is finite, by removing such finitely many \( n \)’s we assume \( U_{n,f} \neq X \) for \( n \in \omega \) and \( f \in A_n \).

Note that each \( \mathcal{U}_n \) is a \( \gamma_F \)-shrinkable \( \gamma \)-covers of \( X \). By \( X \models S_{\text{fin}}(\Gamma_F, \Omega) \), there is a sequence \( \{U_{n,f_1}, \ldots, U_{n,f_{k(n)}}\}_{n \in \omega} \) such that \( U_{n,f_i} \in \mathcal{U}_n \) for each \( n \in \omega \), \( i \in \Gamma, k(n) \) and \( \bigcup_{n \in \omega} \{U_{n,f_1}, U_{n,f_2}, \ldots, U_{n,f_{k(n)}}\} \) is an \( \omega \)-cover of \( X \). We claim a sequence \( \{F_n = \{f_{n,1}, f_{n,2}, \ldots, f_{n,k(n)}\}\}_{n \in \omega} \) such that \( F_n \subset A_n \) for each \( n \in \omega \) and \( \bigcup_{n \in \omega} F_n \in \Omega_0 \).

Let \( K \) be a finite subset of \( X \) and let \( \epsilon \) a positive real number.

Because of \( U_{n,f_i} \neq X \) for \( n \in \omega \) and \( i \in \Gamma, k(n) \), there are \( n' \in \omega \) and \( i' \in 1, k(n') \) such that \( K \subset U_{n', f_{i',n'}} \) and \( \frac{1}{2^n} \epsilon < \epsilon \). Then \( |f_{i',n'}(x)| < \epsilon \) for any \( x \in K \). Thus \( C_p(X) \models S_{\text{fin}}(\Gamma_x, \Omega_x) \).

**Theorem 7.2.** For a space \( X \), the following statements are equivalent:
Corollary 7.3. For a separable metrizable space $X$, the following statements are equivalent:

1. $C_p(X) \models S_{fin}(\mathcal{S}, \mathcal{D})$ and is sequentially separable;
2. $X \models S_{fin}(\Gamma_F, \Omega)$, $X \models V$;
3. $C_p(X) \models S_{fin}(\Gamma_x, \Omega_x)$ and is sequentially separable;
4. $C_p(X) \models S_{fin}(\mathcal{S}, \Omega_x)$ and is sequentially separable.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i\} \subset \Gamma_F$ and $\mathcal{S} = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $C_p(X)$ and $\mathcal{U}_i = \{U^m_i : m \in \omega\}$ where $\mathcal{U}_i \subset \mathcal{V}_i$ and

$$\mathcal{S}_i = \{f^m_i \in C(X) : f^m_i \upharpoonright F(U^m_i) = h_m \text{ and } f^m_i \upharpoonright (X \setminus U^m_i) = 1 \text{ for } m \in \omega\}.$$  

Note that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$.

Since $\mathcal{F}_i = \{F(U^m_i) : m \in \omega\}$ is infinite, it is a $\gamma$-cover of zero subsets of $X$. Since $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a sequence $\{h_m\}_{m \in \omega} \subset \mathcal{S}$ such that $\{h_m\}_{m \in \omega}$ converge to $h$. Let $K$ be a finite subset of $X$, $\epsilon > 0$ and $W = h, K, \epsilon > 0$ be a base neighborhood of $h$, then there is a number $m_0$ such that $K \subset F(U^m_i)$ for $m > m_0$ and $h_m \in W$ for $m > m$. Since $f^m_i \upharpoonright K = h_m \upharpoonright K$ for each $m > m_0$, $f^m_i \in W$ for each $m > m_0$. It follows that a sequence $\{f^m_i\}_{m \in \omega}$ converge to $h$.

By $C(X) \in S_{fin}(\mathcal{S}, \mathcal{D})$, there is a sequence $\{F_i = \{f^i_{m_1}, f^i_{m_2}, \ldots, f^i_{m_k}\}\}_{i \in \omega}$ such that $F_i \subset \mathcal{S}_i$ for each $i \in \omega$ and $\bigcup_{i \in \omega} F_i \in \mathcal{D}$. Then $\bigcup_{i \in \omega} \{U_{m_1}, U_{m_2}, \ldots, U_{m_k}\}$ is an $\omega$-cover of $X$.

Let $K$ be a finite subset of $X$ and $U = h, K, (-1, 1)> 0$ be a base neighborhood of 0, then there is $f^i_{m_1} \in \bigcup_{i \in \omega} F_i$ for some $i \in \omega$ such that $f^i_{m_1} \in U$. It follows that $K \subset U^m_{i(m')}$. We thus get $X \models S_1(\Gamma_F, \Omega)$.

(2) $\Rightarrow$ (3). By Theorem 7.3 and Theorem 2.3.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Suppose that $C_p(X)$ is sequentially separable and $C_p(X) \models S_{fin}(\mathcal{S}, \Omega_x)$.

Let $D = \{d_n : n \in \omega\}$ be a dense subspace of $C_p(X)$. Given a sequence of sequentially dense subspace of $C_p(X)$, enumerate it as $\{S_{n,m} : n, m \in \omega\}$. For each $n \in \omega$, pick $D_{n,m} = \{d_{n,m(1)}, \ldots, d_{n,m(n)}\} \subset S_{n,m}$ so that $d_n \in \bigcup_{m \in \omega} D_{n,m}$. Then $\bigcup_{n,m \in \omega} D_{n,m}$ is dense in $C_p(X)$. 

\[\square\]
1. \( C_p(X) \models S_{\text{fin}}(\mathcal{S}, \mathcal{D}) \);
2. \( X \models S_{\text{fin}}(\Gamma_F, \Omega) \);
3. \( C_p(X) \models S_{\text{fin}}(\Gamma_x, \Omega_x) \);
4. \( C_p(X) \models S_{\text{fin}}(\mathcal{S}, \Omega_x) \).

8. \( S_1(\mathcal{S}, \mathcal{S}) \)

In [29] (Theorem 2.5), M. Sakai proved:

**Theorem 8.1.** (Sakai) For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\Gamma_x, \Gamma_x) \);
2. \( X \models S_1(\Gamma_{\text{cl}}, \Gamma_{\text{cl}}) \) and is strongly zero-dimensional.

**Proposition 8.2.** For a strongly zero-dimensional space \( X \), the following statements are equivalent:

1. \( X \models S_1(\Gamma_F, \Gamma) \);
2. \( X \models S_1(\Gamma_{\text{cl}}, \Gamma_{\text{cl}}) \).

**Proposition 8.3.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\Gamma_x, \Gamma_x) \);
2. \( X \models S_1(\Gamma_F, \Gamma) \).

**Proof.** (1) \( \Rightarrow \) (2). By Theorem 8.1 and Proposition 8.2.

(2) \( \Rightarrow \) (1). Let \( X \models S_1(\Gamma_{\text{cl}}, \Gamma_{\text{cl}}) \) such that \( A_i \in \Gamma_0 \) for each \( i \in \omega \). Consider \( \mathcal{U}_i = \{ f^{-1}[\frac{1}{i+1}, \frac{1}{i}] : f \in A_i \} \) for each \( i \in \omega \). Without loss of generality we can assume that there is \( i' \) that a set \( U \neq X \) for any \( i > i' \) and \( U \in \mathcal{U}_i \). Otherwise there is sequence \( \{ f_{i_k} \}_{k \in \omega} \) uniform converge to 0 and \( \{ f_{i_k} : k \in \omega \} \in \Omega_0 \).

Note that \( \mathcal{F}_i = \{ f^{-1}[\frac{1}{i+1}, \frac{1}{i}] : f \in A_i \} \) is \( \gamma \)-cover of zero-sets of \( X \). It follows that \( \mathcal{U}_i \in \Gamma_F \) for each \( i \in \omega \). By \( X \models S_1(\Gamma_F, \Gamma) \), there is a set \( \{ U_i : i \in \omega \} \) such that for each \( i, U_i \in \mathcal{U}_i \), and \( \{ U_i : i \in \omega \} \) is an element of \( \Gamma \).

We claim that \( \{ f_i : i \in \omega \} \in \Gamma_0 \). Let \( W = \langle 0, K, \epsilon \rangle \) be a base neighborhood of 0 where \( \epsilon > 0 \) and \( K \) is a finite subset of \( X \), then there are \( i_0 \in \omega \) such that \( \frac{1}{i_0} < \epsilon \) and \( U_i \supset K \) for any \( i > i_0 \). It follows that \( f_i \in W \) for \( i > i_0 \), and \( C_p(X) \models S_1(\Gamma_x, \Gamma_x) \).

By Theorem 6.1, we have that \( X \) is strongly zero-dimensional. \( \square \)
Theorem 8.4. For a space $X$, the following statements are equivalent:

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{S})$;
2. $X \models S_1(\Gamma_F, \Gamma)$, $X \models V$ and is strongly zero-dimensional;
3. $X \models S_1(\Gamma_d, \Gamma_d)$, $X \models V$ and is strongly zero-dimensional;
4. $C_p(X) \models S_1(\Gamma_x, \Gamma_x)$ and is sequentially separable;
5. $C_p(X) \models S_1(\mathcal{S}, \Gamma_x)$ and is sequentially separable.

Proof. (1) $\Rightarrow$ (2). Let $\{V_i\} \subset \Gamma_F$ and $\mathcal{S} = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $C_p(X)$ and $U_i = \{U_i^m : m \in \omega\} \subset V_i$ where

$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \omega\}$. Note that $U_i \in \Gamma_F$ for each $i \in \omega$.

Since $F_i = \{F(U_i^m) : m \in \omega\}$ is infinity and it is a $\gamma$-cover of zero-sets of $X$. Since $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a set $\{h_m_s : s \in \omega\} \subset \mathcal{S}$ such that $\{h_m_s\}_{s \in \omega}$ converge to $h$. Let $K$ be a finite subset of $X$, $\epsilon > 0$ and $W = \langle h, K, \epsilon \rangle$ be a base neighborhood of $h$, then there is a number $m_0$ such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_m_s \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_m_s \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s \in \omega}$ converge to $h$.

Since $C(X) \models S_1(\mathcal{S}, \mathcal{S})$, there is a sequence $\{f_i^{m(i)}\}_{i \in \omega}$ such that for each $i$, $f_i^{m(i)} \in \mathcal{S}_i$, and $\{f_i^{m(i)} : i \in \omega\}$ is an element of $\mathcal{S}$.

Consider a set $\{U_i^{m(i)} : i \in \omega\}$.

(a). $U_i^{m(i)} \in U_i$.

(b). $\{U_i^{m(i)} : i \in \omega\}$ is a $\gamma$-cover of $X$.

There is a sequence $\{f_i^{m(i)}\}_{i \in \omega}$ converge to $0$. Let $K$ be a finite subset of $X$ and $U = \langle 0, K, (-1, 1) \rangle$ be a base neighborhood of $0$, then there exists $j_0 \in \omega$ such that $f_j^{m(i)} \in U$ for each $j > j_0$. It follows that $K \subset U_i^{m(i)}$ for $j > j_0$. We thus get $X \models S_f(\Gamma_F, \Gamma)$. But $S_f(\Gamma_F, \Gamma) = S_1(\Gamma_F, \Gamma)$.

By Proposition 6.4, $X \models S_1(\Gamma_F, \Omega)$ implies $C_p(X) \models S_1(\Gamma_x, \Omega_x)$. By Theorem 6.1, $X$ is strongly zero-dimensional.

(2) $\Rightarrow$ (1). Fix $\{S_i : i \in \omega\} \subset \mathcal{S}$ and $\mathcal{S} = \{h_i : i \in \omega\} \subset \mathcal{S}$. For each $i \in \omega$ we consider a set $\{f_k^i : k \in \omega\} \subset S_i$ such that $\{f_k^i\}_{k \in \omega}$ converge to $h_i$. For each $i, k \in \omega$, we put $U_{i,k} = \{x \in X : |f_k^i(x) - h_i(x)| < \frac{1}{i+1}\}$, $Z_{i,k} = \{x \in X : |f_k^i(x) - h_i(x)| \leq \frac{1}{i+1}\}$. Each $U_{i,k}$ (resp., $Z_{i,k}$) is a cozero-set (resp., zero-set)
Theorem 2.3. \( m \in \text{subset of } C \)  

Since \( \gamma \)  

Hence a sequence can easily check that the condition be a finite subset of \( X \). For each \( i, k \in \omega \). We can easily check that the condition \( f_k \rightarrow h_i \) \( (k \rightarrow \infty) \) implies that \( Z_i \) is a \( \gamma \)-cover of \( X \). Since \( X \models S_i(S, \Gamma) \) there is a sequence \( \{U_{i,k(i)}\}_{i \in \omega} \) such that for each \( i, U_{i,k(i)} \in \mathcal{U}_i \), and \( \{U_{i,k(i)} : i \in \omega \} \) is an element of \( \Gamma \). We claim that \( \{f_{k(i)}^i\}_{i \in \omega} \in S \). For \( f \in C(X) \) there is a set \( \{h_{i_s} : s \in \omega \} \subset S \) such that \( \{h_{i_s}\}_{s \in \omega} \) converge to \( f \). Then a sequence \( \{f_{k(i)}^i\}_{s \in \omega} \) converge to \( f \) too. Let \( K \) be a finite subset of \( X, \epsilon > 0 \), and \( U =< f, K, \epsilon > \) be a base neighborhood of \( f \), then there exists \( m \in \omega \) such that \( h_{i_s} \in< f, K, \frac{\epsilon}{2} > \) for each \( i_s \in K \). Since \( \{U_{i,k(i)} : i \in \omega \} \) is an element of \( \Gamma \), there exists \( n > m \) such that \( \frac{1}{n} < \frac{\epsilon}{2} \) and \( K \subset U_{i,k(i)} \) for \( i > n \). It follows that for each \( i_s > n \) and \( x \in K \) we have that \( |f(x) - f_{k(i_s)}^i(x)| \leq |f(x) - h_{i_s}(x)| + |f_{k(i_s)}^i(x) - h_{i_s}(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \). Hence a sequence \( \{f_{k(i_s)}^i\}_{s \in \omega} \) converge to \( f \) and \( \{f_{k(i)}^i\}_{i \in \omega} \in S \).

(2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4). By Theorem 8.1 Proposition 8.2 Proposition 8.3 and Theorem 2.3.

(4) \( \Rightarrow \) (5) is immediate.

(5) \( \Rightarrow \) (2). Let \( C_p(X) \models S_1(S, \Gamma_x) \) and \( C_p(X) \) be a sequentially separable.

Let \( \{V_i\} \subset \Gamma_F \) and \( S = \{h_m : m \in \omega \} \) be a countable sequentially dense subset of \( C_p(X) \). For each \( i \in \omega \) we consider a countable sequentially dense subset \( S_i \) of \( C_p(X) \) and \( \mathcal{U}_i = \{U_i^m\}_{m \in \omega} \subset \mathcal{V}_i \) where \( S_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \omega \} \).

Since \( \mathcal{F}_i = \{F(U_i^m) : m \in \omega \} \) is infinity, it is a \( \gamma \)-cover of zero-subsets in \( X \). Since \( S \) is a countable sequentially dense subset of \( C_p(X) \), we have that \( S_i \) is a countable sequentially dense subset of \( C_p(X) \) for each \( i \in \omega \).

Let \( h \in C(X) \), there is a set \( \{h_{m_s} : s \in \omega \} \subset S \) such that \( \{h_{m_s}\}_{s \in \omega} \) converge to \( h \). Let \( K \) be a finite subset of \( X, \epsilon > 0 \) and \( W =< h, K, \epsilon > \) be a base neighborhood of \( h \), then there is a number \( m_0 \) such that \( K \subset F(U_i^{m_0}) \) for \( m > m_0 \) and \( h_{m_s} \in W \) for \( m_s > m_0 \). Since \( f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K \) for each \( m_s > m_0 \), \( f_i^{m_s} \in W \) for each \( m_s > m_0 \). It follows that a sequence \( \{f_i^{m_s}\}_{s \in \omega} \) converge to \( h \).

By \( C_p(X) \models S_1(S, \Gamma_x) \), there is a sequence \( \{f_i^{m(i)} : i \in \omega \} \) such that for each \( i \), \( f_i^{m(i)} \in S_i \), and \( \{f_i^{m(i)} : i \in \omega \} \) is an element of \( \Gamma_0 \).

Consider a set \( \{U_i^{m(i)} : i \in \omega \} \).

(a). \( U_i^{m(i)} \in \mathcal{U}_i \).

(b). \( \{U_i^{m(i)} : i \in \omega \} \) is a \( \gamma \)-cover of \( X \).

Let \( K \) be a finite subset of \( X \) and \( U =< 0, K, \frac{1}{2} > \) be a base neighborhood.
of 0, then there is \( j_0 \in \omega \) such that \( f^{m(i)}_{i_j} \in U \) for each \( j > j_0 \). It follows that \( K \subset U^{m(i)}_{i_j} \) for each \( j > j_0 \). We thus get \( X \models S_1(\Gamma_F, \Gamma) \). By Theorem 2.3 \( X \models V \). Since \( C_p(X) \models S_1(\mathcal{S}, \Gamma_x) \) implies that \( C_p(X) \models S_1(\mathcal{S}, \Omega_x) \), by Theorem 6.6 we have that \( X \) is strongly zero-dimensional.

**Corollary 8.5.** For a separable metrizable space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\mathcal{S}, \mathcal{S}) \);
2. \( X \models S_1(\Gamma_F, \Gamma) \) and is strongly zero-dimensional;
3. \( X \models S_1(\Gamma_{cl}, \Gamma_{cl}) \) and is strongly zero-dimensional;
4. \( C_p(X) \models S_1(\Gamma_x, \Gamma_x) \);
5. \( C_p(X) \models S_1(\mathcal{S}, \Gamma_x) \).

The proof of fact that \( S_{fin}(\Gamma_F, \Gamma) = S_1(\Gamma_F, \Gamma) \) (or \( S_1(\Gamma_{cl}, \Gamma_{cl}) = S_{fin}(\Gamma_{cl}, \Gamma_{cl}) \)) is analogous to proof of Theorem 1.1 (in [16]) that \( S_1(\Gamma, \Gamma) = S_{fin}(\Gamma, \Gamma) \).

**Proposition 8.6.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_{fin}(\Gamma_F, \Gamma) \);
2. \( X \models S_{fin}(\Gamma_F, \Gamma) \);
3. \( X \models S_{fin}(\Gamma_{cl}, \Gamma_{cl}) \) and is strongly zero-dimensional;
4. \( X \models S_1(\Gamma_F, \Gamma) \).

*Proof.* Note that, by Theorem 2 in [34], \( C_p(X) \models S_1(\Gamma_x, \Gamma_x) \) iff \( C_p(X) \models S_{fin}(\Gamma_x, \Gamma_x) \). By Theorem 8.1 and Proposition 8.2 we obtain the complete proof.

**Theorem 8.7.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_{fin}(\mathcal{S}, \mathcal{S}) \) and is sequentially separable;
2. \( X \models S_1(\Gamma_F, \Gamma), X \models V \) and is strongly zero-dimensional;
3. \( X \models S_{fin}(\Gamma_F, \Gamma), X \models V \) and is strongly zero-dimensional;
4. \( X \models S_{fin}(\Gamma_{cl}, \Gamma_{cl}), X \models V \) and is strongly zero-dimensional;
5. \( C_p(X) \models S_{fin}(\Gamma_x, \Gamma_x) \) and is sequentially separable;
6. \( C_p(X) \models S_{fin}(\mathcal{S}, \Gamma_x) \) and is sequentially separable.

*Proof.* By Proposition 8.6, Theorem 8.4 and Theorem 2.3.
Corollary 8.8. For a separable metrizable space $X$, the following statements are equivalent:

1. $C_p(X) \models S_{\text{fin}}(\mathcal{S}, \mathcal{S})$;
2. $X \models S_1(\Gamma_F, \Gamma)$, and is strongly zero-dimensional;
3. $X \models S_{\text{fin}}(\Gamma_F, \Gamma)$, and is strongly zero-dimensional;
4. $X \models S_{\text{fin}}(\Gamma_{cl}, \Gamma_{cl})$, and is strongly zero-dimensional;
5. $C_p(X) \models S_{\text{fin}}(\Gamma_x, \Gamma_x)$;
6. $C_p(X) \models S_{\text{fin}}(\mathcal{S}, \Gamma_x)$.

9. $U_{\text{fin}}(\mathcal{S}, \mathcal{D})$

Recall that $U_{\text{fin}}(\mathcal{S}, \mathcal{D})$ is the selection hypothesis: whenever $U_1, U_2, \ldots \in \mathcal{S}$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n, n \in \omega$, such that $\bigcup \mathcal{F}_n : n \in \omega \big) \in \mathcal{D}$. For a function space $C(X)$, we can represent the condition $\bigcup \mathcal{F}_n : n \in \omega \big) \in \mathcal{D}$ as $\forall f \in C(X) \forall \mathcal{A} \in D$ a base neighborhood $O(f) = \langle f, K, \epsilon \rangle$ of $f$ where $\epsilon > 0$ and $K = \{x_1, \ldots, x_k\}$ is a finite subset of $X$, there is $n' \in \omega$ such that for each $j \in \{1, \ldots, k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Similarly, $U_{\text{fin}}(\Gamma_0, \Omega_0)$: whenever $\mathcal{S}_1, \mathcal{S}_2, \ldots \in \Gamma_0$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n, n \in \omega$, such that $\bigcup \mathcal{F}_n : n \in \omega \big) \in \Omega_0$, i.e. for a base neighborhood $O(f) = \langle f, K, \epsilon \rangle \big) = 0$ where $\epsilon > 0$ and $K = \{x_1, \ldots, x_k\}$ is a finite subset of $X$, there is $n' \in \omega$ such that for each $j \in \{1, \ldots, k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Theorem 9.1. For a space $X$, the following statements are equivalent:

1. $C_p(X) \models U_{\text{fin}}(\Gamma_x, \Omega_x)$;
2. $X \models U_{\text{fin}}(\Gamma_F, \Omega)$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{V}_i\} \subseteq \Gamma_F$. For each $i \in \omega$ and $\mathcal{U}_i = \{U_i^m\} \in \mathcal{V}_i$ we consider $\mathcal{K}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright U_i^m = 0 \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \omega\}$. Note that $\mathcal{U}_i \in \Gamma_F$ for each $i \in \omega$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega \}$ is a $\gamma$-cover of zero-sets of $X$, we have that $\mathcal{K}_i$ converge to $\mathbf{0}$ for each $i \in \omega$. By $C_p(X) \models U_{\text{fin}}(\Gamma_x, \Omega_x)$, there are finite sets $F_i = \{f_i^m_1, \ldots, f_i^m_{\omega}\} \subseteq \mathcal{K}_i$ such that $\bigcup F_i : i \in \omega \big) \in \Omega_0$. Note that $\{\bigcup \{U_i^{m_1}, U_i^{m_2}\} : i \in \omega \big) \in \Omega$.

(2) $\Rightarrow$ (1). Let $X \models U_{\text{fin}}(\Gamma_F, \Omega)$ and $A_i \in \Omega_0$ for each $i \in \omega$. Consider $\mathcal{U}_i = \{U_i, f = f^{-1}(\frac{1}{i}, 1) : f \in A_i\}$ for each $i \in \omega$. Without loss of generality we can assume that a set $U_i \neq X$ for any $i \in \omega$ and $f \in A_i$. Otherwise
there is sequence \( \{ f_k \}_{k \in \omega} \) such that \( \{ f_k \}_{k \in \omega} \) uniform converge to \( 0 \) and \( \{ f_k : k \in \omega \} \in \Omega_0 \).

Note that \( F_i = \{ F_{i,m} \}_{m \in \omega} = \{ f_i^{-1}[\frac{1}{i+1}, \frac{1}{i+2}] : m \in \omega \} \) is \( \gamma \)-cover of zero-sets of \( X \) and \( F_{i,m} \subset U_{i,m} \) for each \( i, m \in \omega \). It follows that \( \mathcal{U}_i \in \Gamma^F \) for each \( i \in \omega \).

By \( X \models U_{fin}(\Gamma^F, \Omega) \), there is a sequence \( \{ U_{i,m(1)}, U_{i,m(2)}, \ldots, U_{i,m(i)} : i \in \omega \} \) such that for each \( i \) and \( k \in \{ m(1), \ldots, m(i) \} \), \( U_{i,m(k)} \in \mathcal{U}_i \), and

\[
\bigcup \{ U_{i,m(1)}, \ldots, U_{i,m(i)} \} : i \in \omega \} \in \Omega.
\]

We claim that \( \bigcup \{ f_{i,m(1)}, \ldots, f_{i,m(i)} \} : i \in \omega \} \in \Omega_0 \).

Let \( W = \langle 0, K, \epsilon \rangle \) be a base neighborhood of \( 0 \) where \( \epsilon > 0 \) and \( K = \{ x_1, \ldots, x_s \} \) is a finite subset of \( X \), then there are \( i_0, i_1 \in \omega \) such that \( \frac{1}{i_0} < \epsilon, i_1 > i_0 \) and \( \bigcup_{k=m(1)}^{m(i_1)} U_{i_1,k} \supseteq K \). It follows that for each \( j \in \{ 1, \ldots, s \} \) there is \( g \in \{ f_{i_1,m(1)}, \ldots, f_{i_1,m(i_1)} \} \) such that \( g(x_j) \in (\epsilon, \epsilon) \).

\[ \square \]

**Theorem 9.2.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{D}) \) and is sequentially separable;
2. \( X \models U_{fin}(\Gamma^F, \Omega), X \models V; \)
3. \( C_p(X) \models U_{fin}(\Gamma_x, \Omega_x) \) and is sequentially separable;
4. \( C_p(X) \models U_{fin}(\mathcal{S}, \Omega_x) \) and is sequentially separable.

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{D}) \) and is sequentially separable. Let \( \{ \mathcal{V}_i \} \subset \Gamma^F \) and \( \mathcal{S} = \{ h_j : j \in \omega \} \) be a countable sequentially dense subset of \( C_p(X) \).

For each \( i \in \omega \) and \( \mathcal{U}_i = \{ U^j_i : j \in \omega \} \subset \mathcal{V}_i \) we consider \( \mathcal{S}_i = \{ f^j_i \in C(X) : f^j_i \upharpoonright F(U^j_i) = h_j \} \) and \( f^j_i \upharpoonright (X \setminus U^j_i) = 1 \) for \( j \in \omega \).

Since \( \mathcal{F}_i = \{ F(U^m_i) : m \in \omega \} \) is a \( \gamma \)-cover of \( X \), we have that \( \mathcal{S}_i \) is a countable sequentially dense subset of \( C_p(X) \) for each \( i \in \omega \).

By \( C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{D}) \), there are finite sets \( F_i = \{ f^{m_1}_i, \ldots, f^{m_s(i)}_i \} \subset \mathcal{S}_i \) such that \( \bigcup \{ F_i : i \in \omega \} \subset \mathcal{D} \). Note that \( \bigcup \{ U^{m_1}_i, \ldots, U^{m_s(i)}_i \} : i \in \omega \} \in \Omega \).

By Theorem 2.3, \( X \models V \).

(2) \( \Rightarrow \) (3). By Theorem 2.3 and Theorem 9.1.

(3) \( \Rightarrow \) (4) is immediate.

(4) \( \Rightarrow \) (1). Suppose that \( C_p(X) \) is sequentially separable and \( C_p(X) \models U_{fin}(\mathcal{S}, \Omega_x) \).

Let \( D = \{ d_n : n \in \omega \} \) be a dense subspace of \( C_p(X) \). Given a sequence of sequentially dense subspace of \( C_p(X) \), enumerate it as \( \{ S_{n,m} : n, m \in \omega \} \). For each \( n \in \omega \), pick
\[ \mathcal{F}_{n,m} = \{ d_{n,m,1}, ..., d_{n,m,k(n,m)} \} \subset S_{n,m} \text{ so that } d_n \in \{ \bigcup \mathcal{F}_{n,m} : m \in \omega \}, \]
i.e. for a base neighborhood \( O(d_n) = \langle d_n, K, \epsilon \rangle \) of \( d_n \) where \( \epsilon > 0 \) and \( K = \{ x_1, ..., x_k \} \) is a finite subset of \( X \), there is \( m' \in \omega \) such that for each \( j \in \{1, ..., k\} \) there is \( g \in \mathcal{F}_{n,m'} \) such that \( g(x_j) \in (d_n(x_j) - \epsilon, d_n(x_j) + \epsilon) \).

Then \( \bigcup \mathcal{F}_{n,m} : m, n \in \omega \} \in \mathcal{D} \).

\[ \text{Theorem 9.3. For a separable metrizable space } X, \text{ the following statements are equivalent:} \]

1. \( C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{D}) \);
2. \( X \models U_{fin}(\Gamma, \Omega) \);
3. \( C_p(X) \models U_{fin}(\Gamma_x, \Omega_x) \);
4. \( C_p(X) \models U_{fin}(\mathcal{S}, \Omega_x) \).

10. \( U_{fin}(\mathcal{S}, \mathcal{S}) \)

Recall that \( U_{fin}(\mathcal{S}, \mathcal{S}) \) is the selection hypothesis: whenever \( U_1, U_2, ..., \in \mathcal{S} \), there are finite sets \( \mathcal{F}_n \subseteq U_n, n \in \omega \), such that \( \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \mathcal{S} \). For a function space \( C(X) \), we can represent the condition \( \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \mathcal{S} \) as \( \forall f \in C(X) \forall \text{ a base neighborhood of } f O(f) = \langle f, K, \epsilon \rangle > \text{ of } f \) where \( \epsilon > 0 \) and \( K = \{ x_1, ..., x_k \} \) is a finite subset of \( X \), there is \( n' \in \omega \) such that for each \( n > n' \) and \( j \in \{1, ..., k\} \) there is \( g \in \mathcal{F}_n \) such that \( g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon) \).

Similarly, \( U_{fin}(\Gamma_0, \Gamma_0) \): whenever \( S_1, S_2, ..., \in \Gamma_0 \), there are finite sets \( \mathcal{F}_n \subseteq S_n, n \in \omega \), such that \( \{ \bigcup \mathcal{F}_n : n \in \omega \} \in \Gamma_0 \), i.e. for a base neighborhood \( O(f) = \langle f, K, \epsilon \rangle > \text{ of } f = 0 \) where \( \epsilon > 0 \) and \( K = \{ x_1, ..., x_k \} \) is a finite subset of \( X \), there is \( n' \in \omega \) such that for each \( n > n' \) and \( j \in \{1, ..., k\} \) there is \( g \in \mathcal{F}_n \) such that \( g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon) \).

\[ \text{Theorem 10.1. For a space } X, \text{ the following statements are equivalent:} \]

1. \( C_p(X) \models U_{fin}(\Gamma_x, \Gamma_x) \);
2. \( X \models U_{fin}(\Gamma_F, \Gamma) \).

\[ \text{Proof. } (1) \Rightarrow (2). \text{ Let } \{ V_1 \} \subset \Gamma_F. \text{ For each } i \in \omega \text{ we consider a subset } \mathcal{S}_i \text{ of } \]

\[ C_p(X) \text{ and } \mathcal{U}_i = \{ U_i^m \}_{m \in \omega} \subset V_1 \text{ where } \]

\[ \mathcal{S}_i = \{ f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = 0 \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \omega \}. \]

Since \( \mathcal{F}_i = \{ F(U_i^m) : m \in \omega \} \) is a \( \gamma \)-cover of \( X \), we have that \( \mathcal{S}_i \) converge to \( 0 \), i.e. \( \mathcal{S}_i \in \Gamma_0 \text{ for each } i \in \omega. \)

22
Since $C(X) \models U_{fin}(\Gamma_x, \Gamma_x)$, there is a sequence $\{F_i\}_{i \in \omega} = \{f_i^{m_1}, ..., f_i^{m_k(i)} : i \in \omega\}$ such that for each $i$, $F_i \subseteq S_i$, and $\bigcup_{i \in \omega} F_i \in \Gamma_0$.

Consider a sequence $\{W_i\}_{i \in \omega} = \{U_i^{m_1}, ..., U_i^{m_k(i)} : i \in \omega\}$.

(a) $W_i \subset U_i$.

(b) $\{\bigcup W_i : i \in \omega\}$ is a $\gamma$-cover of $X$.

Let $K = \{x_1, ..., x_s\}$ be a finite subset of $X$ and $U = \{0\}$, $\frac{1}{2} >$ be a base neighborhood of $0$, then there exists $i_0 \in \omega$ such that for each $i > i_0$ and $j \in \{1, ..., s\}$ there is $g \in F_i$ such that $g(x_j) \in (-\frac{1}{2}, \frac{1}{2})$.

It follows that $K \subset \bigcup_{j=1}^{k(i)} U_i^{m_j}$ for $i > i_0$. We thus get $X \models U_{fin}(\Gamma_F, \Gamma)$.

(2) $\Rightarrow$ (1). Fix $\{S_i : i \in \omega\} \subset \Gamma_0$ where $S_i = \{f_i^k : k \in \omega\}$ for each $i \in \omega$.

For each $i, k \in \omega$, we put $U_{i,k} = \{x \in X : |f_i^k(x)| < \frac{1}{i}\}$, $Z_{i,k} = \{x \in X : |f_i^k(x)| \leq \frac{1}{i+1}\}$.

Each $U_{i,k}$ (resp., $Z_{i,k}$) is a cozero-set (resp., zero-set) in $X$ with $Z_{i,k} \subset U_{i,k}$.

Let $U_i = \{U_{i,k} : k \in \omega\}$ and $Z_i = \{Z_{i,k} : k \in \omega\}$. So, without loss of generality, we may assume $U_{i,k} \neq X$ for each $i, k \in \omega$. We can easily check that the condition $f_i^k \to 0$ ($k \to \infty$) implies that $Z_i$ is a $\gamma$-cover of $X$.

Since $X \models U_{fin}(\Gamma_F, \Gamma)$ there is a sequence $\{F_i\}_{i \in \omega} = \{U_{i,k_1}, ..., U_{i,k_i} : i \in \omega\}$ such that for each $i$, $F_i \subset U_i$, and $\bigcup F_i : i \in \omega\}$ is an element of $\Gamma$.

Let $K = \{x_1, ..., x_s\}$ be a finite subset of $X$, $\epsilon > 0$, and $U = \{0\}$, $K, \epsilon >$ be a base neighborhood of $0$, then there exists $i' \in \omega$ such that for each $i > i'$ $K \subset \bigcup F_i$. It follows that for each $i > i'$ and $j \in \{1, ..., s\}$ there is $g \in S_i$ such that $g(x_j) \in (-\epsilon, \epsilon)$. So $C_p(X) \models U_{fin}(\Gamma_x, \Gamma_x)$.

\[ \Box \]

**Theorem 10.2.** For a space $X$, the following statements are equivalent:

1. $C_p(X) \models U_{fin}(S, S)$ and is sequentially separable;
2. $X \models U_{fin}(\Gamma_F, \Gamma)$, $X \models V$;
3. $C_p(X) \models U_{fin}(\Gamma_x, \Gamma_x)$ and is sequentially separable;
4. $C_p(X) \models U_{fin}(S, \Gamma_x)$ and is sequentially separable.

**Proof.** (1) $\Rightarrow$ (2). Let $\{V_i\} \subset \Gamma_F$ and $S = \{h_m : m \in \omega\}$ be a countable sequentially dense subset of $C_p(X)$. For each $i \in \omega$ we consider a countable sequentially dense subset $S_i$ of $C_p(X)$ and $U_i = \{U_i^m\}_{m \in \omega} \subset V_i$ where $S_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \omega\}$. 

23
Since $\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}$ is a $\gamma$-cover of zero-sets of $X$ and $\mathcal{S}$ is a countable sequentially dense subset of $C_p(X)$, we have that $\mathcal{S}_i$ is a countable sequentially dense subset of $C_p(X)$ for each $i \in \omega$. Let $h \in C(X)$, there is a sequence $\{h_{m_s} : s \in \omega\} \subset \mathcal{S}$ such that $\{h_{m_s}\}_{s \in \omega}$ converge to $h$. Let $K$ be a finite subset of $X$, $\epsilon > 0$ and $W = \langle h, K, \epsilon \rangle$ be a base neighborhood of $h$, then there is a number $m_0$ such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s \in \omega}$ converge to $h$.

Since $C(X) \models U_{fin}(\mathcal{S}, \mathcal{S})$, there is a sequence $\{F_i\} = \{f_i^{m_1}, ..., f_i^{m_k} : i \in \omega\}$ such that for each $i$, $F_i \subset \mathcal{S}_i$, and $\{\bigcup F_i : i \in \omega\}$ is an element of $\mathcal{S}$, i.e. for any $f \in C(X)$ and a base neighborhood $O(f) = \langle f, K, \epsilon \rangle$ of $f$ where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of $X$, there is $i' \in \omega$ such that for each $i > i'$ and $j \in \{1, ..., k\}$ there is $g \in F_i$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

Consider a sequence $\{Q_i\}_{i \in \omega} = \{U_i^{m_1}, ..., U_i^{m_k} : i \in \omega\}$.

- (a). $Q_i \subset \mathcal{U}_i$.
- (b). $\bigcup Q_i : i \in \omega\}$ is a $\gamma$-cover of $X$.

We thus get $X \models U_{fin}(\Gamma_F, \Gamma)$. By Theorem 2.3 $X \models V$.

\[\square\]

**Theorem 10.3.** For a separable metrizable space $X$, the following statements are equivalent:

1. $C_p(X) \models U_{fin}(\mathcal{S}, \mathcal{S})$;
2. $X \models U_{fin}(\Gamma, \Gamma)$ [Hurewicz property];
3. $C_p(X) \models U_{fin}(\Gamma_x, \Gamma_x)$;
4. $C_p(X) \models U_{fin}(\mathcal{S}, \Gamma_x)$.

11. $S_1(\mathcal{A}, \mathcal{A})$

**Definition 11.1.** A set $A \subseteq C_p(X)$ will be called $n$-dense in $C_p(X)$, if for each $n$-finite set $\{x_1, ..., x_n\} \subset X$ such that $x_i \neq x_j$ for $i \neq j$ and an open sets $W_1, ..., W_n$ in $\mathbb{R}$ there is $f \in A$ such that $f(x_i) \in W_i$ for $i \in \overline{1,n}$.

Obviously, that if $A$ is a $n$-dense set of $C_p(X)$ for each $n \in \omega$ then $A$ is a dense set of $C_p(X)$.

For a space $C_p(X)$ we denote:

- $\mathcal{A}_n$ — the family of a $n$-dense subsets of $C_p(X)$.
- If $n = 1$, then we denote $\mathcal{A}$ instead of $\mathcal{A}_1$.
Definition 11.2. Let \( f \in C(X) \). A set \( B \subseteq C_p(X) \) will be called \( n \)-dense at point \( f \), if for each \( n \)-finite set \( \{ x_1, \ldots, x_n \} \subseteq X \) and \( \epsilon > 0 \) there is \( h \in B \) such that \( h(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon) \) for \( i \in 1, n \).

Obviously, that if \( B \) is a \( n \)-dense at point \( f \) for each \( n \in \omega \) then \( f \in \overline{B} \).

For a space \( C_p(X) \) we denote:
\( \mathcal{B}_{n,f} \) — the family of \( n \)-dense at point \( f \) subsets of \( C_p(X) \).

If \( n = 1 \), then we denote \( \mathcal{B}_f \) instead of \( \mathcal{B}_{1,f} \).

Let \( \mathcal{U} \) be an open cover of \( X \) and \( n \in \mathbb{N} \).

\( x \) is cover of \( X \) if for each \( F \subseteq X \) with \( |F| \leq n \), there is \( U \in \mathcal{U} \) such that \( F \subseteq U \).

1. \( \mathcal{O}_n = \) the family of open \( n \)-covers of \( X \).
2. \( S_1(\mathcal{O}, \mathcal{O}) = \overline{S_1(\Omega, \mathcal{O})} \) [Rothberger property];
3. \( C_p(X) \models S_1(\mathcal{A}, \mathcal{A}) \);
4. \( C_p(X) \models S_1(\mathcal{A}, \mathcal{B}_f) \);
5. \( C_p(X) \models S_1(\mathcal{D}, \mathcal{A}) \);
6. \( C_p(X) \models S_1(\{ \mathcal{A}_n \}_{n \in \mathbb{N}}, \mathcal{A}) \);
7. \( C_p(X) \models S_1(\{ \mathcal{B}_{n,f} \}_{n \in \mathbb{N}}, \mathcal{B}_f) \);
8. \( C_p(X) \models S_1(\{ \mathcal{A}_n \}_{n \in \mathbb{N}}, \mathcal{B}_f) \).

Theorem 11.3. For a space \( X \), the following statements are equivalent:

Proof. (1) \( \Rightarrow \) (2). Let \( \{ \mathcal{O}_n \}_{n \in \omega} \) be a sequence of open covers of \( X \). We set \( A_n = \{ f \in C(X) : f \upharpoonright (X \setminus U) = 1 \} \) and \( f \upharpoonright K = q \) for some \( U \in \mathcal{O}_n \), a finite set \( K \subseteq U \) and \( q \in \mathbb{Q} \). It is not difficult to see that each \( A_n \) is 1-dense subset of \( C_p(X) \) since each \( \mathcal{O}_n \) is a cover of \( X \) and \( X \) is Tychonoff.

By the assumption there exist \( f_n \in A_n \) such that \( \{ f_n : n \in \omega \} \in \mathcal{A} \).

For each \( f_n \) we take \( U_n \in \mathcal{O}_n \) such that \( f_n \upharpoonright (X \setminus U_n) = 1 \).

Set \( \mathcal{U} = \{ U_n : n \in \omega \} \). For \( x \in X \) we consider the basic open neighborhood of \( 0 \) \( [x, W] \), where \( W = \left(-\frac{1}{2}, \frac{1}{2}\right) \).

Note that there is \( m \in \omega \) such that \( [x, W] \) contains \( f_m \in \{ f_n : n \in \omega \} \).

This means \( x \in U_m \). Consequently \( \mathcal{U} \) is cover of \( X \).

(2) \( \Rightarrow \) (3). Let \( B_n \in \mathcal{B}_f \) for each \( n \in \omega \). We renumber \( \{ B_n \}_{n \in \omega} \) as \( \{ B_{i,j} \}_{i,j \in \omega} \). Since \( C(X) \) is homogeneous, we may think that \( f = 0 \). We set
$U_{i,j} = \{ g^{-1}(-1/i, 1/i) : g \in B_{i,j} \}$ for each $i, j \in \omega$. Since $B_{i,j} \in \mathcal{B}_0$, $U_{i,j}$ is an open cover of $X$ for each $i, j \in \omega$. In case the set $M = \{ i \in \omega : X \in U_{i,j} \}$ is infinite, choose $g_m \in B_{m,j} \ m \in M$ so that $g^{-1}(-1/m, 1/m) = X$, then $\{g_m : m \in \omega \} \in \mathcal{B}_f$.

So we may assume that there exists $i' \in \omega$ such that for each $i \geq i'$ and $g \in B_{i,j} g^{-1}(-1/i, 1/i)$ is not $X$.

For the sequence $V_i = \{ U_{i,j} : j \in \omega \}$ of open covers there exist $f_{i,j} \in B_{i,j}$ such that $U_i = \{ f_{i,j}^{-1}(-1/i, 1/i) : j \in \omega \}$ is a cover of $X$. Let $[x, W]$ be any basic open neighborhood of $0$, where $W = (-\epsilon, \epsilon)$, $\epsilon > 0$. There exists $m \geq i'$ and $j \in \omega$ such that $1/m < \epsilon$ and $x \in f_m^{-1}(-1/m, 1/m)$. This means $\{ f_{i,j} : i, j \in \omega \} \in \mathcal{B}_f$.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Let $A_n \in \mathcal{A}$ for each $n \in \omega$. We renumber $\{ A_n \}_{n \in \omega}$ as $\{ A_{i,j} \}_{i, j \in \omega}$. Renumber the rational numbers $\mathbb{Q}$ as $\{ q_i : i \in \omega \}$. Fix $i \in \omega$. By the assumption there exist $f_{i,j} \in A_{i,j}$ such that $\{ f_{i,j} : j \in \omega \} \in \mathcal{A}_n$, where $q_i$ is the constant function to $q_i$. Then $\{ f_{i,j} : i, j \in \omega \} \in \mathcal{A}$.

(1) $\Rightarrow$ (5). Since a dense set of $C_p(X)$ is a 1-dense set of $C_p(X)$, we have $C_p(X) \models S_1(\mathcal{D}, \mathcal{A})$.

(5) $\Rightarrow$ (6). Let $D_n \in \mathcal{A}_n$ for each $n \in \omega$. We renumber $\{ D_n \}_{n \in \omega}$ as $\{ D_{i,j} \}_{i, j \in \omega}$. Then $P_j = \{ D_{i,j} : i \in \omega \}$ is a dense subset of $C_p(X)$ for each $j \in \omega$. By (5), there is $p_j \in P_j$ for each $j \in \omega$ such that $\{ p_j : j \in \omega \} \in \mathcal{A}$.

Hence, we have $C_p(X) \models S_1(\{ A_n \}_{n \in \omega}, \mathcal{A})$.

(6) $\Rightarrow$ (8) is immediate.

(8) $\Rightarrow$ (2). Claim that $X \models S_1(\{ \mathcal{O}_n \}_{n \in \omega}, \mathcal{O})$. Fix $\{ \mathcal{O}_n \}_{n \in \omega}$. For every $n \in \omega$ a set $S_n = \{ f \in C(X) : f \upharpoonright (X \setminus U) = 1 \text{ and } f(x_i) \in \mathbb{Q} \text{ for each } i = 1, ..., n \}$ for $U \in \mathcal{O}_n$ and a finite set $K = \{ x_1, ..., x_n \} \subset U$. Note that $S_n \in \mathcal{A}_n$ for each $n \in \omega$. By (8), there is $f_n \in S_n$ for each $n \in \omega$ such that $\{ f_n : n \in \omega \} \in \mathcal{B}_0$. Then $\{ U_n : n \in \omega \} \in \mathcal{O}$.

(3) $\Rightarrow$ (7) is immediate.

(7) $\Rightarrow$ (2). The proof is analogous to proof of implication (8) $\Rightarrow$ (2).

\[ \square \]

12. $S_{fin}(\mathcal{A}, \mathcal{A})$

Theorem 12.1. For a space $X$, the following statements are equivalent:

1. $C_p(X) \models S_{fin}(\mathcal{A}, \mathcal{A})$;
2. $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ [Menger property];
3. \( C_p(X) \models S_{\text{fin}}(B_f, B_f) \);
4. \( C_p(X) \models S_{\text{fin}}(A, B_f) \);
5. \( C_p(X) \models S_{\text{fin}}(D, A) \);
6. \( C_p(X) \models S_{\text{fin}}(\{A_n\}_{n \in \mathbb{N}}, A) \);
7. \( C_p(X) \models S_{\text{fin}}(\{B_n,f\}_{n \in \mathbb{N}}, B_f) \);
8. \( C_p(X) \models S_{\text{fin}}(\{A_n\}_{n \in \mathbb{N}}, B_f) \).

**Proof.** The proof is analogous to proof of Theorem \[ \underline{13.3} \]

\[ \square \]

13. \( S_1(S, A) \)

**Proposition 13.1.** For a space \( X \), the following statements are equivalent:

1. \( C_p(X) \models S_1(\Gamma_x, B_f) \);
2. \( X \models S_1(\Gamma_F, O) \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( \{U_i\} \subseteq \Gamma_F \). For each \( i \in \omega \) we consider the set \( S_i = \{ f \in C(X) : f \upharpoonright F(U) = 0 \text{ and } f \upharpoonright (X \setminus U) = 1 \text{ for } U \in U_i \} \).

Since \( F_i = \{ F(U) : U \in U_i \} \) is a \( \gamma \)-cover of \( X \), we have that \( S_i \) converge to \( 0 \), i.e. \( S_i \in \Gamma_0 \) for each \( i \in \omega \).

Since \( C_p(X) \models S_1(\Gamma_x, B_f) \), there is a sequence \( \{f_i\}_{i \in \omega} \) such that for each \( i, f_i \in S_i \), and \( \{f_i : i \in \omega\} \subseteq B_0 \).

Consider \( V = \{U_i : U_i \in U_i \text{ such that } f_i \upharpoonright F(U_i) = 0 \text{ and } f_i \upharpoonright (X \setminus U_i) = 1\} \). Let \( x \in X \) and \( W = [x, (-1, 1)] \) be a neighborhood of \( 0 \), then there exists \( i_0 \in \omega \) such that \( f_{i_0} \in W \).

It follows that \( x \in U_{i_0} \) and \( V \in O \). We thus get \( X \models S_1(\Gamma_F, O) \).

(2) \( \Rightarrow \) (1). Fix \( \{S_n : n \in \omega\} \subseteq \Gamma_0 \). We renumber \( \{S_n : n \in \omega\} \) as \( \{S_{i,j} : i, j \in \omega\} \).

For each \( i, j \in \omega \) and \( f \in S_{i,j} \), we put \( U_{i,j,f} = \{x \in X : |f(x)| < \frac{1}{i+j+1}\} \), \( Z_{i,j,f} = \{x \in X : |f(x)| \leq \frac{1}{i+j+1}\} \).

Each \( U_{i,j,f} \) (resp., \( Z_{i,j,f} \)) is a cozero-set (resp., zero-set) in \( X \) with \( Z_{i,j,f} \subseteq U_{i,j,f} \). Let \( U_{i,j,f} = \{U_{i,j,f} : f \in S_{i,j}\} \) and \( Z_{i,j,f} = \{Z_{i,j,f} : f \in S_{i,j}\} \). So without loss of generality, we may assume \( U_{i,j,f} \neq X \) for each \( i, j \in \omega \) and \( f \in S_{i,j} \).

We can easily check that the condition \( S_{i,j} \in \Gamma_0 \) implies that \( Z_{i,j} \) is a \( \gamma \)-cover of \( X \).

Since \( X \models S_1(\Gamma_F, O) \) for each \( j \in \omega \) there is a sequence \( \{U_{i,j,f_{i,j}} : i \in \omega\} \) such that for each \( i, U_{i,j,f_{i,j}} \subseteq U_{i,j} \), and \( \{U_{i,j,f_{i,j}} : i \in \omega\} \subseteq O \). Claim that
\{f_{ij} : i, j \in \omega\} \in \mathcal{B}_0$. Let \(x \in X, \epsilon > 0\), and \(W = [x, (-\epsilon, \epsilon)]\) be a base neighborhood of 0, then there exists \(j' \in \omega\) such that \(\frac{1}{1+j'} < \epsilon\). It follow that there exists \(i'\) such that \(f_{i',j'}(x) \in (-\epsilon, \epsilon)\). So \(C_p(X) \models S_1(\Gamma_x, \mathcal{B}_f)\).

\[\square\]

**Theorem 13.2.** For a space \(X\), the following statements are equivalent:

1. \(C_p(X) \models S_1(\mathcal{S}, \mathcal{A})\) and is sequentially separable;
2. \(X \models S_1(\Gamma_F, \mathcal{O})\), \(X \models V\);
3. \(C_p(X) \models S_1(\Gamma_x, \mathcal{B}_f)\) and is sequentially separable;
4. \(C_p(X) \models S_1(\mathcal{S}, \mathcal{B}_f)\) and is sequentially separable.

**Proof.** (1) \(\Rightarrow\) (2). Let \(\{\mathcal{V}_i : i \in \omega\} \subset \Gamma_F\) and \(\mathcal{S} = \{h_m : m \in \omega\}\) be a countable sequentially dense subset of \(C_p(X)\). For each \(i \in \omega\) we consider a countable sequentially dense subset \(\mathcal{S}_i\) of \(C_p(X)\) and \(\mathcal{U}_i = \{U_i^m : m \in \omega\} \subset \mathcal{V}_i\) where

\[\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = h_m\} \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \omega\}.

Since \(\mathcal{F}_i = \{F(U_i^m) : m \in \omega\}\) is a \(\gamma\)-cover of zero subsets of \(X\) and \(\mathcal{S}\) is a countable sequentially dense subset of \(C_p(X)\), we have that \(\mathcal{S}_i\) is a countable sequentially dense subset of \(C_p(X)\) for each \(i \in \omega\). Let \(h \in C(X)\), there is a sequence \(\{h_m : s \in \omega\} \subset \mathcal{S}\) such that \(\{h_m\}_{s \in \omega}\) converge to \(h\). Let \(K\) be a finite subset of \(X, \epsilon > 0\) and \(W = \langle h, K, \epsilon \rangle\) be a base neighborhood of \(h\), then there is a number \(m_0\) such that \(K \subset F(U_i^m)\) for \(m > m_0\) and \(h_m \in W\) for \(m_s > m_0\). Since \(f_i^{m_s} \upharpoonright K = h_m \upharpoonright K\) for each \(m_s > m_0, f_i^{m_s} \in W\) for each \(m_s > m_0\). It follows that a sequence \(\{f_i^{m_s}\}_{s \in \omega}\) converge to \(h\).

By \(C_p(X) \in S_1(\mathcal{S}, \mathcal{A})\), there is a set \(\{f_i^{m(i)} : i \in \omega\}\) such that for each \(i, f_i^{m(i)} \in \mathcal{S}_i\), and \(\{f_i^{m(i)} : i \in \omega\}\) is an element of \(\mathcal{A}\).

Consider a set \(\{U_i^{m(i)} : i \in \omega\}\).

(a). \(U_i^{m(i)} \in \mathcal{U}_i\).
(b). \(\{U_i^{m(i)} : i \in \omega\}\) is a cover of \(X\).

Let \(x \in X\) and \(U = \langle 0, x, \frac{1}{2} \rangle\) be a base neighborhood of \(0\), then there is \(f_i^{m(j_0)} \in U\) for some \(j_0 \in \omega\). It follows that \(x \in U_i^{m(i)j_0}\). We thus get \(X \models S_1(\Gamma_F, \mathcal{O})\).

(2) \(\Leftrightarrow\) (3). By Proposition 13.1.

(3) \(\Rightarrow\) (4) is immediate.

(4) \(\Rightarrow\) (1). Let \(S_n \in \mathcal{S}\) for each \(n \in \omega\). We renumber \(\{S_n\}_{n \in \omega}\) as \(\{S_{i,j}\}_{i,j \in \omega}\). Renumber the rational numbers \(\mathbb{Q}\) as \(\{q_i : i \in \omega\}\). Fix \(i \in \omega\). By
the assumption there exist $f_{i,j} \in S_{i,j}$ such that $\{f_{i,j} : j \in \omega\} \in \mathcal{B}_{q_i}$, where $q_i$ is the constant function to $q_i$. Then $\{f_{i,j} : i, j \in \omega\} \in \mathcal{A}$.

\[\square\]

**Theorem 13.3.** For a separable metrizable space $X$, the following statements are equivalent:

1. $C_p(X) \models S_1(\mathcal{S}, \mathcal{A})$;
2. $X \models S_1(\Gamma_F, \mathcal{O})$;
3. $C_p(X) \models S_1(\Gamma_x, \mathcal{B}_f)$;
4. $C_p(X) \models S_1(\mathcal{S}, \mathcal{B}_f)$.

### 14. Critical cardinalities

For a collection $\mathcal{J}$ of spaces $C_p(X)$, let $\text{non}C_p(\mathcal{J})$ denote the minimal cardinality for $X$ which $C_p(X)$ is not a member of $\mathcal{J}$.

The critical cardinalities in the Scheepers Diagram \cite{40} are equal to the critical cardinalities of selectors for sequences of countable dense and countable sequentially subsets of $C_p(X)$.

**Theorem 14.1.** For a collection $C_p(X)$ of all real-valued continuous functions, defined on Tychonoff spaces $X$ with $i\omega(X) = \aleph_0$,

1. $\text{non}C_p(S_1(\mathcal{D}, \mathcal{S})) = \mathfrak{p}$.
2. $\text{non}C_p(S_1(\mathcal{S}, \mathcal{S})) = \text{non}C_p(U_{\text{fin}}(\mathcal{S}, \mathcal{S})) = \mathfrak{b}$.
3. $\text{non}C_p(S_{\text{fin}}(\mathcal{D}, \mathcal{D})) = \text{non}C_p(S_1(\mathcal{S}, \mathcal{D})) = \text{non}C_p(S_1(\mathcal{S}, \mathcal{A})) = \mathfrak{d}$.
4. $\text{non}C_p(U_{\text{fin}}(\mathcal{S}, \mathcal{D})) = \text{non}C_p(U_{\text{fin}}(\mathcal{S}, \mathcal{A})) = \text{non}C_p(S_{\text{fin}}(\mathcal{S}, \mathcal{D})) = \mathfrak{d}$.
5. $\text{non}C_p(S_1(\mathcal{D}, \mathcal{D})) = \text{non}C_p(S_1(\mathcal{A}, \mathcal{A})) = \text{cov}(\mathcal{M})$. 

29
We can summarize the relationships between considered notions in next diagrams.

Fig. 2. The Diagram of selectors for sequences of dense sets of $C_p(X)$.

Fig. 3. The Diagram of selection principles for metrizable separable space $X$ corresponding to selectors for sequences of dense sets of $C_p(X)$. 
Acknowledgment. The author express gratitude to Boaz Tsaban for useful discussions.

References


