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On some properties of the space of upper semicontinuous functions

Alexander V. Osipov[a], Evgenii G. Pytkeev[b]

[a]*Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, Ural State University of Economics, 620219, Ekaterinburg, Russia*
[b]*Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, 620219, Ekaterinburg, Russia*

Abstract

For a Tychonoff space X, we will denote by $USC_p(X)$ ($B_1(X)$) the set of all real-valued upper semicontinuous functions (the set of all Baire functions of class 1) defined on X endowed with the pointwise convergence topology.

In this paper we describe a class of Tychonoff spaces X for which the space $USC_p(X)$ is sequentially separable. Unexpectedly, it turns out that this class coincides with the class of spaces for which a stronger form of the sequential separability for the space $B_1(X)$ holds.

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1. Introduction

If X is a topological space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. If D is a countable sequentially dense subset of X then X is called sequentially separable space [7, 9].

Let X be a Tychonoff space. We consider the following function spaces.

• $C_p(X)$ is the set of all real-valued continuous functions defined on X endowed with the topology of pointwise convergence.

URL: OAB@list.ru (Alexander V. Osipov), pyt@imm.uran.ru (Evgenii G. Pytkeev)

• $B_1(X)$ is the set of all Baire functions class 1 (i.e., pointwise limits of continuous functions) defined on X endowed with the pointwise convergence topology.

• $USC_p(X)$ is the set $USC(X) = \{f \in \mathbb{R}^X : f^{-1}((-\infty, r))$ is an open set of X for any $r \in \mathbb{R}\}$ (i.e. the set all upper semicontinuous functions defined on X) endowed with the topology of pointwise convergence.

Note that $C_p(X) \subseteq USC_p(X) \subseteq B_1(X)$ for a separable metrizable space X. It follows that $USC_p(X)$ is sequentially separable for a separable metrizable space X (Theorem 3.2).

It is well known that $f : X \to \mathbb{R}$ is a Baire function if and only if there exists a continuous mapping $\varphi : X \to M$ from X onto a separable metrizable space M and a Borel function $g : M \to \mathbb{R}$ such that $f = g \circ \varphi$. If we replace the Borel function by upper semicontinuous function in this characterization, we obtain the function $f : X \to \mathbb{R}$ such that $f^{-1}((-\infty, r)) = \varphi^{-1}(g^{-1}((-\infty, r)))$ is a cozero-set of X for any $r \in \mathbb{R}$.

Define the function space $USC^f(X) := \{f \in \mathbb{R}^X : f^{-1}((-\infty, r))$ is a cozero set of X for any $r \in \mathbb{R}\}$. Clearly, that if X is a perfectly normal space then $USC^f(X) = USC(X)$. We denote by $USC^f_p(X)$ the set $USC^f(X)$ endowed with the topology of pointwise convergence.

We claim that $USC^f_p(X)$ is sequentially separable if and only if there exists a countable subset S of $C(X)$ such that $[S]_{seq} = B_1(X)$ (i.e., when a stronger form of the sequential separability for the space $B_1(X)$ holds).

2. Main definitions and notation

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set of X if $X \setminus O$ is a zero-set. If a set $Z = \bigcup_{i \in \mathbb{N}} Z_i$ where Z_i is a zero-set of X for any $i \in \mathbb{N}$, then Z is called Z_σ-set of X. Note that if a space X is a perfectly normal space, then class of Z_σ-sets of X coincides with class of F_σ-sets of X. It is well known [6], that $f \in B_1(X)$ if and only if $f^{-1}(G)$ - Z_σ-set for any open set G of real line $\mathbb{R}$.

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.

Theorem 2.1. *(Noble's Theorem in [2]) Let X be a Tychonoff space. A space $C_p(X)$ is separable if and only if $iw(X) = \aleph_0$.*

Theorem 2.2. *(Pestriakov's Theorem in [5]). Let X be a Tychonoff space. A space $B_1(X)$ is separable if and only if $iw(X) = \aleph_0$.*

Definition 2.3. A Tychonoff space X has the Velichko property ($X \models V$), if there exists a condensation (one-to-one continuous mapping) $f : X \mapsto Y$ from the space X on a separable metric space Y, such that $f(U)$ is an F_σ-set of Y for any cozero-set U of X.

Theorem 2.4. *(Velichko [8]). Let X be a Tychonoff space. A space $C_p(X)$ is sequentially separable if and only if $X \models V$.*

Theorem 2.5. *([8]) A space $B_1(X)$ is sequentially separable for any separable metric space X.*

Note that proof of this theorem gives more, namely that there exists a countable subset $S \subset C_p(X)$, such that $[S]_{seq} = B_1(X)$.

Hence, a space $USC_p(X)$ is sequentially separable for any separable metric space X.

In [4], Osipov and Pytkeev have established criterion for $B_1(X)$ to be sequentially separable.

Definition 2.6. A space X has OP**-property** ($X \models OP$), if there exists a bijection $\varphi : X \mapsto Y$ from a space X onto a separable metrizable space Y, such that

1. $\varphi^{-1}(U)$ is a Z_σ-set of X for any open set U of Y;
2. $\varphi(T)$ is an F_σ-set of Y for any zero-set T of X.

Theorem 2.7. *(Theorem 3.1 in [4]) A function space $B_1(X)$ is sequentially separable if and only if $X \models OP$.*

Theorem 2.8. *(Example 3.3 in [4]) There is a Tychonoff space X such that $C_p(X)$ is sequentially separable, but $B_1(X)$ is not.*

In the above theorem, the promised space X could be, for example, if X is the Sorgenfrey line (or the Niemytzki plane) [4].

3. Main results

Definition 3.1. A space X has U**-property** ($X \models U$), if there exists a condensation $f : X \mapsto Y$ from the space X onto a separable metric space Y, such that $f(D)$ is a Z_σ-set of Y for any zero-set D of X.

Theorem 3.2. *Let X be a Tychonoff space. Then the following statements are equivalent:*

1. $USC_p^f(X)$ *is sequentially separable;*
2. $X \models U$*;*
3. *there exists a countable subset S of $C(X)$ such that $[S]_{seq} = B_1(X)$.*

Proof. (1) $\Rightarrow$ (2). Assume that $USC_p^f(X)$ is sequentially separable. Let $A = \{f_i : i \in \mathbb{N}\}$ be a sequentially dense subset of $USC_p^f(X)$. Note that $f^{-1}(W)$ is a Z_σ-set of X for an open set W of $\mathbb{R}$ and $f \in USC_p^f(X)$. It follows that $USC_p^f(X)$ is a dense subset of $B_1(X)$ and hence $B_1(X)$ is separable. By Theorem 2.2, $iw(X) = \aleph_0$. Hence there exists a condensation from the space X on a separable metric space M. Let β be a countable base of the space M. Let $\alpha = \{f_i^{-1}(-\infty, r) : r \in \mathbb{Q} \text{ and } i \in \mathbb{N}\} \bigcup \beta$ and let τ be a topology on X generating α. Denote $Y = (X, \tau)$. Note that there exists a condensation $f : X \mapsto Y$ from the space X onto a separable metric space Y. By definition of α, $f_i \in USC(Y)$ for each $i \in \mathbb{N}$.

We will prove that $f(D)$ is an F_σ-set of Y for any Z_σ-set D of X. Fix a Z_σ-set $D = \bigcup\limits_{i\in\mathbb{N}} D_i$ of X where D_i is a zero-set of X and $D_i \subset D_{i+1}$ for each $i \in \mathbb{N}$. Define the function h: $h(D_1) = 1$, $h(D_{i+1} \setminus D_i) = \frac{1}{i+1}$ for each $i \in \mathbb{N}$ and $h(X \setminus D) = 0$. By construction of h, $D = h^{-1}((0, +\infty))$.

Note that $h \in USC^f(X)$ and hence there are $\{f_{i_k} : k \in \mathbb{N}\} \subset A$ such that $f_{i_k} \to h$ ($k \to \infty$). It follows that $D = h^{-1}((0, +\infty)) = \bigcup\limits_{j\in\mathbb{N}} \bigcap\limits_{i_k > j} f_{i_k}^{-1}([\frac{1}{j}, +\infty))$ and hence $f(D)$ is an F_σ-set of Y.

(2) $\Rightarrow$ (1). Assume that $X \models U$, i.e. there is a condensation (one-to-one continuous mapping) $f : X \mapsto Y$ from the space X on a separable metric space Y, such that $f(D)$ is an F_σ-set of Y for any Z_σ-set D of X. Then $USC_p^f(X) \subset USC_p(Y) \subset B_1(Y)$. By Velichko's Theorem 2.5, there is $A = \{f_i : i \in \mathbb{N}\} \subset C_p(Y)$ such that $[A]_{seq} = B_1(Y)$. Note that $C_p(Y) \subset C_p(X) \subset USC_p^f(X) \subset B_1(Y)$. It follows that A is a countable sequentially dense subset of $USC_p^f(X)$.

(3) $\Rightarrow$ (2). Suppose that exists a countable subset S of $C(X)$ such that $[S]_{seq} = B_1(X)$. Consider a topology τ generated by the family $\alpha = \{f^{-1}(G) : G$ is an open subset of $\mathbb{R}$ and $f \in S\}$. Denote $Y = (X, \tau)$. Let φ be a identity map from X onto Y. By Theorem 3.1 in [4], φ is a bijection such that $\varphi(D)$ is a Z_σ-set of Y for any zero-set D of X. Since $S \subset C(X)$, φ is a condensation.

(2) $\Rightarrow$ (3). Let $X \models U$. By Theorem 2.5, there exists a countable dense subset L of $C_p(Y)$ such that $[L]_{seq} = B_1(Y)$. Then $S = \{f \circ \varphi : f \in L\}$ is a countable subset of $C(X)$. Let $\varphi^*(h) := h \circ \varphi$ for $h \in B_1(Y)$. Then $\varphi^* : B_1(Y) \mapsto B_1(X)$ is a first-level Baire isomorphism. It follows that $[S]_{seq} = B_1(X)$.

□

Corollary 3.3. Let X be a Tychonoff space and let $USC_p^f(X)$ be sequentially separable. Then $C_p(X)$ and $B_1(X)$ are sequentially separable.

Proof. By Theorem 2.4, $C_p(X)$ is sequentially separable. By Theorem 2.7, $B_1(X)$ is sequentially separable. □

Corollary 3.4. Let X be a perfectly normal space. Then the following statements are equivalent:

1. $USC_p(X)$ is sequentially separable;
2. $X \models U$;
3. there exists a countable subset S of $C(X)$ such that $[S]_{seq} = B_1(X)$.

A continuous image of sequentially separable space is sequentially separable. Hence *cosmic* spaces - the continuous images of separable metric spaces (space with a countable network) - are sequentially separable. So, for any separable metric space X (or more generally, cosmic X), $C_p(X)$ is cosmic, and hence sequentially separable.

Corollary 3.5. Let X be a separable metrizable space. Then:

1. $USC_p(X)$ is sequentially separable;
2. $B_1(X)$ is sequentially separable;
3. there exists a countable subset S of $C(X)$ such that $[S]_{seq} = B_1(X)$.

Recall that an analytic space is a metrizable space that is a continuous image of a Polish space.

A map $f : X \to Y$ be called Z_σ-map, if $f^{-1}(Z)$ is a Z_σ-set of X for any zero-set Z of Y.

We need the following theorem as a special case of the Theorem 1 in [1].

Theorem 3.6. *([1]) Suppose that $\varphi : L \mapsto S$ be a Z_σ-mapping from an analytic space L onto a cosmic space S. Then φ is piecewise continuous.*

Theorem 3.7. *There exists a Tychonoff space X such that $C_p(X)$ and $B_1(X)$ are sequentially separable, but $USC_p^f(X)$ is not.*

Proof. Let $X = Z^{\aleph_0}$ where $Z = \mathbb{N} \cup \{p\}$ for $p \in \mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$.

Assume that $USC_p^f(X)$ is sequentially separable. Then, by Theorem 3.2, there exists a condensation $f : X \mapsto Y$ from the space X on a separable metrizable space Y, such that $f(D)$ is an F_σ-set of Y for any Z_σ-set D of X. Since Z is a continuous image of $\mathbb{N}$, X is a continuous image of irrational numbers $\mathbb{P}$, i.e. there is a continuous mapping $\alpha : \mathbb{P} \mapsto X$ from $\mathbb{P}$ onto the space X. It follows that $f \circ \alpha : \mathbb{P} \mapsto Y$ is a continuous mapping, and hence Y is an analytic space. By Theorem 3.6, $f^{-1} : Y \mapsto X$ is a piecewise continuous function (i.e. Y admits a closed and disjoint cover $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$, such that for each $F_n \in \mathcal{F}$ the restriction $f^{-1}|F_n$ is continuous function). It follows that $f^{-1}|F_n : F_n \mapsto f^{-1}(F_n)$ is a homeomorphism, and hence $X = \bigcup\limits_{F_n \in \mathcal{F}} f^{-1}(F_n)$ where $f^{-1}(F_n)$ is a separable metrizable space for each $n \in \mathbb{N}$. Since non-empty open set of X is not metrizable, $f^{-1}(F_n)$ is a closed nowhere dense subset of X for each $n \in \mathbb{N}$. But X is a Baire space, a contradiction.

□

4. Open questions

Question 1. Suppose that $B_1(X)$ is sequentially separable. Is then $C_p(X)$ sequentially separable ?

Question 2. Suppose that $B_1(X)$ is sequentially separable. Is then exist first-level Baire isomorphism $F : X \to M$ between X and a separable metrizable space M ?

Question 3. Suppose that $f : X \mapsto Y$ is a first-level Baire isomorphism between X and a separable metrizable space Y. Is then $C_p(X)$ sequentially separable ?

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