CHAIN VARIETIES OF MONOIDS

S. V. GUSEV AND B. M. VERNIKOV

Abstract. A variety of universal algebras is called a chain variety if its subvariety lattice is a chain. Non-group chain varieties of semigroups were completely classified by Sukhanov in 1982. Here we completely determine non-group chain varieties of monoids as algebras of type $(2,0)$.

Contents

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1. Introduction and summary

There are many articles devoted to the examination of the lattice **SEM** of all semigroup varieties. An overview of this area is contained in the detailed survey [21]; see also the recent work [23] devoted to elements of **SEM** satisfying some special properties. In sharp contrast, the lattice **MON** of all monoid varieties has received much less attention over the years; when referring to monoid varieties, we consider monoids as algebras with an associative binary operation and the nullary operation that fixes the identity element. As far as we know, there are only three papers containing substantial results on this subject. We have in mind the article [7] where the lattice of commutative monoid varieties is completely described, the article [24] which contains a complete description of the lattice of band monoid varieties, and the article [19] where an example of a monoid variety without covers in the lattice **MON** is found.

Recently, the situation began to change gradually. The papers [8, 9, 12–16] are devoted principally to an examination of identities of monoids but contain also some intermediate results about lattices of varieties. Moreover, the article [9] contains some results about the lattice **MON** that are of undoubted independent interest.

Thus nowadays, interest in the lattice **MON** has grown. Nevertheless, many questions in this area still remain open. For example, it is known that the lattice **MON** is not modular (see, e.g., [12, Proposition 4.1] or Fig. 2.1b) below), but it was unknown up to the recent time whether this lattice satisfies some non-trivial identity. Only recently the first author gave a negative answer to this question [6]. In contrast, the fact that the lattice **SEM** does not satisfy any non-trivial lattice identity is known since the beginning of 1970's [3, 4].

The problem of describing monoid varieties with modular or even distributive subvariety lattice seems to be quite difficult. As a first step on this direction, it seems natural to consider the extreme strengthening of the distributive law, namely the property of being a chain. Varieties whose subvariety lattice is a chain are called *chain* varieties. Non-group chain varieties of semigroups were listed by Sukhanov in [22] (see Fig. 7.2 in Section 7 below), while locally finite chain group varieties were completely determined by Artamonov in [2]. Note that the problem of completely describing arbitrary chain varieties of groups seems to be extremely difficult. This is confirmed by the fact that there are uncountably many periodic non-locally finite varieties of groups with the 3-element subvariety lattice [11].

Some non-trivial examples of chain varieties of monoids appeared in [8, 12, 15]. We introduce here one of these examples. To do this, we need some notation. We denote by F the free semigroup over a countably infinite alphabet A. Elements of both F and A are denoted by small Latin letters. However, elements of F unlike elements of A are written in bold. As usual, elements of F and A are called *words* and *letters* respectively. The symbol F^1 stands for the semigroup F with a new identity element adjoined. We treat this identity element as the empty word and denote it by λ. The following notion was introduced by Perkins [18] and often appeared in the literature (see [8–10,12,15], for instance; in [9, Remark 2.4] there is a number of other references). Let W

be a set of possibly empty words. We denote by $\overline{W}$ the set of all subwords of words from W and by $I\big(\overline{W}\big)$ the set $F^1 \setminus \overline{W}$. It is clear that $I\big(\overline{W}\big)$ is an ideal of F^1. Then $S(W)$ denotes the Rees quotient monoid $F^1/I\big(\overline{W}\big)$. If $W = \{\mathbf{w}\}$ then we will write $S(\mathbf{w})$ rather than $S\big(\{\mathbf{w}\}\big)$. It is verified in [8, Lemmas 4.4 and 5.10] that the variety generated by the monoid $S(xzxyty)$ is a chain variety. Besides that, this variety turns out to be non-finitely based [8, Lemma 5.5].

However, chain monoid varieties were not studied systematically so far. In this paper we obtain a complete description of non-group chain varieties of monoids. In order to formulate the main result of the article, we need some new notation. Two sides of identities we connect by the symbol $\approx$, while the symbol $=$ denotes the equality relation on F^1. One can introduce notation for the following three identities:

$$\begin{aligned}
\sigma_1 &: \ xyzxty \approx yxzxty,\\
\sigma_2 &: \ xtyzxy \approx xtyzyx,\\
\gamma_1 &: \ y_1y_0x_1y_1x_0x_1 \approx y_1y_0y_1x_1x_0x_1.
\end{aligned}$$

Note that the identities σ_1 and σ_2 are dual to each other. The identity γ_1 belongs to a countably infinite series of identities γ_k that will be defined in Subsection 6.1. For an identity system Σ, we denote by var Σ the variety of monoids given by Σ. Let us fix notation for the following varieties:

$$\begin{aligned}
&\mathbf{C}_n = \mathrm{var}\{x^n \approx x^{n+1},\, xy \approx yx\} \text{ where } n \ge 2,\\
&\mathbf{D} = \mathrm{var}\{x^2 \approx x^3,\, x^2y \approx yx^2,\, \sigma_1,\, \sigma_2,\, \gamma_1\},\\
&\mathbf{K} = \mathrm{var}\{xyx \approx xyx^2,\, x^2y^2 \approx y^2x^2,\, x^2y \approx x^2yx\},\\
&\mathbf{LRB} = \mathrm{var}\{xy \approx xyx\},\\
&\mathbf{N} = \mathrm{var}\{x^2y \approx yx^2,\, x^2yz \approx xyxzx,\, \sigma_2,\, \gamma_1\},\\
&\mathbf{RRB} = \mathrm{var}\{yx \approx xyx\}.
\end{aligned}$$

To define one more variety, we need some additional notation. For an arbitrary natural number n, we denote by S_n the full symmetric group on the set $\{1, 2, \dots, n\}$. For arbitrary permutations $\pi, \tau \in S_n$, we put

$$\begin{aligned}
\mathbf{w}_n(\pi,\tau) &= \Big(\prod_{i=1}^{n} z_it_i\Big) x \Big(\prod_{i=1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Big) x \Big(\prod_{i=n+1}^{2n} t_iz_i\Big),\\
\mathbf{w}'_n(\pi,\tau) &= \Big(\prod_{i=1}^{n} z_it_i\Big) x^2 \Big(\prod_{i=1}^{n} z_{\pi(i)}z_{n+\tau(i)}\Big) \Big(\prod_{i=n+1}^{2n} t_iz_i\Big).
\end{aligned}$$

Note that the words $\mathbf{w}_n(\pi,\tau)$ and $\mathbf{w}'_n(\pi,\tau)$ with the trivial permutations π and τ appeared earlier in [8, proof of Proposition 5.5]. Put

$$\mathbf{L} = \mathrm{var}\{x^2y \approx yx^2, xyxzx \approx x^2yz, \sigma_1, \sigma_2, \mathbf{w}_n(\pi,\tau) \approx \mathbf{w}'_n(\pi,\tau) \mid n \in \mathbb{N}, \pi,\tau \in S_n\}.$$

If $\mathbf{X}$ is a monoid variety then we denote by $\overleftarrow{\mathbf{X}}$ the variety *dual to* $\mathbf{X}$, i.e. the variety consisting of monoids antiisomorphic to monoids from $\mathbf{X}$.

The main result of the paper is the following

Theorem 1.1. *A non-group monoid variety is a chain variety if and only if it is contained in one of the varieties* $\mathbf{C}_n$ *for some* $n \ge 2$, $\mathbf{D}$, $\mathbf{K}$, $\overleftarrow{\mathbf{K}}$, $\mathbf{L}$, $\mathbf{LRB}$, $\mathbf{N}$, $\overleftarrow{\mathbf{N}}$ *and* $\mathbf{RRB}$.

As we will see below, the variety $\mathbf{L}$ is generated by the monoid $S(xzxyty)$ (see Lemma 4.6). In view of the results of [8] mentioned above, the variety $\mathbf{L}$ is non-finitely based. Our Theorem implies that $\mathbf{L}$ is the unique non-finitely based non-group chain variety of monoids (see Corollary 7.1 below). For comparison, we note that all non-group chain semigroup varieties and locally finite chain group varieties are finitely based. This follows from the above-mentioned results of [2, 22]. Note also that following the above-mentioned result of [11], there exist non-finitely based non-locally finite chain varieties of groups. But explicit examples of such varieties have not yet been specified anywhere.

The complete list of all non-group chain varieties of monoids will be given in Corollary 7.1 below. The unique non-finitely based non-group chain variety of monoids mentioned above is the variety $\mathbf{L}$ (see Corollary 4.8 below).

A minimal non-chain variety is called a *just non-chain* variety. It is noted in [22, Corollary 2] that, among non-group varieties in $\mathbf{SEM}$, any chain variety is contained in some maximal chain variety and any non-chain variety contains some just non-chain subvariety. However, similar results do not hold for non-group varieties in $\mathbf{MON}$. Specifically, the varieties $\mathbf{C}_3$, $\mathbf{C}_4$, ... are not contained in any maximal chain variety (see Fig. 7.1 in Section 7), while it follows from Theorem 1.1 that there is a non-chain variety of monoids that does not contain any just non-chain subvariety (see Corollary 7.4 below).

In [22] non-group chain varieties of semigroups were described in two ways. The first one is a description in the identity language. Theorem 1.1 is an analogue of this result in the case of monoids. The second way is by presenting the full list of non-group just non-chain varieties of semigroups; this gives a characterization of chain varieties because, in view of [22, Corollary 2], a non-group variety of semigroups is a chain variety if and only if it does not contain any just non-chain subvariety. As we have mentioned in the preceding paragraph, an analogous claim is false for monoids. Therefore, the second way of describing chain varieties is not applicable in the case of monoids. Due to this reason, we do not consider just non-chain monoid varieties here.

The article consists of seven sections. Section 2 contains definitions, notation and auxiliary results. In Section 3 we introduce a series of new notions and notation and prove a number of results of technical character concerning these notions. These notions and results play a valuable role in the proof of Theorem 1.1. Section 4 is devoted to the proof of the "only if" part of Theorem 1.1, while the "if" part is verified in Sections 5 and 6. Finally, in Section 7 some corollaries of Theorem 1.1 and its proof are established.

2. Preliminaries

A word is called a *semigroup* one if it does not contain the symbol of nullary operation 1. An identity is called a *semigroup* one if both its sides are semigroup words. Note that an identity of the form $\mathbf{w} \approx 1$ is equivalent to the pair of identities $\mathbf{w}x \approx x\mathbf{w} \approx x$ where the letter x does not occur in the word $\mathbf{w}$.

Further, any monoid satisfies the identities $\mathbf{u}\cdot 1 \approx 1\cdot \mathbf{u} \approx \mathbf{u}$ for any word $\mathbf{u}$. These observations allow us to assume that all identities that appear below are semigroup ones.

The *content* of a word $\mathbf{w}$, i.e., the set of all letters occurring in $\mathbf{w}$, is denoted by $\mathrm{con}(\mathbf{w})$. We denote by $\mathbf{SL}$ the variety of all semilattice monoids. The following statement is well known in fact. But it never appeared anywhere in this form, as far as we know. For the sake of completeness, we give its proof here.

Lemma 2.1. *For a monoid variety $\mathbf{V}$, the following are equivalent:*

a) $\mathbf{V}$ *is a group variety;*
b) $\mathbf{V}$ *satisfies an identity $\mathbf{u} \approx \mathbf{v}$ with $\mathrm{con}(\mathbf{u}) \ne \mathrm{con}(\mathbf{v})$;*
c) $\mathbf{SL} \nsubseteq \mathbf{V}$.

Proof. The implication a) $\longrightarrow$ c) is obvious.

The implication c) $\longrightarrow$ b) immediately follows from the evident fact that the variety $\mathbf{SL}$ satisfies any identity $\mathbf{u} \approx \mathbf{v}$ with $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$.

b) $\longrightarrow$ a) By the hypothesis, there is a letter x that occurs in precisely one of the words $\mathbf{u}$ and $\mathbf{v}$. Let y be a letter with $y \notin \mathrm{con}(\mathbf{uv})$. Clearly, the identities $\mathbf{u}y \approx \mathbf{v}y$ and $y\mathbf{u} \approx y\mathbf{v}$ hold in $\mathbf{V}$. One can substitute 1 for all letters occurring in these identities except x and y. Then we obtain $\mathbf{V}$ satisfies $x^n y \approx y$ and $yx^n \approx y$ for some n. Hence $\mathbf{V}$ is a group variety. □

A letter is called *simple* [*multiple*] *in a word* $\mathbf{w}$ if it occurs in $\mathbf{w}$ once [at least twice]. The set of all simple [multiple] letters in a word $\mathbf{w}$ is denoted by $\mathrm{sim}(\mathbf{w})$ [respectively $\mathrm{mul}(\mathbf{w})$]. The following statement is well known and can be easily verified.

Proposition 2.2. *A non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety $\mathbf{C}_2$ if and only if the claim*

$$\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v}) \text{ and } \mathrm{mul}(\mathbf{u}) = \mathrm{mul}(\mathbf{v}) \tag{2.1}$$

is true. □

A word $\mathbf{w}$ is called an *isoterm* for a class of semigroups if no semigroup in the class satisfies any non-trivial identity of the form $\mathbf{w} \approx \mathbf{w}'$. The following statement is known in fact and plays an important role below.

Lemma 2.3. *Let $\mathbf{V}$ be a monoid variety and W a set of possibly empty words. Then $S(W)$ lies in $\mathbf{V}$ if and only if each word in W is an isoterm for $\mathbf{V}$.*

Proof. It is easy to verify that it suffices to consider the case when W consists of one word (see the paragraph after Lemma 3.3 in [8]). Then necessity is obvious, while sufficiency is proved in [10, Lemma 5.3]. □

The variety generated by a monoid M is denoted by $\mathrm{var}\, M$.

Lemma 2.4 ([1, Corollary 6.1.5])**.** $\mathbf{C}_{n+1} = \mathrm{var}\, S(x^n)$ *for any natural n.* □

Lemma 2.5. *Let $\mathbf{V}$ be a monoid variety and n a natural number. If $\mathbf{C}_{n+1} \nsubseteq \mathbf{V}$ then $\mathbf{V}$ satisfies an identity $x^n \approx x^{n+m}$ for some m.*

Proof. We can assume that $\mathbf{V}$ is not a group variety because the required conclusion is evident otherwise. Lemmas 2.3 and 2.4 apply with the conclusion that the variety $\mathbf{V}$ satisfies a non-trivial identity of the form $x^n \approx \mathbf{w}$. Then $\mathrm{con}(\mathbf{w}) = \{x\}$ by Lemma 2.1, whence $\mathbf{w} = x^k$ for some $k \neq n$. Clearly, the identity $x^n \approx x^k$ implies an identity $x^n \approx x^{n+m}$ for some m. Thus, the variety $\mathbf{V}$ satisfies the identity $x^n \approx x^{n+m}$. □

As in the case of semigroups, a variety of monoids is called *completely regular* if it consists of *completely regular monoids* (i.e., unions of groups). It is well known that a variety is completely regular if and only if it satisfies an identity $x \approx x^{m+1}$ for some m. This observation, together with Lemma 2.5 and the evident fact that the variety $\mathbf{C}_2$ is non-completely regular, implies the following

Corollary 2.6. *A monoid variety* $\mathbf{V}$ *is completely regular if and only if* $\mathbf{C}_2 \nsubseteq \mathbf{V}$. □

For any natural number k, we denote by $\mathbf{D}_k$ the subvariety of the variety $\mathbf{D}$ given within $\mathbf{D}$ by the identity $x^2y_1y_2\cdots y_k \approx xy_1xy_2x\cdots xy_kx$. The proof of Proposition 4.1 in [15] implies the following

Lemma 2.7. $\mathbf{D}_1 = \mathrm{var}\, S(xy)$ *and* $\mathbf{D}_{n+1} = \mathrm{var}\, S(xy_1xy_2x\cdots xy_nx)$ *for any natural* n. □

We denote by $\mathbf{T}$ the trivial variety of monoids. The subvariety lattice of a monoid variety $\mathbf{X}$ is denoted by $L(\mathbf{X})$. Proposition 4.1 of [15] and its proof readily imply also the following

Lemma 2.8. *The lattice* $L(\mathbf{D})$ *is the chain* $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \cdots \subset \mathbf{D}_k \subset \cdots \subset \mathbf{D}$. □

The following statement immediately follows from [24, Proposition 4.7].

Lemma 2.9. (i) *Every variety of band monoids either contains the variety* $\mathbf{LRB} \vee \mathbf{RRB}$ *or is contained in this variety.*
(ii) *The lattice* $L(\mathbf{LRB} \vee \mathbf{RRB})$ *has the form shown in Fig.* 2.1a). □

Put

$$\mathbf{E} = \mathrm{var}\{x^2 \approx x^3,\, x^2y \approx xyx,\, x^2y^2 \approx y^2x^2\}.$$

The following lemma is verified in [12, Proposition 4.1(i) and Lemma 3.3(iv)].

Lemma 2.10. (i) $\mathbf{LRB} \vee \mathbf{C}_2 = \mathrm{var}\{x^2 \approx x^3,\, x^2y \approx xyx\}$.
(ii) *The lattice* $L(\mathbf{LRB} \vee \mathbf{C}_2)$ *has the form shown in Fig.* 2.1b). □

Let $\mathbf{w}$ be a word and x a letter. We denote by $\mathrm{occ}_x(\mathbf{w})$ the number of occurrences of x in $\mathbf{w}$. If $x \in \mathrm{con}(\mathbf{w})$ and $i \le \mathrm{occ}_x(\mathbf{w})$ then $\ell_i(\mathbf{w}, x)$ denotes the length of the minimal prefix $\mathbf{p}$ of $\mathbf{w}$ with $\mathrm{occ}_x(\mathbf{p}) = i$.

Example 2.11. If $\mathbf{w} = xyx^2zy$ then, evidently, $\mathrm{occ}_x(\mathbf{w}) = 3$, $\mathrm{occ}_y(\mathbf{w}) = 2$ and $\mathrm{occ}_z(\mathbf{w}) = 1$. Further, the shortest prefixes $\mathbf{p}$ of $\mathbf{w}$ with $\mathrm{occ}_x(\mathbf{p}) = 1$, $\mathrm{occ}_x(\mathbf{p}) = 2$ and $\mathrm{occ}_x(\mathbf{p}) = 3$ are x, xyx and xyx^2 respectively, whence $\ell_1(\mathbf{w}, x) = 1$, $\ell_2(\mathbf{w}, x) = 3$ and $\ell_3(\mathbf{w}, x) = 4$. Analogously, $\ell_1(\mathbf{w}, y) = 2$, $\ell_2(\mathbf{w}, y) = 6$ and $\ell_1(\mathbf{w}, z) = 5$.

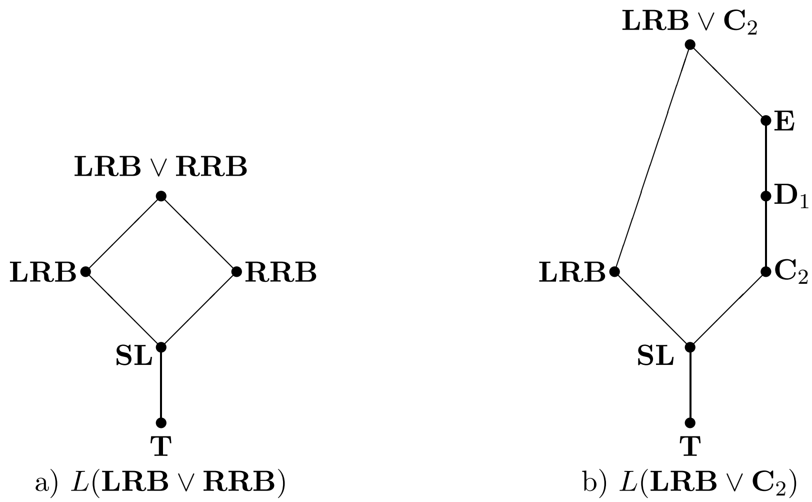

FIGURE 2.1. The lattices $L(\mathbf{LRB} \vee \mathbf{RRB})$ and $L(\mathbf{LRB} \vee \mathbf{C}_2)$

Below we often deal with inequalities like $\ell_i(\mathbf{w}, x) < \ell_j(\mathbf{w}, y)$. Clearly, this inequality means simply that ith occurrence of x in $\mathbf{w}$ precedes jth occurrence of y in $\mathbf{w}$.

If $\mathbf{w}$ is a word and X is a set of letters then $\mathbf{w}_X$ denotes the word obtained from $\mathbf{w}$ by deleting all letters from X. If $X = \{x\}$ then we write $\mathbf{w}_x$ rather than $\mathbf{w}_{\{x\}}$.

Lemma 2.12. *If a non-commutative variety of monoids* $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$ *such that the claim* (2.1) *holds then*

$$\mathbf{u}_{\mathrm{mul}(\mathbf{u})} = \mathbf{v}_{\mathrm{mul}(\mathbf{u})}. \tag{2.2}$$

Proof. According to the claim (2.1), $\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v})$ and $\mathrm{mul}(\mathbf{u}) = \mathrm{mul}(\mathbf{v})$. It is evident that the claim (2.2) holds whenever the set $\mathrm{sim}(\mathbf{u})$ contains < 2 letters. Suppose now that $\mathrm{sim}(\mathbf{u})$ contains at least two different letters and the claim (2.2) is false. Then there are letters $x, y \in \mathrm{sim}(\mathbf{u})$ such that $\ell_1(\mathbf{u}, x) < \ell_1(\mathbf{u}, y)$ and $\ell_1(\mathbf{v}, x) > \ell_1(\mathbf{v}, y)$. One can substitute 1 for all letters occurring in the identity $\mathbf{u} \approx \mathbf{v}$ except x and y. Then we obtain $xy \approx yx$ contradicting the fact that $\mathbf{V}$ is non-commutative. □

Proposition 2.13. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{D}_1$ *if and only if the claims* (2.1) *and* (2.2) *are true.*

Proof. Necessity. The inclusion $\mathbf{C}_2 \subseteq \mathbf{D}_1$ and Proposition 2.2 imply that the identity $\mathbf{u} \approx \mathbf{v}$ satisfies the claim (2.1). Since the variety $\mathbf{D}_1$ is non-commutative, Lemma 2.12 implies that the claim (2.2) holds too.

Sufficiency. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ satisfies the claims (2.1) and (2.2). Let $\mathrm{sim}(\mathbf{u}) = \{y_1, y_2, \dots, y_m\}$. We may assume without loss of generality that

$$\mathbf{u} = \mathbf{u}_0 y_1 \mathbf{u}_1 y_2 \mathbf{u}_2 \cdots y_m \mathbf{u}_m$$

where $\mathrm{con}(\mathbf{u}_0 \mathbf{u}_1 \cdots \mathbf{u}_m) = \mathrm{mul}(\mathbf{u})$. It follows from the claim (2.1) that $\mathrm{sim}(\mathbf{v}) = \{y_1, y_2, \ \dots, y_m\}$. Moreover, $\mathbf{v} = \mathbf{v}_0 y_1 \mathbf{v}_1 y_2 \mathbf{v}_2 \cdots y_m \mathbf{v}_m$ by the claim (2.2).

We can apply the claim (2.1) again and conclude that $\mathrm{con}(\mathbf{u}_0\mathbf{u}_1\cdots\mathbf{u}_m) = \mathrm{con}(\mathbf{v}_0\mathbf{v}_1\cdots\mathbf{v}_m)$. Now it is easy to see that the identity system $\{x^2 \approx x^3, x^2y \approx xyx \approx yx^2\}$ implies the identities

$$\mathbf{u} = \mathbf{u}_0 y_1 \mathbf{u}_1 y_2 \mathbf{u}_2 \cdots y_m \mathbf{u}_m \approx \mathbf{v}_0 y_1 \mathbf{v}_1 y_2 \mathbf{v}_2 \cdots y_m \mathbf{v}_m = \mathbf{v},$$

whence $\mathbf{D}_1$ satisfies $\mathbf{u} \approx \mathbf{v}$. □

Lemma 2.14. *If a variety of monoids* $\mathbf{V}$ *is non-completely regular and non-commutative then* $\mathbf{D}_1 \subseteq \mathbf{V}$*.*

Proof. Suppose that $\mathbf{D}_1 \nsubseteq \mathbf{V}$. Then there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{V}$ but is false in $\mathbf{D}_1$. Corollary 2.6 implies that $\mathbf{C}_2 \subseteq \mathbf{V}$. Then $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{C}_2$, whence the claim (2.1) holds by Proposition 2.2. Now Lemma 2.12 and the claim that the variety $\mathbf{V}$ is non-commutative imply that the equality (2.2) is true. Now Proposition 2.13 applies and we conclude that the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{D}_1$, a contradiction. □

Lemma 2.15. *If* $\mathbf{X}$ *is a non-completely regular variety of monoids and* $\mathbf{D}_{n+1} \nsubseteq \mathbf{X}$ *for some* n *then* $\mathbf{X}$ *satisfies an identity of the form*

$$xy_1xy_2x\cdots xy_nx \approx x^{k_1}y_1x^{k_2}y_2x^{k_2}\cdots x^{k_n}y_nx^{k_{n+1}} \tag{2.3}$$

where $k_i > 1$ *for some* i*.*

Proof. If the variety $\mathbf{X}$ is commutative then it satisfies the identity

$$xy_1xy_2x\cdots xy_nx \approx x^{n+1}y_1y_2\cdots y_n,$$

and we are done. Suppose now that $\mathbf{X}$ is non-commutative. Then $\mathbf{X}$ satisfies a non-trivial identity of the form $xy_1xy_2x\cdots xy_nx \approx \mathbf{w}$ by Lemmas 2.3 and 2.7. Now Lemma 2.14 applies with the conclusion that $\mathbf{D}_1 \subseteq \mathbf{X}$. According to Proposition 2.13,

$$\mathbf{w} = x^{k_1}y_1x^{k_2}y_2x^{k_2}\cdots y_nx^{k_{n+1}}.$$

If $k_i > 1$ for some i then we are done. Suppose that $k_i \le 1$ for all i. There is $1 \le i \le n+1$ with $k_i = 0$ because the identity $xy_1xy_2x\cdots xy_nx \approx \mathbf{w}$ is trivial otherwise. Substitute xy_i for y_i in this identity for all i such that $k_i = 0$. If $k_{n+1} = 0$ then we multiply the resulted identity by x on the right. Thus, we obtain an identity of the form (2.3) where $k_i > 1$ for some i. □

3. k-decomposition of a word and related notions

Here we introduce a series of notions and examine their properties. These notions and results play a key role in the most complicated part of the proof of Theorem 1.1 in Section 6.

For a word $\mathbf{u}$ and letters $x_1, x_2, \dots, x_k \in \mathrm{con}(\mathbf{u})$, let $\mathbf{u}(x_1, x_2, \dots, x_k)$ denote the word obtained from $\mathbf{u}$ by retaining the letters $x_1, x_2, \dots, x_k$. Equivalently,

$$\mathbf{u}(x_1, x_2, \dots, x_k) = \mathbf{u}_{\mathrm{con}(\mathbf{u})\setminus\{x_1, x_2, \dots, x_k\}}.$$

Let $\mathbf{w}$ be a word and $\mathrm{sim}(\mathbf{w}) = \{t_1, t_2, \dots, t_m\}$. We can assume without loss of generality that $\mathbf{w}(t_1, t_2, \dots, t_m) = t_1t_2\cdots t_m$. Then

$$\mathbf{w} = t_0\mathbf{w}_0t_1\mathbf{w}_1\cdots t_m\mathbf{w}_m \tag{3.1}$$

where $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ are possibly empty words and $t_0 = \lambda$. The words $\mathbf{w}_0$, $\mathbf{w}_1$, ..., $\mathbf{w}_m$ are called 0-*blocks* of a word $\mathbf{w}$, while $t_0, t_1, \dots, t_m$ are said to be 0-*dividers* of $\mathbf{w}$. The representation of the word $\mathbf{w}$ as a product of alternating 0-dividers and 0-blocks, starting with the 0-divider t_0 and ending with the 0-block $\mathbf{w}_m$ is called a 0-*decomposition* of the word $\mathbf{w}$.

Let now k be a natural number. We define a k-decomposition of $\mathbf{w}$ by induction on k. Let (3.1) be a $(k-1)$-decomposition of the word $\mathbf{w}$ with $(k-1)$-blocks $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ and $(k-1)$-dividers $t_0, t_1, \dots, t_m$. For any $i = 0, 1, \dots, m$, let $s_{i1}, s_{i2}, \dots, s_{ir_i}$ be all simple in the word $\mathbf{w}_i$ letters that does not occur in the word $\mathbf{w}$ to the left of $\mathbf{w}_i$. We can assume that $\mathbf{w}_i(s_{i1}, s_{i2}, \dots, s_{ir_i}) = s_{i1}s_{i2}\cdots s_{ir_i}$. Then

$$\mathbf{w}_i = \mathbf{v}_{i0}s_{i1}\mathbf{v}_{i1}s_{i2}\mathbf{v}_{i2}\cdots s_{ir_i}\mathbf{v}_{ir_i} \tag{3.2}$$

for possibly empty words $\mathbf{v}_{i0}, \mathbf{v}_{i1}, \dots, \mathbf{v}_{ir_i}$. Put $s_{i0} = t_i$. The words $\mathbf{v}_{i0}, \mathbf{v}_{i1}, \dots, \mathbf{v}_{ir_i}$ are called k-*blocks* of the word $\mathbf{w}$, while the letters $s_{i0}, s_{i1}, \dots, s_{ir_i}$ are said to be k-*dividers* of $\mathbf{w}$.

Remark 3.1. Note that only first occurrence of a letter in a given word might be a k-divider of this word for some k. In view of this observation, below we use expressions like"a letter x is (or is not) a k-divider of a word $\mathbf{w}$" meaning that first occurrence of x in $\mathbf{w}$ has the specified property.

For any $i = 0, 1, \dots, m$, we represent $(k-1)$-block $\mathbf{w}_i$ in the form (3.2). As a result, we obtain the representation of the word $\mathbf{w}$ as a product of alternating k-dividers and k-blocks, starting with the k-divider $s_{00} = t_0$ and ending with the k-block $\mathbf{v}_{mr_m}$. This representation is called a *k-decomposition of the word* $\mathbf{w}$.

Remark 3.2. Since the length of the word $\mathbf{w}$ is finite, there is a number k such that the k-decomposition of $\mathbf{w}$ coincides with its n-decompositions for all $n > k$.

For reader convenience, we illustrate the notions of k-blocks, k-dividers and k-decomposition of a word by the following

Example 3.3. Let $\mathbf{w} = xyxzytszxs$. A unique simple letter in $\mathbf{w}$ is t. Therefore, the 0-decomposition of $\mathbf{w}$ has the form

$$\lambda \cdot \underline{xyxzy} \cdot t \cdot \underline{szxs} \tag{3.3}$$

(here and below throughout this example we underline blocks to distinguish them from dividers). A unique simple letter of the most left 0-block $xyxzy$ is z; the 0-block $szxs$ contains two simple letters, namely z and x but both these letters occur in $\mathbf{w}$ to the left of this block. Therefore, the 1-decomposition of $\mathbf{w}$ has the form

$$\lambda \cdot \underline{xyx} \cdot z \cdot \underline{y} \cdot t \cdot \underline{szxs}.$$

Analogous arguments show that the 2-decomposition of $\mathbf{w}$ has the form

$$\lambda \cdot \underline{x} \cdot y \cdot \underline{x} \cdot z \cdot \underline{y} \cdot t \cdot \underline{szxs}$$

and if $k \ge 3$ then the k-decomposition of $\mathbf{w}$ has the form

$$\lambda \cdot \underline{\lambda} \cdot x \cdot \underline{\lambda} \cdot y \cdot \underline{x} \cdot z \cdot \underline{y} \cdot t \cdot \underline{szxs}.$$

For a given word $\mathbf{w}$, a letter $x \in \operatorname{con}(\mathbf{w})$, a natural number $i \le \operatorname{occ}_x(\mathbf{w})$ and an integer $k \ge 0$, we denote by $h_i^k(\mathbf{w}, x)$ the right-most k-divider of $\mathbf{w}$ that precedes ith occurrence of x in $\mathbf{w}$. The (possibly empty) letter $h_i^k(\mathbf{w}, x)$ is called an (i, k)-*restrictor* of the letter x in the word $\mathbf{w}$. This notion is illustrated by the following

Example 3.4. Let $\mathbf{w}$ be the same word as in Example 3.3. The 0-decomposition of $\mathbf{w}$ has the form (3.3). We see that the right-most 0-divider of $\mathbf{w}$ that precedes the first two occurrences of x, the two occurrences of y, and the first occurrences of z and t is λ, while the right-most 0-divider of $\mathbf{w}$ that precedes third occurrence of x, second occurrence of z and both occurrences of s is t. This means that $h_1^0(\mathbf{w}, x) = h_2^0(\mathbf{w}, x) = \lambda$, $h_3^0(\mathbf{w}, x) = t$, $h_1^0(\mathbf{w}, y) = h_2^0(\mathbf{w}, y) = \lambda$, $h_1^0(\mathbf{w}, z) = \lambda$, $h_2^0(\mathbf{w}, z) = t$, $h_1^0(\mathbf{w}, s) = h_2^0(\mathbf{w}, s) = t$ and $h_1^0(\mathbf{w}, t) = \lambda$. Analogously, based on Example 3.3, it is easy to find all other restrictors of letters in the word $\mathbf{w}$. Results are presented in Table 3.1.

TABLE 3.1. Restrictors of letters in the word $xyxzytszxs$

a	k	i	$h_i^k(\mathbf{w}, a)$	a	k	i	$h_i^k(\mathbf{w}, a)$
x	0	1	λ	z	0	1	λ
		2	λ			2	t
		3	t		1	1	λ
	1	1	λ			2	t
		2	λ		2	1	y
		3	t			2	t
	2	1	λ		≥ 3	1	y
		2	y			2	t
		3	t	s	0	1	t
	≥ 3	1	λ			2	t
		2	y		1	1	t
		3	t			2	t
y	0	1	λ		2	1	t
		2	λ			2	t
	1	1	λ		≥ 3	1	t
		2	z			2	t
	2	1	λ	t	0	1	λ
		2	z		1	1	z
	≥ 3	1	x		2	1	z
		2	z		≥ 3	1	z

Lemma 3.5. *Let* $\mathbf{w}$ *be a word,* t *be a letter and* k, r *be numbers with* $r < k$.

(i) *If* t *is an* r*-divider of* $\mathbf{w}$ *then* t *is a* k*-divider of* $\mathbf{w}$ *too.*
(ii) *If* $h_1^k(\mathbf{w}, x) = h_2^k(\mathbf{w}, x)$ *then* $h_1^r(\mathbf{w}, x) = h_2^r(\mathbf{w}, x)$ *as well.*
(iii) *If* $t_0\mathbf{w}_0t_1\mathbf{w}_1 \cdots t_m\mathbf{w}_m$ *is the* k*-decomposition of* $\mathbf{w}$ *and* $m > 0$ *then* $t_m \in \operatorname{sim}(\mathbf{w})$.

Proof. The claims (i) and (ii) are obvious. One can verify the claim (iii). Suppose that $t_m \in \mathrm{mul}(\mathbf{w})$. Then t_m is not a 0-divider of $\mathbf{w}$. Let p be the least natural number such that t_m is a p-divider but not a $(p-1)$-divider of $\mathbf{w}$. Evidently, $p \le k$.

Suppose that $h_1^{p-1}(\mathbf{w}, t_m) = h_2^{p-1}(\mathbf{w}, t_m)$. This means that there are no $(p-1)$-dividers in $\mathbf{w}$ between the first and the second occurrences of t_m in $\mathbf{w}$. In other words, the first and the second occurrences of t_m in $\mathbf{w}$ lie in the same $(p-1)$-block of $\mathbf{w}$. Therefore, t_m is not simple in this $(p-1)$-block. In particular, t_m is not a p-divider of $\mathbf{w}$, contradicting the choice of t_m. Thus, $h_1^{p-1}(\mathbf{w}, t_m) \ne h_2^{p-1}(\mathbf{w}, t_m)$. Note that the arguments of this paragraph is very typical. Below we use arguments like these many times, without repeating them explicitly.

Note that $t_m \ne h_2^{p-1}(\mathbf{w}, t_m)$ because t_m is not a $(p-1)$-divider of $\mathbf{w}$. Put $t_{m+1} = h_2^{p-1}(\mathbf{w}, t_m)$. Since $p-1 < k$, the claim (i) implies that t_{m+1} is a k-divider of $\mathbf{w}$. The last k-divider of $\mathbf{w}$ is t_m. Therefore, first occurrence of t_{m+1} in $\mathbf{w}$ precedes first occurrence of t_m in $\mathbf{w}$. Therefore, $h_1^{p-1}(\mathbf{w}, t_m) = t_{m+1} = h_2^{p-1}(\mathbf{w}, t_m)$, a contradiction. □

For a given word $\mathbf{w}$ and a letter $x \in \mathrm{con}(\mathbf{w})$, we define some number that is called a *depth* of x in $\mathbf{w}$ and is denoted by $D(\mathbf{w}, x)$. If $x \in \mathrm{sim}(\mathbf{w})$ then we put $D(\mathbf{w}, x) = 0$. Suppose now that $x \in \mathrm{mul}(\mathbf{w})$. If there is a natural k such that the first and the second occurrences of x in $\mathbf{w}$ lie in different $(k-1)$-blocks of $\mathbf{w}$ then the depth of x in $\mathbf{w}$ equals the minimal number k with this property. Finally, if, for any natural k, the first and the second occurrences of x in $\mathbf{w}$ lie in the same k-block of $\mathbf{w}$ then we put $D(\mathbf{w}, x) = \infty$. In other words, $D(\mathbf{w}, x) = k$ if and only if $h_1^{k-1}(\mathbf{w}, x) \ne h_2^{k-1}(\mathbf{w}, x)$ and k is the least number with this property, while $D(\mathbf{w}, x) = \infty$ if and only if $h_1^{k-1}(\mathbf{w}, x) = h_2^{k-1}(\mathbf{w}, x)$ for any k. The definition of the depth of a letter in a word is illustrated by the following

Example 3.6. As in Examples 3.3 and 3.4, put $\mathbf{w} = xyxzytszxs$. Here we systematically use information about restrictors of letters in the word $\mathbf{w}$ indicated in Table 3.1. In particular, in view of this table, $h_1^k(\mathbf{w}, x) = \lambda$ for all k, while $h_2^0(\mathbf{w}, x) = h_2^1(\mathbf{w}, x) = \lambda$ and $h_2^2(\mathbf{w}, x) = y$. Therefore, $D(\mathbf{w}, x) = 3$. Further, $h_1^0(\mathbf{w}, y) = h_2^0(\mathbf{w}, y) = \lambda$, $h_1^1(\mathbf{w}, y) = \lambda$ and $h_2^1(\mathbf{w}, y) = z$. Hence $D(\mathbf{w}, y) = 2$. The equalities $h_1^0(\mathbf{w}, z) = \lambda$ and $h_2^0(\mathbf{w}, z) = t$ imply that $D(\mathbf{w}, z) = 1$. Further, $h_1^k(\mathbf{w}, s) = h_2^k(\mathbf{w}, s) = t$ for each $k \ge 0$, whence $D(\mathbf{w}, s) = \infty$. Finally, $D(\mathbf{w}, t) = 0$ because $t \in \mathrm{sim}(\mathbf{w})$.

The following criterion for a letter of a word to be a k-divider is often used in the proof of Theorem 1.1.

Lemma 3.7. *A letter t is a k-divider of a word $\mathbf{w}$ if and only if $D(\mathbf{w}, t) \le k$.*

Proof. This statement is evident whenever $k = 0$ because both the property of t to be a 0-divider of $\mathbf{w}$ and the equality $D(\mathbf{w}, t) = 0$ are equivalent to the claim that t is simple in $\mathbf{w}$. Further, if $k > 0$ then a property of a letter t to be a k-divider of $\mathbf{w}$ is equivalent to the claim that the first and the second occurrences of t lie in different $(k-1)$-blocks of $\mathbf{w}$. In turn, the last claim is equivalent to the non-equality $h_1^{k-1}(\mathbf{w},t) \ne h_2^{k-1}(\mathbf{w},t)$, i.e., to the required statement that $D(\mathbf{w},t) \le k$. □

The words $\mathbf{u}$ and $\mathbf{v}$ are said to be *k-equivalent* if these words have the same set of k-dividers and these k-dividers appear in $\mathbf{u}$ and in $\mathbf{v}$ in the same order.

Lemma 3.8. *Let k be a non-negative integer. Words $\mathbf{u}$ and $\mathbf{v}$ are k-equivalent if and only if the claim* (2.1) *is true and, for any $x \in \mathrm{con}(\mathbf{uv})$, $h_1^k(\mathbf{u},x) = h_1^k(\mathbf{v},x)$ whenever either $D(\mathbf{u},x) \le k$ or $D(\mathbf{v},x) \le k$.*

Proof. Sufficiency. Suppose that

$$t_0\mathbf{u}_0t_1\mathbf{u}_1\cdots t_m\mathbf{u}_m \tag{3.4}$$

and $s_0\mathbf{v}_0s_1\mathbf{v}_1\cdots s_r\mathbf{v}_r$ are k-decompositions of the words $\mathbf{u}$ and $\mathbf{v}$, respectively. Evidently, $t_0 = s_0 = \lambda$. If $m = r = 0$ then the required fact is evident. Let now $m > 0$. In view of Lemma 3.7, $D(\mathbf{u},t_i) \le k$ for any $1 \le i \le m$. By the hypothesis, this implies that $t_{i-1} = h_1^k(\mathbf{u},t_i) = h_1^k(\mathbf{v},t_i)$ for any $1 \le i \le m$, whence t_{i-1} is a k-divider of the word $\mathbf{v}$. According to Lemma 3.5(iii), $t_m \in \mathrm{sim}(\mathbf{u})$. Then the claim (2.1) implies that $t_m \in \mathrm{sim}(\mathbf{v})$, whence t_m is a 0-divider of $\mathbf{v}$. Now Lemma 3.5(i) applies with the conclusion that t_m is a k-divider of $\mathbf{v}$. So, the letters $t_1, t_2, \dots, t_m$ are k-dividers of the word $\mathbf{v}$, whence $m \le r$. By symmetry, $r \le m$. We prove that $m = r$. Further, t_1 coincides with s_p for some p. If $p \ne 1$ then $h_1^k(\mathbf{v},t_1) \ne t_0$. This contradicts the fact that $h_1^k(\mathbf{v},t_1) = h_1^k(\mathbf{u},t_1) = t_0$. So, $p = 1$ and therefore, $t_1 = s_1$. By induction, we can verify that $t_j = s_j$ for any $j \le m$.

Necessity. Suppose that (3.4) is the k-decomposition of the word $\mathbf{u}$. Then the k-decomposition of $\mathbf{v}$ has the form

$$t_0\mathbf{v}_0t_1\mathbf{v}_1\cdots t_m\mathbf{v}_m. \tag{3.5}$$

Let $x \in \mathrm{con}(\mathbf{u})$ and $D(\mathbf{u},x) \le k$. Lemma 3.7 implies that $x = t_i$ for some $1 \le i \le m$. Therefore, $h_1^k(\mathbf{v},x) = h_1^k(\mathbf{u},x) = t_{i-1}$. Analogously, we verify that if $x \in \mathrm{con}(\mathbf{v})$ and $D(\mathbf{v},x) \le k$ then $h_1^k(\mathbf{v},x) = h_1^k(\mathbf{u},x)$. □

Lemma 3.9. *Let $\mathbf{w}$ be a word, x be a letter multiple in $\mathbf{w}$ with $D(\mathbf{w},x) = k$ and t be a $(k-1)$-divider of $\mathbf{w}$.*

(i) *If $t = h_2^{k-1}(\mathbf{w},x)$ then $\ell_1(\mathbf{w},x) < \ell_1(\mathbf{w},t)$.*
(ii) *If $\ell_1(\mathbf{w},x) < \ell_1(\mathbf{w},t) < \ell_2(\mathbf{w},x)$ then $D(\mathbf{w},t) = k-1$; if besides that $k > 1$ then $\ell_2(\mathbf{w},x) < \ell_2(\mathbf{w},t)$.*

Proof. (i) Suppose that $\ell_1(\mathbf{w},t) < \ell_1(\mathbf{w},x)$. Then the equality $t = h_2^{k-1}(\mathbf{w},x)$ implies that $t = h_1^{k-1}(\mathbf{w},x)$. Thus, $h_1^{k-1}(\mathbf{w},x) = h_2^{k-1}(\mathbf{w},x)$. This contradicts the assumption that $D(\mathbf{w},x) = k$. So, $\ell_1(\mathbf{w},x) \le \ell_1(\mathbf{w},t)$. Since t is a $(k-1)$-divider, Lemma 3.7 implies that $D(\mathbf{w},t) \le k-1$. In particular, $D(\mathbf{w},t) \ne D(\mathbf{w},x)$, whence $t \ne x$. Therefore, $\ell_1(\mathbf{w},x) < \ell_1(\mathbf{w},t)$.

(ii) Suppose now that $\ell_1(\mathbf{w},x) < \ell_1(\mathbf{w},t) < \ell_2(\mathbf{w},x)$. Put $r = D(\mathbf{w},t)$. By Lemma 3.7, $r \le k-1$. If $D(\mathbf{w},t) = r < k-1$ then t is an r-divider by Lemma 3.7. Therefore, $t = h_2^r(\mathbf{w},x)$. Further, $t \ne h_1^r(\mathbf{w},x)$ because $\ell_1(\mathbf{w},x) < \ell_1(\mathbf{w},t)$. Thus, $h_1^r(\mathbf{w},x) \ne h_2^r(\mathbf{w},x)$. This means that $D(\mathbf{w},x) \le r+1 < k$, a contradiction. So, $D(\mathbf{w},t) = k-1$.

Let now $k > 1$. Then $t \in \mathrm{mul}(\mathbf{w})$. Suppose that $\ell_2(\mathbf{w},t) < \ell_2(\mathbf{w},x)$. Put $s = h_2^{k-2}(\mathbf{w},t)$. In view of the claim (i), $\ell_1(\mathbf{w},t) < \ell_1(\mathbf{w},s)$. Arguments similar to those from the previous paragraph imply that $D(\mathbf{w},s) = k-2$. According to Lemma 3.7, s is a $(k-2)$-divider of $\mathbf{w}$. The choice of s guarantees that first occurrence of s in $\mathbf{w}$ precedes second occurrence of t. On the other hand, second occurrence of t precedes second occurrence of x. Thus, first occurrence of s precedes second occurrence of x. At the same time, first occurrence of x precedes first occurrence of s because $\ell_1(\mathbf{w},x) < \ell_1(\mathbf{w},t) < \ell_1(\mathbf{w},s)$. Therefore, first and second occurrences of x in $\mathbf{w}$ lie in different $(k-2)$-blocks. Hence, $D(\mathbf{w},x) \le k-1$, a contradiction. □

Lemma 3.10. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be words and* ℓ *be a natural number. Suppose that the claims* (2.1) *and*

$$h_i^{\ell-1}(\mathbf{u},x) = h_i^{\ell-1}(\mathbf{v},x) \text{ for } i = 1,2 \text{ and all } x \in \mathrm{con}(\mathbf{u}) \tag{3.6}$$

are true. Then the words $\mathbf{u}$ *and* $\mathbf{v}$ *have the same set of* ℓ*-dividers.*

Proof. Let t be an arbitrary ℓ-divider of $\mathbf{u}$. If $t \in \mathrm{sim}(\mathbf{u})$ then $t \in \mathrm{sim}(\mathbf{v})$ by the claim (2.1). Therefore, t is a 0-divider of $\mathbf{v}$. According to Lemma 3.5(i), t is an ℓ-divider of $\mathbf{v}$. Suppose now that $t \in \mathrm{mul}(\mathbf{u})$. The claim (2.1) implies that $t \in \mathrm{mul}(\mathbf{v})$. Since t is an ℓ-divider of $\mathbf{u}$, $h_1^{\ell-1}(\mathbf{u},t) \ne h_2^{\ell-1}(\mathbf{u},t)$. Then $h_1^{\ell-1}(\mathbf{v},t) \ne h_2^{\ell-1}(\mathbf{v},t)$ by the claim (3.6). This implies that t is an ℓ-divider of $\mathbf{v}$. Similarly we prove that if s is an ℓ-divider of $\mathbf{v}$ then s is an ℓ-divider of $\mathbf{u}$. □

Lemma 3.11. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be words and* k *be a natural number. Suppose that the claims* (2.1) *and* (3.6) *with* $\ell = k$ *are true. Then the claim* (3.6) *with* $\ell = s$ *is true for any* $1 \le s \le k$.

Proof. If $k = 1$ then the assertion is valid by the hypothesis. Suppose now that $k > 1$. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$. In view of Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). Let $s < k$ be the least number such that (3.6) with $\ell = s$ is false. Then there exists a letter x such that $h_i^{s-1}(\mathbf{u},x) \ne h_i^{s-1}(\mathbf{v},x)$ for some $i \in \{1,2\}$. By the definition of $(i,s-1)$-restrictors, $h_i^{s-1}(\mathbf{u},x)$ and $h_i^{s-1}(\mathbf{v},x)$ are some $(s-1)$-dividers of $\mathbf{u}$ and $\mathbf{v}$ respectively. Lemma 3.5(i) implies that $(s-1)$-dividers of $\mathbf{u}$ and $\mathbf{v}$ are $(k-1)$-dividers of these words. Therefore, $h_i^{s-1}(\mathbf{u},x) = t_p$ and $h_i^{s-1}(\mathbf{v},x) = t_q$ for some $p \ne q$. We may assume without loss of generality that $p < q$. By the hypothesis, $h_i^{k-1}(\mathbf{u},x) = h_i^{k-1}(\mathbf{v},x)$, whence this $(i,k-1)$-restrictor of x coincide with t_n for some n. Clearly, $n \ge q$ because $s < k$. Since t_n precedes ith occurrence of x in $\mathbf{u}$, we have $\ell_1(\mathbf{u},t_q) < \ell_i(\mathbf{u},x)$. Since t_p is an $(i,s-1)$-restrictor of x in $\mathbf{u}$, there are no $(s-1)$-dividers of $\mathbf{u}$ between first occurrence of t_p and ith occurrence of x in $\mathbf{u}$. In particular, t_q is not an $(s-1)$-divider of $\mathbf{u}$. Further, Lemma 3.7 implies that $D(\mathbf{u},t_q) > s-1$. In particular, $D(\mathbf{u},t_q) > 0$, whence $t_q \in \mathrm{mul}(\mathbf{u})$. If $s = 1$ then t_q is a 0-divider of $\mathbf{v}$, whence t_q is simple in $\mathbf{v}$. This contradicts the claim (2.1). Thus, $s > 1$. This means that $h_1^{s-2}(\mathbf{u},t_q) = h_2^{s-2}(\mathbf{u},t_q)$. Since the claim (3.6) with $\ell = s-1$ is true, we obtain $h_1^{s-2}(\mathbf{v},t_q) = h_2^{s-2}(\mathbf{v},t_q)$. According to Lemma 3.5(ii), $h_1^{r-2}(\mathbf{v},t_q) = h_2^{r-2}(\mathbf{v},t_q)$

for all $r \leq s$. Then $D(\mathbf{v}, t_q) > s-1$. Lemma 3.7 implies that t_q is not an $(s-1)$-divider of $\mathbf{v}$, a contradiction with the equality $t_q = h_i^{s-1}(\mathbf{v}, x)$. □

Lemma 3.12. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be words and* k *be a natural number. Suppose that the claims* (2.1) *and* (3.6) *with* $\ell = k$ *are true. Then, for any letter* $x \in \mathrm{con}(\mathbf{u})$, $D(\mathbf{u}, x) = k$ *if and only if* $D(\mathbf{v}, x) = k$.

Proof. In view of Lemma 3.11, the claim (3.6) with $\ell = s$ is true for any $1 \leq s \leq k$. Suppose that $D(\mathbf{u}, x) = k$. This implies that

$$h_1^{s-1}(\mathbf{v}, x) = h_1^{s-1}(\mathbf{u}, x) = h_2^{s-1}(\mathbf{u}, x) = h_2^{s-1}(\mathbf{v}, x)$$

whenever $1 \leq s < k$ but

$$h_1^{k-1}(\mathbf{v}, x) = h_1^{k-1}(\mathbf{u}, x) \neq h_2^{k-1}(\mathbf{u}, x) = h_2^{k-1}(\mathbf{v}, x).$$

This implies that $D(\mathbf{v}, x) = k$. By symmetry, if $D(\mathbf{v}, x) = k$ then $D(\mathbf{u}, x) = k$. □

Lemma 3.13. *Let* $\mathbf{w}$ *be a word,* $r > 1$ *be a number and* y *be a letter such that* $D(\mathbf{w}, y) = r-2$. *Then if* $\ell_1(\mathbf{w}, z) < \ell_1(\mathbf{w}, y)$ *for some letter* z *with* $D(\mathbf{w}, z) \geq r$ *then* $\ell_2(\mathbf{w}, z) < \ell_1(\mathbf{w}, y)$.

Proof. Let z be a letter with $\ell_1(\mathbf{w}, z) < \ell_1(\mathbf{w}, y)$ and $D(\mathbf{u}, z) \geq r$. Lemma 3.7 implies that y is an $(r-2)$-divider of $\mathbf{w}$. Then if $\ell_1(\mathbf{u}, y) < \ell_2(\mathbf{u}, z)$ then the $(r-2)$-divider y is located between the first and the second occurrences of z in $\mathbf{u}$. This contradicts the equality $h_1^{r-2}(\mathbf{u}, z) = h_2^{r-2}(\mathbf{u}, z)$. The case $\ell_1(\mathbf{u}, y) = \ell_2(\mathbf{u}, z)$ also is impossible. Therefore, $\ell_2(\mathbf{w}, z) < \ell_1(\mathbf{w}, y)$. □

Below, in order to facilitate understanding of our considerations, we will sometimes write the number in brackets over a letter to indicate the number of the occurrences of this letter in the given word; for instance, we may write

$$\mathbf{w} = \overset{(1)}{x_1}\,\overset{(1)}{x_2}\,\overset{(2)}{x_1}\,\overset{(1)}{x_3}\,\overset{(2)}{x_2}\,\overset{(3)}{x_1}\,.$$

Lemma 3.14. *Let* $\mathbf{u} \approx \mathbf{v}$ *be an identity and* s *be a natural number. Suppose that the claims* (2.1) *and* (3.6) *with* $\ell = s$ *are true and there is a letter* x_s *such that* $D(\mathbf{u}, x_s) = s$. *Then there exist letters* $x_0, x_1, \ldots, x_{s-1}$ *such that* $D(\mathbf{u}, x_r) = D(\mathbf{v}, x_r) = r$ *for any* $0 \leq r < s$ *and the identity* $\mathbf{u} \approx \mathbf{v}$ *has the form*

$$\begin{aligned}
&\mathbf{u}_{2s+1}\,\overset{(1)}{x_s}\,\mathbf{u}_{2s}\,\overset{(1)}{x_{s-1}}\,\mathbf{u}_{2s-1}\,\overset{(2)}{x_s}\,\mathbf{u}_{2s-2}\,\overset{(1)}{x_{s-2}}\,\mathbf{u}_{2s-3}\,\overset{(2)}{x_{s-1}}\,\mathbf{u}_{2s-4}\,\overset{(1)}{x_{s-3}}\\
&\cdot\mathbf{u}_{2s-5}\,\overset{(2)}{x_{s-2}}\cdots\mathbf{u}_4\,\overset{(1)}{x_1}\,\mathbf{u}_3\,\overset{(2)}{x_2}\,\mathbf{u}_2\,\overset{(1)}{x_0}\,\mathbf{u}_1\,\overset{(2)}{x_1}\,\mathbf{u}_0\\
\approx\ &\mathbf{v}_{2s+1}\,\overset{(1)}{x_s}\,\mathbf{v}_{2s}\,\overset{(1)}{x_{s-1}}\,\mathbf{v}_{2s-1}\,\overset{(2)}{x_s}\,\mathbf{v}_{2s-2}\,\overset{(1)}{x_{s-2}}\,\mathbf{v}_{2s-3}\,\overset{(2)}{x_{s-1}}\,\mathbf{v}_{2s-4}\,\overset{(1)}{x_{s-3}}\\
&\cdot\mathbf{v}_{2s-5}\,\overset{(2)}{x_{s-2}}\cdots\mathbf{v}_4\,\overset{(1)}{x_1}\,\mathbf{v}_3\,\overset{(2)}{x_2}\,\mathbf{v}_2\,\overset{(1)}{x_0}\,\mathbf{v}_1\,\overset{(2)}{x_1}\,\mathbf{v}_0
\end{aligned} \tag{3.7}$$

for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{2s+1}$ *and* $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{2s+1}$.

Proof. In view of Lemma 3.11, the claim (3.6) with $\ell = r$ is true for any $1 \leq r \leq s$. We use this fact below without references.

Put $x_{s-1} = h_2^{s-1}(\mathbf{u}, x_s)$. The claim (3.6) with $\ell = s$ implies that $h_2^{s-1}(\mathbf{v}, x_s) = h_2^{s-1}(\mathbf{u}, x_s) = x_{s-1}$. According to Lemma 3.9, $D(\mathbf{u}, x_{s-1}) = s-1$ and $\ell_j(\mathbf{u}, x_s) < \ell_j(\mathbf{u}, x_{s-1})$ for any $j = 1, 2$. Recall that $D(\mathbf{u}, x_s) = s$. According to Lemma 3.12,

$D(\mathbf{v}, x_s) = s$. Now we apply Lemma 3.9 again and obtain $D(\mathbf{v}, x_{s-1}) = s-1$ and $\ell_j(\mathbf{v}, x_s) < \ell_j(\mathbf{v}, x_{s-1})$ for any $j = 1, 2$.

Further, put $x_{s-2} = h_2^{s-2}(\mathbf{u}, x_{s-1})$. According to Lemma 3.9, $D(\mathbf{u}, x_{s-2}) = s-2$ and $\ell_j(\mathbf{u}, x_{s-1}) < \ell_j(\mathbf{u}, x_{s-2})$ for any $j = 1, 2$. The claim (3.6) with $\ell = s-1$ implies that $h_2^{s-2}(\mathbf{v}, x_{s-1}) = h_2^{s-2}(\mathbf{u}, x_{s-1}) = x_{s-2}$. Now we apply Lemma 3.9 again and obtain $D(\mathbf{v}, x_{s-2}) = s-2$ and $\ell_j(\mathbf{v}, x_{s-1}) < \ell_j(\mathbf{v}, x_{s-2})$ for any $j = 1, 2$. Since $\ell_1(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, x_{s-1}) < \ell_1(\mathbf{u}, x_{s-2})$, we have $\ell_2(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, x_{s-2})$ by Lemma 3.13. Analogously, $\ell_2(\mathbf{v}, x_s) < \ell_1(\mathbf{v}, x_{s-2})$.

Continuing these considerations, we define one by one the letters $x_r = h_2^r(\mathbf{u}, x_{r+1})$ for $r = s-3, s-4, \dots, 1$ and prove that $D(\mathbf{u}, x_r) = D(\mathbf{v}, x_r) = r$, $\ell_j(\mathbf{u}, x_{r+1}) < \ell_j(\mathbf{u}, x_r)$, $\ell_j(\mathbf{v}, x_{r+1}) < \ell_j(\mathbf{v}, x_r)$ for any $j = 1, 2$, $\ell_2(\mathbf{u}, x_{r+2}) < \ell_1(\mathbf{u}, x_r)$ and $\ell_2(\mathbf{v}, x_{r+2}) < \ell_1(\mathbf{v}, x_r)$.

Finally, put $x_0 = h_2^0(\mathbf{u}, x_1)$. According to Lemma 3.9, $D(\mathbf{u}, x_0) = 0$ and $\ell_1(\mathbf{u}, x_1) < \ell_1(\mathbf{u}, x_0)$. The claim (3.6) with $\ell = 1$ implies that $h_2^0(\mathbf{v}, x_1) = h_2^0(\mathbf{u}, x_1) = x_0$. Now we apply Lemma 3.9 again and obtain $D(\mathbf{v}, x_0) = 0$ and $\ell_1(\mathbf{v}, x_1) < \ell_1(\mathbf{v}, x_0)$. Since $\ell_1(\mathbf{u}, x_2) < \ell_1(\mathbf{u}, x_1) < \ell_1(\mathbf{u}, x_0)$, we have $\ell_2(\mathbf{u}, x_2) < \ell_1(\mathbf{u}, x_0)$ by Lemma 3.13. Analogously, $\ell_2(\mathbf{v}, x_2) < \ell_1(\mathbf{v}, x_0)$.

In view of the above, we have the identity $\mathbf{u} \approx \mathbf{v}$ has the form (3.7) for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2s+1}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2s+1}$. □

Lemma 3.15. *Let $\mathbf{w} = y_1 y_2 \cdots y_n$ where the letters $y_1, y_2, \dots, y_n$ are not necessarily pairwise different. Further, let $\mathbf{u} = \mathbf{u}'\xi(\mathbf{w})\mathbf{u}''$ for some possibly empty words $\mathbf{u}'$ and $\mathbf{u}''$ and some endomorphism ξ of F^1. Put $\xi(y_i) = \mathbf{w}_i$ for all $i = 1, 2, \dots, n$. If $D(\mathbf{w}, y_i) > 0$ then the subword $\mathbf{w}_i$ of $\mathbf{u}$ does not contain any r-divider of $\mathbf{u}$ for all $r < D(\mathbf{w}, y_i)$.*

Proof. Let $1 \le i \le n$ and $D(\mathbf{w}, y_i) > 0$. Then $y_i \in \mathrm{mul}(\mathbf{w})$, whence $\mathrm{con}(\mathbf{w}_i) \subseteq \mathrm{mul}\big(\xi(\mathbf{w})\big) \subseteq \mathrm{mul}(\mathbf{u})$. This implies that $\mathbf{w}_i$ does not contain any 0-divider of $\mathbf{u}$. Let now $r > 0$ be the least number such that there exists i with $D(\mathbf{w}, y_i) > r$ but $\mathbf{w}_i$ contains some r-divider t of $\mathbf{u}$. The choice of r and Lemma 3.7 imply that $D(\mathbf{u}, t) = r$. Clearly, $t \notin \mathrm{con}(\mathbf{w}_1 \mathbf{w}_2 \cdots \mathbf{w}_{i-1})$, whence y_i differs from $y_1, y_2, \dots, y_{i-1}$. Since $y_i \in \mathrm{mul}(\mathbf{w})$, there is some $j \ge i$ such that $\mathbf{w}_j$ contains second occurrence of t in $\mathbf{u}$. Put $x = h_2^{r-1}(\mathbf{u}, t)$. In view of Lemma 3.9(i), $\ell_1(\mathbf{u}, t) < \ell_1(\mathbf{u}, x)$. Then there is $i \le \ell \le j$ such that $\mathbf{w}_\ell$ contains the $(r-1)$-divider x of $\mathbf{u}$. In view of the choice of r, $D(\mathbf{w}, y_\ell) \le r-1$. This implies that $y_i \ne y_\ell$, whence $\ell_1(\mathbf{w}, y_i) < \ell_1(\mathbf{w}, y_\ell)$. Further, since $y_i \in \mathrm{mul}(\mathbf{w})$, there is $p \ge j$ such that $y_i = y_p$. We note that $\ell < p$ because $y_p = y_i \ne y_\ell$. So, we obtain $\ell_1(\mathbf{w}, y_i) < \ell_1(\mathbf{w}, y_\ell) < \ell_2(\mathbf{w}, y_i)$. Lemma 3.7 implies that y_ℓ is an $(r-1)$-divider of $\mathbf{w}$, whence $h_1^{r-1}(\mathbf{w}, y_i) \ne h_2^{r-1}(\mathbf{w}, y_i)$. We have a contradiction with the fact that $D(\mathbf{w}, y_i) > r$. □

4. The proof of the "only if" part

Throughout this section, $\mathbf{V}$ denotes a fixed non-group chain variety of monoids. We aim to verify that $\mathbf{V}$ is contained in one of the varieties listed in Theorem 1.1. The section is divided into three subsections.

4.1. **Reduction to the case when $\mathbf{D}_2 \subseteq \mathbf{V}$.** A variety of monoids is called *aperiodic* if all its groups are singletons. Lemma 2.1 implies that $\mathbf{SL} \subseteq \mathbf{V}$.

If $\mathbf{V}$ contains a non-trivial group then the variety generated by this group is incomparable with $\mathbf{SL}$. This contradicts the fact that $\mathbf{V}$ is a chain variety. Therefore, $\mathbf{V}$ is aperiodic, whence it satisfies the identity $x^n \approx x^{n+1}$ for some n. If $\mathbf{V}$ is commutative then $\mathbf{V} \subseteq \mathbf{SL} \subseteq \mathbf{C}_2$ whenever $n = 1$ and $\mathbf{V} \subseteq \mathbf{C}_n$ otherwise.

Further, if $\mathbf{V}$ is a variety of band monoids then Lemma 2.9 and the observation that $\mathbf{V}$ cannot contain simultaneously the incomparable varieties $\mathbf{LRB}$ and $\mathbf{RRB}$ imply that $\mathbf{V}$ is contained in one of these two varieties.

Suppose now that $\mathbf{V}$ is non-commutative and is not a variety of band monoids. Then $\mathbf{V}$ is non-completely regular because every aperiodic completely regular variety consists of bands. Then Lemma 2.14 implies that $\mathbf{D}_1 \subseteq \mathbf{V}$. To continue our considerations, we need several assertions.

Lemma 4.1. *Let $\mathbf{X}$ be a monoid variety such that $\mathbf{D}_1 \subseteq \mathbf{X}$. Then either $\mathbf{X}$ satisfies an identity of the form*

$$x^s y x^t \approx y x^r \tag{4.1}$$

where $s \ge 1$, $t \ge 0$, $s + t \ge 2$ and $r \ge 2$ or, for any identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$, the claim

$$h_1^0(\mathbf{u}, x) = h_1^0(\mathbf{v}, x) \text{ for all } x \in \mathrm{con}(\mathbf{u}) \tag{4.2}$$

is true.

Proof. Let $\mathbf{u} \approx \mathbf{v}$ be an identity that holds in $\mathbf{X}$. The inclusion $\mathbf{D}_1 \subseteq \mathbf{X}$ and Proposition 2.13 imply that the claims (2.1) and (2.2) are true. Hence if (3.4) is the 0-decomposition of $\mathbf{u}$ then the 0-decomposition of $\mathbf{v}$ has the form (3.5). Suppose that the claim (4.2) is false. Then there is a letter $x \in \mathrm{mul}(\mathbf{u})$ such that $h_1^0(\mathbf{u}, x) \ne h_1^0(\mathbf{v}, x)$. The claim (2.1) implies that $x \in \mathrm{mul}(\mathbf{v})$. Further, we may assume without loss of generality that there are $i < j$ such that $t_i = h_1^0(\mathbf{u}, x)$ and $t_j = h_1^0(\mathbf{v}, x)$. Substituting y for t_j and 1 for all letters occurring in the identity $\mathbf{u} \approx \mathbf{v}$ except x and t_j, we obtain $\mathbf{X}$ satisfies an identity of the form (4.1) where $s \ge 1$, $t \ge 0$, $s + t \ge 2$ and $r \ge 2$. □

When we make simultaneously several substitutions in some identity, say, substitute $\mathbf{u}_i$ for x_i for $i = 1, 2, \dots, k$, then we will say for brevity that we perform the substitution

$$(x_1, x_2, \dots, x_k) \mapsto (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k)$$

in this identity.

Proposition 4.2. *A non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety $\mathbf{E}$ if and only if the claims (2.1) and (4.2) are true.*

Proof. Necessity. Suppose that $\mathbf{E}$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$. The inclusion $\mathbf{D}_1 \subseteq \mathbf{E}$ and Proposition 2.13 imply that this identity satisfies the claim (2.1). Suppose that the claim (4.2) is false. Then Lemma 4.1 applies with the conclusion that $\mathbf{E}$ satisfies an identity of the form (4.1) where $s \ge 1$, $t \ge 0$, $s + t \ge 2$ and $r \ge 2$. Let us consider the semigroup

$$P = \langle e, a \mid e^2 = e,\ ae = a,\ ea = 0 \rangle = \{e, a, 0\}.$$

Note that $\mathbf{E}$ contains the monoid P^1, i.e., the semigroup P with a new identity element adjoined. Making the substitution $(x, y) \mapsto (e, a)$ in the identity (4.1) results in the contradiction $0 = a$. Thus, P^1 and therefore, $\mathbf{E}$ violates (4.1), a contradiction.

Sufficiency. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ satisfies the claims (2.1) and (4.2). Let (3.4) be the 0-decomposition of $\mathbf{u}$. In view of Lemma 3.8, the 0-decomposition of $\mathbf{v}$ has the form (3.5). We are going to verify that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{E}$. Recall that the variety $\mathbf{E}$ is given by the identity system

$$\{x^2 \approx x^3,\, x^2y \approx xyx,\, x^2y^2 \approx y^2x^2\}. \tag{4.3}$$

Put $X = \mathrm{con}(\mathbf{u}_0) = \{x_1, x_2, \dots, x_k\}$. Clearly, any block of an arbitrary word $\mathbf{w}$ does not contain letters simple in $\mathbf{w}$. Therefore, we may assume without loss of generality that $\mathbf{u}_0 = x_1^2x_2^2 \cdots x_k^2$.

We will use induction on the parameter m from (3.4) and (3.5).

Induction base. Let $m = 0$. The claim (2.1) implies that $\mathrm{con}(\mathbf{u}_0) = \mathrm{con}(\mathbf{v}_0)$. Since the variety $\mathbf{E}$ satisfies the identity

$$x^2y^2 \approx y^2x^2, \tag{4.4}$$

it also satisfies the identity $\mathbf{v}_0 \approx x_1^2x_2^2 \cdots x_k^2$. Therefore, the identities

$$\mathbf{u} = t_0\mathbf{u}_0 = t_0x_1^2x_2^2 \cdots x_k^2 \approx t_0\mathbf{v}_0 = \mathbf{v}$$

hold in $\mathbf{E}$.

Induction step. Let now $m > 0$. The identity system (4.3) implies the identity

$$\mathbf{u} \approx t_0x_1^2x_2^2 \cdots x_k^2t_1(\mathbf{u}_1)_X \cdots t_m(\mathbf{u}_m)_X.$$

By the claim (4.2), $\mathrm{con}(\mathbf{u}_0) = \mathrm{con}(\mathbf{v}_0)$, whence the identity system (4.3) implies the identity

$$\mathbf{v} \approx t_0x_1^2x_2^2 \cdots x_k^2t_1(\mathbf{v}_1)_X \cdots t_m(\mathbf{v}_m)_X.$$

Put $\mathbf{u}' = (\mathbf{u}_1)_X \cdots t_m(\mathbf{u}_m)_X$ and $\mathbf{v}' = (\mathbf{v}_1)_X \cdots t_m(\mathbf{v}_m)_X$. It is easy to verify that the identity $\mathbf{u}' \approx \mathbf{v}'$ satisfies the claims (2.1) and (4.2). By the induction assumption, the identity $\mathbf{u}' \approx \mathbf{v}'$ holds in $\mathbf{E}$, whence this variety satisfies

$$\mathbf{u} \approx t_0x_1^2x_2^2 \cdots x_k^2t_1\mathbf{u}' \approx t_0x_1^2x_2^2 \cdots x_k^2t_1\mathbf{v}' \approx \mathbf{v}.$$

Thus, $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{E}$. □

Lemma 4.3. *Let $\mathbf{X}$ be a non-completely regular variety of monoids. If $\mathbf{E} \nsubseteq \mathbf{X}$ and $\mathbf{X}$ satisfies the identity*

$$x^2 \approx x^3 \tag{4.5}$$

then $\mathbf{X}$ satisfies also the identity

$$yx^2 \approx x^2yx^2. \tag{4.6}$$

Proof. If $\mathbf{X}$ is commutative then we apply the identity (4.5) and obtain $\mathbf{X}$ satisfies the identities $yx^2 \approx yx^4 \approx x^2yx^2$. Suppose now that $\mathbf{X}$ is non-commutative. Then Lemma 2.14 implies that $\mathbf{D}_1 \subseteq \mathbf{X}$. Since $\mathbf{E} \nsubseteq \mathbf{X}$, there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but fails in $\mathbf{E}$. Then Proposition 4.2 applies with the conclusion that either (2.1) or (4.2) is false. Proposition 2.2 implies that (2.1) is true because $\mathbf{C}_2 \subseteq \mathbf{D}_1 \subseteq \mathbf{X}$. Therefore, the claim (4.2) is false. Now Lemma 4.1

applies and we conclude that $\mathbf{X}$ satisfies an identity of the form (4.1) where $s \geq 1$, $t \geq 0$, $s+t \geq 2$ and $r \geq 2$. Substitute x^2 for x in this identity. Since $\mathbf{X}$ satisfies the identity (4.5), we obtain the identity (4.6) holds in $\mathbf{X}$. □

Let us return to an examination of a chain variety $\mathbf{V}$. Recall that we reduce considerations to the case when $\mathbf{D}_1 \subseteq \mathbf{V}$. Hence $\mathbf{C}_3 \nsubseteq \mathbf{V}$ because the varieties $\mathbf{C}_3$ and $\mathbf{D}_1$ are incomparable. Then Lemma 2.5 and the fact that $\mathbf{V}$ is aperiodic imply that the identity (4.5) holds in $\mathbf{V}$. Suppose now that $\mathbf{D}_2 \nsubseteq \mathbf{V}$. The variety $\mathbf{V}$ does not contain at least one of the incomparable varieties $\mathbf{E}$ and $\overleftarrow{\mathbf{E}}$. Assume without loss of generality that $\overleftarrow{\mathbf{E}} \nsubseteq \mathbf{V}$. The dual of Lemma 4.3 implies then that $\mathbf{V}$ satisfies the identity

$$x^2y \approx x^2yx^2. \tag{4.7}$$

Further, Lemma 2.15 implies that the identity

$$xyx \approx x^qyx^r \tag{4.8}$$

with $q > 1$ or $r > 1$ holds in $\mathbf{V}$.

If $\mathbf{u}$ and $\mathbf{v}$ are words and ε is an identity then we will write $\mathbf{u} \stackrel{\varepsilon}{\approx} \mathbf{v}$ in the case when the identity $\mathbf{u} \approx \mathbf{v}$ follows from ε. If $q > 1$ then $\mathbf{V}$ satisfies the identities

$$xyx \stackrel{(4.8)}{\approx} x^qyx^r \stackrel{(4.7)}{\approx} x^qyx^{r+2} \stackrel{(4.5)}{\approx} x^2yx^2 \stackrel{(4.7)}{\approx} x^2y.$$

Recall that $\mathbf{V}$ satisfies the identity (4.5) too. Then Lemma 2.10(i) applies and we conclude that $\mathbf{V} \subseteq \mathbf{LRB} \vee \mathbf{C}_2$. Since $\mathbf{V}$ is non-idempotent and chain, $\mathbf{V} \subseteq \mathbf{E}$ by Lemma 2.10(ii). Therefore, $\mathbf{V} \subseteq \mathbf{K}$.

Suppose now that $q \leq 1$. Then $r > 1$. If $q = 0$ then $\mathbf{V} \subseteq \mathbf{RRB} \vee \mathbf{C}_2$ by the dual of Lemma 2.10(i) because $\mathbf{V}$ satisfies the identity (4.5). Since $\overleftarrow{\mathbf{E}} \nsubseteq \mathbf{V}$ and $\mathbf{V}$ is not a variety of band monoids, it follows from the dual of Lemma 2.10(ii) that $\mathbf{V} \subseteq \mathbf{D}_1 \subseteq \mathbf{D}$.

Let now $q = 1$. Then $\mathbf{V}$ satisfies the identity

$$xyx \approx xyx^2 \tag{4.9}$$

because it satisfies (4.5). Therefore, the identities $x^2yx \stackrel{(4.9)}{\approx} x^2yx^2 \stackrel{(4.7)}{\approx} x^2y$ hold in $\mathbf{V}$. Thus, $\mathbf{V}$ satisfies

$$x^2y \approx x^2yx. \tag{4.10}$$

Recall that $\mathbf{D}_1 \subseteq \mathbf{V}$. Therefore, $\mathbf{LRB} \nsubseteq \mathbf{V}$. Hence there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{V}$ but fails in $\mathbf{LRB}$. The *initial part* of a word $\mathbf{w}$, denoted by $\mathrm{ini}(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining first occurrence of each letter. It is evident that an identity $\mathbf{a} \approx \mathbf{b}$ holds in the variety $\mathbf{LRB}$ if and only if $\mathrm{ini}(\mathbf{a}) = \mathrm{ini}(\mathbf{b})$. Hence $\mathrm{ini}(\mathbf{u}) \neq \mathrm{ini}(\mathbf{v})$. Proposition 2.2 implies that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. Therefore, we can assume without any loss of generality that there are letters $x, y \in \mathrm{con}(\mathbf{u})$ such that $\mathbf{u}(x, y) = x^sy\mathbf{w}_1$ and $\mathbf{v}(x, y) = y^tx\mathbf{w}_2$ where $s, t > 0$ and $\mathrm{con}(\mathbf{w}_1) = \mathrm{con}(\mathbf{w}_2) = \{x, y\}$. Let us substitute 1 for all letters except x and y in $\mathbf{u} \approx \mathbf{v}$. We obtain that $\mathbf{V}$ satisfies the identity $x^sy\mathbf{w}_1 \approx y^tx\mathbf{w}_2$. If $s = 1$ then we substitute x^2 for x in this identity and obtain an identity of the form $x^2y\mathbf{w}_1' \approx y^tx^2\mathbf{w}_2'$. Thus, we can assume that $s \geq 2$. Analogously, we can assume that $t \geq 2$. Moreover, the identity (4.5) allows us to assume that $s = t = 2$. Now we can apply the identity (4.10) and deduce an identity of the form $x^2y^k \approx y^2x^m$ where $k, m > 1$. Moreover, the identity (4.5) allows us to assume that $k = m = 2$. We prove that the identity (4.4) holds in $\mathbf{V}$. This means that $\mathbf{V} \subseteq \mathbf{K}$.

It remains to consider the case when $\mathbf{D}_2 \subseteq \mathbf{V}$.

4.2. **Reduction to the case when $\mathbf{L} \subseteq \mathbf{V}$.** Here we need some notation and a series of auxiliary assertions. Let n and m be arbitrary non-negative integers such that $n + m > 0$. For any $\theta \in S_{n+m}$, we put

$$\mathbf{w}_{n,m}(\theta) = \biggl(\prod_{i=1}^{n} z_i t_i\biggr) x \biggl(\prod_{i=1}^{n+m} z_{\theta(i)}\biggr) x \biggl(\prod_{i=n+1}^{n+m} t_i z_i\biggr),$$

$$\mathbf{w}'_{n,m}(\theta) = \biggl(\prod_{i=1}^{n} z_i t_i\biggr) x^2 \biggl(\prod_{i=1}^{n+m} z_{\theta(i)}\biggr) \biggl(\prod_{i=n+1}^{n+m} t_i z_i\biggr).$$

Note that the words $\mathbf{w}_n(\pi, \tau)$ and $\mathbf{w}'_n(\pi, \tau)$ introduced in Section 1 are words of the form $\mathbf{w}_{n,n}(\theta)$ and $\mathbf{w}'_{n,n}(\theta)$ respectively for an appropriate permutation $\theta \in S_{2n}$.

Lemma 4.4. *The variety* $\mathbf{L}$ *satisfies the identities of the form*

$$\mathbf{w}_{n,m}(\theta) \approx \mathbf{w}'_{n,m}(\theta) \tag{4.11}$$

for all n, m *and* $\theta \in S_{n+m}$.

Proof. It suffices to verify that each identity of the form (4.11) follows from some identity of the form

$$\mathbf{w}_n(\pi, \tau) \approx \mathbf{w}'_n(\pi, \tau). \tag{4.12}$$

To do this, we fix an identity of the form (4.11). It has the form

$$\mathbf{p}_0 x \mathbf{q}_0 x \mathbf{r}_0 \approx \mathbf{p}_0 x^2 \mathbf{q}_0 \mathbf{r}_0$$

where $\mathbf{p}_0 = z_1 t_1 \cdots z_n t_n$, $\mathbf{q}_0 = z_{\theta(1)} \cdots z_{\theta(n+m)}$ and $\mathbf{r}_0 = t_{n+1} z_{n+1} \cdots t_{n+m} z_{n+m}$. The word $\mathbf{q}_0$ may be uniquely represented in the form

$$\mathbf{q}_0 = \mathbf{u}_1 \mathbf{v}_1 \cdots \mathbf{u}_k \mathbf{v}_k$$

where $\operatorname{con}(\mathbf{u}_1 \cdots \mathbf{u}_k) = \{z_1, \dots, z_n\}$ and $\operatorname{con}(\mathbf{v}_1 \cdots \mathbf{v}_k) = \{z_{n+1}, \dots, z_{n+m}\}$ (we mean here that $\mathbf{u}_1 = \lambda$ whenever $\theta(1) > n$, and $\mathbf{v}_k = \lambda$ whenever $\theta(n+m) \le n$). Each of the words $\mathbf{u}_1, \dots, \mathbf{u}_k$ (except $\mathbf{u}_1$ whenever $\mathbf{u}_1 = \lambda$) has the form $z_{j_1} \cdots z_{j_s}$ where $j_1, \dots, j_s \le n$, while each of the words $\mathbf{v}_1, \dots, \mathbf{v}_k$ (except $\mathbf{v}_k$ whenever $\mathbf{v}_k = \lambda$) has the form $z_{j_1} \cdots z_{j_s}$ where $j_1, \dots, j_s > n$.

Suppose at first that $\mathbf{u}_1 = \lambda$. Let z and t be letters that do not occur in the word $\mathbf{p}_0 \mathbf{q}_0 \mathbf{r}_0 x$. Put $\mathbf{p}' = zt\mathbf{p}_0$, $\mathbf{q}' = z\mathbf{q}_0$ and $\mathbf{r}' = \mathbf{r}_0$. The identity $\mathbf{p}' x \mathbf{q}' x \mathbf{r}' \approx \mathbf{p}' x^2 \mathbf{q}' \mathbf{r}'$ evidently implies the identity (4.11). Up to the evident renaming of letters, the identity $\mathbf{p}' x \mathbf{q}' x \mathbf{r}' \approx \mathbf{p}' x^2 \mathbf{q}' \mathbf{r}'$ has the form indicated in the previous paragraph with $\mathbf{u}_1 \ne \lambda$. Thus, we can assume that $\mathbf{u}_1 \ne \lambda$. Analogous arguments allow us to suppose that $\mathbf{v}_k \ne \lambda$.

Let now $\mathbf{u}_1 = z_{j_1} \cdots z_{j_s}$ with $j_1, \dots, j_s \le n$. Let z'_{j_1}, t'_{j_1}, $\dots$, $z'_{j_{s-1}}$, $t'_{j_{s-1}}$ be letters that do not occur in the word $\mathbf{p}_0 \mathbf{q}_0 \mathbf{r}_0 x$. Put $\mathbf{p}_1 = \mathbf{p}_0$. Denote by $\mathbf{q}_1$ the word that is obtained from $\mathbf{q}_0$ by replacing of the word $\mathbf{u}_1$ with

$z_{j_1}z'_{j_1}\cdots z_{j_{s-1}}z'_{j_{s-1}}z_{j_s}$. Finally, we put $\mathbf{r}_1=\mathbf{r}_0t'_{j_1}z'_{j_1}\cdots t'_{j_{s-1}}z'_{j_{s-1}}$. The identity $\mathbf{p}_0x\mathbf{q}_0x\mathbf{r}_0\approx\mathbf{p}_0x^2\mathbf{q}_0\mathbf{r}_0$ follows from $\mathbf{p}_1x\mathbf{q}_1x\mathbf{r}_1\approx\mathbf{p}_1x^2\mathbf{q}_1\mathbf{r}_1$ because the former identity may be obtained by substitution of 1 for the letters $z'_{j_1},t'_{j_1},\dots,z'_{j_{s-1}}$, $t'_{j_{s-1}}$ in the latter identity.

Further, let $\mathbf{v}_1=z_{j_1}\cdots z_{j_s}$ where $j_1,\dots,j_s>n$. Let $z'_{j_1},t'_{j_1},\dots,z'_{j_{s-1}},t'_{j_{s-1}}$ be letters that do not occur in $\mathbf{p}_1\mathbf{q}_1\mathbf{r}_1x$. Put $\mathbf{p}_2=z'_{j_1}t'_{j_1}\cdots z'_{j_{s-1}}t'_{j_{s-1}}\mathbf{p}_1$. Further, we denote by $\mathbf{q}_2$ the word that is obtained from $\mathbf{q}_1$ by replacing of the word $\mathbf{v}_1$ with $z_{j_1}z'_{j_1}\cdots z_{j_{s-1}}z'_{j_{s-1}}z_{j_s}$. Finally, we put $\mathbf{r}_2=\mathbf{r}_1$. The identity $\mathbf{p}_1x\mathbf{q}_1x\mathbf{r}_1\approx\mathbf{p}_1x^2\mathbf{q}_1\mathbf{r}_1$ follows from $\mathbf{p}_2x\mathbf{q}_2x\mathbf{r}_2\approx\mathbf{p}_2x^2\mathbf{q}_2\mathbf{r}_2$ because the former identity may be obtained by substitution of 1 for the letters z'_{j_1}, t'_{j_1}, $\dots$, $z'_{j_{s-1}}$, $t'_{j_{s-1}}$ in the latter identity.

We continue this process and apply the analogous modifications of our identity with the use of the words $\mathbf{u}_2,\mathbf{v}_2,\dots,\mathbf{u}_k,\mathbf{v}_k$. As a result, we obtain an identity of the form

$$\mathbf{p}_{2k}x\mathbf{q}_{2k}x\mathbf{r}_{2k}\approx\mathbf{p}_{2k}x^2\mathbf{q}_{2k}\mathbf{r}_{2k} \tag{4.13}$$

that implies the identity of the form (4.11) fixed at the beginning of the proof. We can evidently rename the letters and assume that $\mathbf{p}_{2k}=z_1t_1\cdots z_pt_p$, $\mathbf{q}_{2k}=z_{\xi(1)}\cdots z_{\xi(p+q)}$ and $\mathbf{r}_{2k}=t_{p+1}z_{p+1}\cdots t_{p+q}z_{p+q}$ for some natural numbers p,q and some permutation $\xi\in S_{p+q}$ with $\xi(i)\le p$ for all odd i and $\xi(i)>p$ for all even i. It remains to verify that $p=q$. For $i=1,\dots,k$, we denote the length of the word $\mathbf{u}_i$ by n_i and the length of the word $\mathbf{v}_i$ by m_i. Then $n_1+\cdots+n_k=n$ and $m_1+\cdots+m_k=m$. It is easy to see that

$$\begin{aligned}p&=n+(m_1-1)+\cdots+(m_k-1)=n+m-k\\&=m+n-k=m+(n_1-1)+\cdots+(n_k-1)=q.\end{aligned}$$

Therefore, the identity (4.13) has the form (4.12). □

Lemma 4.5. *Suppose that a monoid variety* $\mathbf{X}$ *satisfies the identities*

$$xyxzx\approx x^2yz, \tag{4.14}$$

$$x^2y\approx yx^2 \tag{4.15}$$

and (4.11) *for all* n, m *and* $\theta\in S_{n+m}$. *Let* $\mathbf{u}$ *be a word. If there is a letter* $x\in\mathrm{mul}(\mathbf{u})$ *such that* $\mathbf{u}(x,y)\ne xyx$ *for any letter* y *then* $\mathbf{X}$ *satisfies the identity*

$$\mathbf{u}\approx x^2\mathbf{u}_x. \tag{4.16}$$

Proof. Suppose at first that $\mathrm{occ}_x(\mathbf{u})>2$. Then $\mathbf{u}=\mathbf{u}_1x\mathbf{u}_2x\mathbf{u}_3\cdots\mathbf{u}_nx\mathbf{u}_{n+1}$ where $n>2$ and $\mathbf{u}_1,\mathbf{u}_2,\dots,\mathbf{u}_{n+1}$ are possibly empty words with $x\notin\mathrm{con}(\mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_{n+1})$. Clearly, $\mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_{n+1}=\mathbf{u}_x$. Then $\mathbf{X}$ satisfies the identities

$$\mathbf{u}=\mathbf{u}_1x\mathbf{u}_2x\mathbf{u}_3\cdots\mathbf{u}_nx\mathbf{u}_{n+1}\stackrel{(4.14)}{\approx}\mathbf{u}_1x^2\mathbf{u}_2\mathbf{u}_3\cdots\mathbf{u}_{n+1}\stackrel{(4.15)}{\approx}x^2\mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_{n+1}=x^2\mathbf{u}_x,$$

whence (4.16) holds in $\mathbf{X}$.

It remains to consider the case when $\mathrm{occ}_x(\mathbf{u}) = 2$. Then $\mathbf{u} = \mathbf{u}_1 x \mathbf{u}_2 x \mathbf{u}_3$ and $x \notin \mathrm{con}(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3)$. If $\mathbf{u}_2 = \lambda$ then $\mathbf{u} = \mathbf{u}_1 x^2 \mathbf{u}_3 \stackrel{(4.15)}{\approx} x^2\mathbf{u}_1\mathbf{u}_3 = x^2\mathbf{u}_x$ hold in $\mathbf{X}$, and we are done. Let now $\mathbf{u}_2 \ne \lambda$.

If $y \in \mathrm{con}(\mathbf{u}_2)$ and $y \in \mathrm{sim}(\mathbf{u})$ then $\mathbf{u}(x, y) = xyx$, a contradiction. Thus, $y \in \mathrm{mul}(\mathbf{u})$ for any $y \in \mathrm{con}(\mathbf{u}_2)$. Suppose that $\mathrm{occ}_y(\mathbf{u}) > 2$ for some $y \in \mathrm{con}(\mathbf{u}_2)$. Then we can use the same arguments as in the first paragraph of the proof and conclude that the variety $\mathbf{X}$ satisfies the identity $\mathbf{u} \approx y^2\mathbf{u}_y$. This identity can be rewritten in the form $\mathbf{u} \approx \mathbf{u}_1' x \mathbf{u}_2' x \mathbf{u}_3'$ where $\mathbf{u}_1' = y^2\mathbf{u}_1$, $\mathbf{u}_2' = (\mathbf{u}_2)_y$ and $\mathbf{u}_3' = (\mathbf{u}_3)_y$. Thus, we can remove from $\mathbf{u}_2$ all letters y with $\mathrm{occ}_y(\mathbf{u}) > 2$. In other words, we can assume that either $\mathbf{u}_2 = \lambda$ or $\mathrm{occ}_y(\mathbf{u}) = 2$ for all $y \in \mathrm{con}(\mathbf{u}_2)$. The former case is already considered in the previous paragraph. Now we examine the latter case.

Recall that a word $\mathbf{w}$ is called *linear* if $\mathrm{occ}_x(\mathbf{w}) \le 1$ for any letter x. Suppose that the word $\mathbf{u}_2$ is linear, say, $\mathbf{u}_2 = y_1 y_2 \cdots y_k$ for some letters $y_1, y_2, \dots, y_k$. Then either $y_i \in \mathrm{con}(\mathbf{u}_1) \setminus \mathrm{con}(\mathbf{u}_3)$ or $y_i \in \mathrm{con}(\mathbf{u}_3) \setminus \mathrm{con}(\mathbf{u}_1)$ for any $1 \le i \le k$. Renaming, if necessary, the letters $y_1, y_2, \dots, y_k$, we may assume that $y_1, y_2, \dots, y_n \in \mathrm{con}(\mathbf{u}_1) \setminus \mathrm{con}(\mathbf{u}_3)$ and $y_{n+1}, \dots, y_{n+m} \in \mathrm{con}(\mathbf{u}_3) \setminus \mathrm{con}(\mathbf{u}_1)$ for some n and m with $n + m = k$. Then

$$\mathbf{u} = \mathbf{u}_1 x y_{\theta(1)} y_{\theta(2)} \cdots y_{\theta(n+m)} x \mathbf{u}_3$$

for an appropriate permutation $\theta \in S_{n+m}$. We have also

$$\mathbf{u}_1 = \mathbf{w}_0 y_1 \mathbf{w}_1 y_2 \mathbf{w}_2 \cdots y_n \mathbf{w}_n, \mathbf{u}_3 = \mathbf{w}_{n+1} y_{n+1} \mathbf{w}_{n+2} y_{n+2} \cdots \mathbf{w}_{n+m} y_{n+m} \mathbf{w}_{n+m+1}$$

for some possibly empty words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_{n+m+1}$. Then $\mathbf{X}$ satisfies the identities

$$\begin{aligned}
\mathbf{u} &= \mathbf{w}_0 \Bigl(\prod_{i=1}^{n} y_i \mathbf{w}_i\Bigr) x \Bigl(\prod_{i=1}^{n+m} y_{\theta(i)}\Bigr) x \Bigl(\prod_{i=n+1}^{n+m} \mathbf{w}_i y_i\Bigr) \mathbf{w}_{n+m+1}\\
&\stackrel{(4.11)}{\approx} \mathbf{w}_0 \Bigl(\prod_{i=1}^{n} y_i \mathbf{w}_i\Bigr) x^2 \Bigl(\prod_{i=1}^{n+m} y_{\theta(i)}\Bigr) \Bigl(\prod_{i=n+1}^{n+m} \mathbf{w}_i y_i\Bigr) \mathbf{w}_{n+m+1}\\
&\stackrel{(4.15)}{\approx} x^2 \mathbf{w}_0 \Bigl(\prod_{i=1}^{n} y_i \mathbf{w}_i\Bigr) \Bigl(\prod_{i=1}^{n+m} y_{\theta(i)}\Bigr) \Bigl(\prod_{i=n+1}^{n+m} \mathbf{w}_i y_i\Bigr) \mathbf{w}_{n+m+1}\\
&= x^2 \mathbf{u}_x.
\end{aligned}$$

We have $\mathbf{X}$ satisfies the identity (4.16) again.

It remains to consider the case when the word $\mathbf{u}_2$ is not linear. Then there is a letter $y \in \mathrm{con}(\mathbf{u}_2)$ such that $\mathbf{u}_2 = \mathbf{v}_1 y \mathbf{v}_2 y \mathbf{v}_3$ where $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are possibly empty words, $y \notin \mathrm{con}(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3)$ and the word $\mathbf{v}_2$ is either empty or linear. If $\mathbf{v}_2$ is linear then the same arguments as in the previous paragraph show that

$$\mathbf{u} = \mathbf{u}_1 x \mathbf{v}_1 y \mathbf{v}_2 y \mathbf{v}_3 x \mathbf{u}_3 \approx y^2 \mathbf{u}_y = y^2 \mathbf{u}_1 x \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 x \mathbf{u}_3 = \mathbf{u}_1' x \mathbf{u}_2' x \mathbf{u}_3$$

hold in $\mathbf{X}$ where $\mathbf{u}_1' = y^2\mathbf{u}_1$ and $\mathbf{u}_2' = \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$. If $\mathbf{v}_2 = \lambda$ then

$$\mathbf{u} = \mathbf{u}_1 x \mathbf{v}_1 y^2 \mathbf{v}_3 x \mathbf{u}_3 \stackrel{(4.15)}{\approx} y^2 \mathbf{u}_1 x \mathbf{v}_1 \mathbf{v}_3 x \mathbf{u}_3 = \mathbf{u}_1' x \mathbf{u}_2' x \mathbf{u}_3$$

is valid in $\mathbf{X}$ where $\mathbf{u}_1' = y^2\mathbf{u}_1$ and $\mathbf{u}_2' = \mathbf{v}_1\mathbf{v}_3$. In both the cases $y \notin \mathrm{con}(\mathbf{u}_2')$. In other words, we can remove the letter y from $\mathbf{u}_2$. Further, we will repeat these arguments as long as the word $\mathbf{u}_2$ will remain non-empty and non-linear. In other words, we may assume that the word $\mathbf{u}_2$ is either empty or linear. Both these cases have been already considered above. Thus, we prove that $\mathbf{X}$ satisfies the identity (4.16) always. □

Lemma 4.6. $\mathbf{L} = \mathrm{var}\, S(xzxyty)$.

Proof. Put $\mathbf{Z} = \mathrm{var}\, S(xzxyty)$. First, we are going to verify that $\mathbf{Z} \subseteq \mathbf{L}$. In view of Lemma 2.3, to achieve this aim it suffices to check that the word $xzxyty$ is an isoterm for $\mathbf{L}$. Put

$$\Psi = \{x^2y \approx yx^2,\ xyxzx \approx x^2yz,\ \sigma_1,\ \sigma_2,\ \mathbf{w}_n(\pi,\tau) \approx \mathbf{w}_n'(\pi,\tau) \mid n \in \mathbb{N},\ \pi,\tau \in S_n\}.$$

We recall that $\mathbf{L} = \mathrm{var}\,\Psi$. We suppose that $\mathbf{L}$ satisfies a non-trivial idenity $xzxyty \approx \mathbf{w}$ for some word $\mathbf{w}$. Therefore, there exists a *deduction* of the identity $xzxyty \approx \mathbf{w}$ from the identity system Ψ, i.e., a sequence of words

$$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m \tag{4.17}$$

such that $\mathbf{v}_0 = xzxyty$, $\mathbf{v}_m = \mathbf{w}$ and, for any $0 \le i < m$, there exist words $\mathbf{a}_i$, $\mathbf{b}_i$, the identity $\mathbf{s}_i \approx \mathbf{t}_i \in \Psi$ and endomorphism ξ_i of F^1 such that either $\mathbf{v}_i = \mathbf{a}_i\xi_i(\mathbf{s}_i)\mathbf{b}_i$ and $\mathbf{v}_{i+1} = \mathbf{a}_i\xi_i(\mathbf{t}_i)\mathbf{b}_i$ or $\mathbf{v}_i = \mathbf{a}_i\xi_i(\mathbf{t}_i)\mathbf{b}_i$ and $\mathbf{v}_{i+1} = \mathbf{a}_i\xi_i(\mathbf{s}_i)\mathbf{b}_i$. We can assume without loss of generality that the sequence (4.17) is the shortest deduction of the identity $xzxyty \approx \mathbf{w}$ from the identity system Ψ. In particular, this means that $xzxyty \ne \mathbf{v}_1$. We note that if $\xi_0(x) = \lambda$ then $\xi_0(\mathbf{s}_0) = \xi_0(\mathbf{t}_0)$ for any $\mathbf{s}_0 \approx \mathbf{t}_0 \in \Psi$. The latest equality implies that $xzxyty = \mathbf{v}_1$, but this is impossible. Thus, we can assume that $\xi_0(x) \ne \lambda$.

Suppose that $xzxyty = \mathbf{v}_0 = \mathbf{a}_0\xi_0(\mathbf{s}_0)\mathbf{b}_0$ and $\mathbf{v}_1 = \mathbf{a}_0\xi_0(\mathbf{t}_0)\mathbf{b}_0$. The case when $\mathbf{s}_0 = x^2y$ is impossible because the word $\xi_0(\mathbf{s}_0)$ contains the square of a non-empty word, while the word $xzxyty$ is square-free. The case when $\mathbf{s}_0 = xyxzx$ is also impossible because there is a letter that occurs in the word $\xi_0(\mathbf{s}_0)$ at least three times, while every letter from $\mathrm{con}(xzxyty)$ occurs in the word $xzxyty$ no more than twice. Finally, the case when $\mathbf{s}_0 = \mathbf{w}_n(\pi,\tau)$ for some $n \in \mathbb{N}$, $\pi,\tau \in S_n$ is impossible because there exists a letter $c \in \xi_0(x)$ such that c is multiple in $\xi_0(\mathbf{s}_0)$ and every letter located between the first and the second occurrences of c in $\xi_0(\mathbf{s}_0)$ is multiple, while for every $d \in \mathrm{mul}(xzxyty)$ there is a letter $e \in \mathrm{sim}(xzxyty)$ such that e lies between the first and the second occurrences of d in $xzxyty$. So, the identity $\mathbf{s}_0 \approx \mathbf{t}_0$ is either σ_1 or σ_2. By symmetry, we can consider only the case when $\mathbf{s}_0 \approx \mathbf{t}_0$ is equal to σ_1. Then $\mathbf{s}_0 = xyzxty$ and $\mathbf{t}_0 = yxzxty$. Since $\xi_0(x) \ne \lambda$, we have $\mathrm{con}(\xi_0(x))$ contains a letter a. Then $a \in \{x,y\}$ because $a \in \mathrm{mul}(\xi_0(\mathbf{s}_0))$. Suppose that $a = x$. Then $\xi_0(y) = \lambda$ because

$$xzxyty = \mathbf{a}_0\xi_0(\mathbf{s}_0)\mathbf{b}_0 = \mathbf{a}_0\xi_0(x)\xi_0(y)\xi_0(z)\xi_0(x)\xi_0(t)\xi_0(y)\mathbf{b}_0.$$

Therefore, $\xi_0(\mathbf{t}_0) = \xi_0(x)\xi_0(z)\xi_0(x)\xi_0(t) = \xi_0(\mathbf{s}_0)$. Then

$$\mathbf{v}_1 = \mathbf{a}_0\xi_0(\mathbf{t}_0)\mathbf{b}_0 = \mathbf{a}_0\xi_0(\mathbf{s}_0)\mathbf{b}_0 = xzxyty,$$

contradicting the choice of the sequence (4.17). The case when $a = y$ is considered similarly.

Suppose now that $xzxyty = \mathbf{v}_0 = \mathbf{a}_0\xi_0(\mathbf{t}_0)\mathbf{b}_0$. The case when

$$\mathbf{t}_0 \in \{yx^2,\ x^2yz,\ \mathbf{w}_n'(\pi,\tau) \mid n \in \mathbb{N},\ \pi,\tau \in S_n\}$$

is impossible because the word $\xi_0(\mathbf{t}_0)$ contains the square of a non-empty word in this case, while the word $xzxyty$ is square-free. So, the identity $\mathbf{s}_0 \approx \mathbf{t}_0$ is either σ_1 or σ_2. Arguments similar to those from the previous paragraph allow us to obtain a contradiction with the fact that the words $xzxyty$ and $\mathbf{v}_1$ are distinct.

Thus, we have verified that $xzxyty$ is an isoterm for $\mathbf{L}$ and therefore, $\mathbf{Z} \subseteq \mathbf{L}$. It remains to verify the opposite inclusion. Suppose that $\mathbf{Z}$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$. We need to prove that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{L}$. Lemma 4.4 allows us to use Lemma 4.5 below. Let x be a letter multiple in $\mathbf{u}$ and $\mathbf{u}(x,y) \ne xyx$ for any letter y. By Lemma 4.5, the variety $\mathbf{L}$ satisfies the identity (4.16). Obviously, $\mathbf{C}_2 \subseteq \mathbf{Z}$, whence $x \in \mathrm{mul}(\mathbf{v})$ by Proposition 2.2. Since the word $xzxyty$ is an isoterm for $\mathbf{Z}$, the word xyx is an isoterm for $\mathbf{Z}$ too. Therefore, $\mathbf{v}(x,y) \ne xyx$ for any letter y. We apply Lemma 4.5 again and conclude that the identity $\mathbf{v} \approx x^2\mathbf{v}_x$ holds in $\mathbf{L}$. Thus, if the identity $\mathbf{u}_x \approx \mathbf{v}_x$ holds in the variety $\mathbf{L}$ then this variety satisfies the identities $\mathbf{u} \approx x^2\mathbf{u}_x \approx x^2\mathbf{v}_x \approx \mathbf{v}$. So, we can remove from $\mathbf{u} \approx \mathbf{v}$ all multiple letters x such that $\mathbf{u}(x,y) \ne xyx$ for any y. In other words, we may assume without loss of generality that for any letter $x \in \mathrm{mul}(\mathbf{u})$ there is a letter y such that $\mathbf{u}(x,y) = xyx = \mathbf{v}(x,y)$. In particular, this means that $\mathrm{occ}_x(\mathbf{u}), \mathrm{occ}_x(\mathbf{v}) \le 2$ for any letter x.

Lemma 2.1 and the evident inclusion $\mathbf{C}_2 \subseteq \mathbf{Z}$ imply that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. It is evident that for any letters $a, b \notin \mathrm{con}(\mathbf{u})$, the identities $\mathbf{u} \approx \mathbf{v}$ and $a\mathbf{u}b \approx a\mathbf{v}b$ are equivalent in the class of monoids. Therefore, we can assume without loss of generality that the first and the last letters in each of the words $\mathbf{u}$ and $\mathbf{v}$ are simple in this word. Let $\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v}) = \{t_0, t_1, \dots, t_m\}$. We can assume without any loss that $\mathbf{v}(t_1, t_2, \dots, t_m) = t_1t_2\cdots t_m$. In view of Lemma 2.7, $\mathbf{D}_1 \subseteq \mathbf{Z}$. Then Proposition 2.13 implies that

$$\mathbf{u} = t_0\mathbf{a}_1t_1\mathbf{a}_2t_2\cdots t_{m-1}\mathbf{a}_mt_m \text{ and } \mathbf{v} = t_0\mathbf{b}_1t_1\mathbf{b}_2t_2\cdots t_{m-1}\mathbf{b}_mt_m$$

for some possibly empty words $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ and $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$.

Let $0 \le i \le m-1$. Then $\mathbf{u} = \mathbf{w}_1t_i\mathbf{a}_{i+1}t_{i+1}\mathbf{w}_2$ where

$$\mathbf{w}_1 = \begin{cases} t_0\mathbf{a}_1t_1\cdots t_{i-1}\mathbf{a}_i & \text{if } 0 < i \le m-1, \\ \lambda & \text{if } i = 0 \end{cases}$$

and

$$\mathbf{w}_2 = \begin{cases} \mathbf{a}_{i+2}t_{i+2}\cdots \mathbf{a}_mt_m & \text{if } 0 \le i < m-1, \\ \lambda & \text{if } i = m-1. \end{cases}$$

We are going to check that

$$\mathbf{a}_{i+1} = \mathbf{u}_1\mathbf{u}_1'\mathbf{u}_2\mathbf{u}_2'\cdots\mathbf{u}_k\mathbf{u}_k' \tag{4.18}$$

and therefore, $\mathbf{u} = \mathbf{w}_1t_i\mathbf{u}_1\mathbf{u}_1'\mathbf{u}_2\mathbf{u}_2'\cdots\mathbf{u}_k\mathbf{u}_k't_{i+1}\mathbf{w}_2$ for some possibly empty words $\mathbf{u}_1, \mathbf{u}_k'$ and non-empty words $\mathbf{u}_1', \mathbf{u}_2, \mathbf{u}_2', \dots, \mathbf{u}_k$ such that $\mathrm{con}(\mathbf{u}_j) \subseteq \mathrm{con}(\mathbf{w}_1)$ and $\mathrm{con}(\mathbf{u}_j') \subseteq \mathrm{con}(\mathbf{w}_2)$ for all $j = 1, \dots, k$. If $\mathbf{a}_{i+1} = \lambda$ then the equality (4.18) holds with $k = 1$ and $\mathbf{u}_1 = \mathbf{u}_1' = \lambda$. Suppose now that $\mathbf{a}_{i+1} \ne \lambda$. Let $x \in \mathrm{con}(\mathbf{a}_{i+1})$. Then $x \in \mathrm{mul}(\mathbf{u})$. There is a letter $y \in \mathrm{sim}(\mathbf{u})$ with $\mathbf{u}(x,y) = xyx$. Suppose that $x \in \mathrm{mul}(\mathbf{a}_{i+1})$. Therefore, xyx is a subword of $\mathbf{a}_{i+1}$. This means that y is simple in $\mathbf{a}_{i+1}$. But this is not the case because $y \ne t_j$ for any $0 \le j \le m$. Thus, x is simple in $\mathbf{a}_{i+1}$, whence $x \in \mathrm{con}(\mathbf{w}_1\mathbf{w}_2)$. We prove that every letter from $\mathrm{con}(\mathbf{a}_{i+1})$ is simple in $\mathbf{a}_{i+1}$ and occurs either in $\mathbf{w}_1$ or in $\mathbf{w}_2$.

Let $\mathbf{u}_1$ be the maximal prefix of $\mathbf{a}_{i+1}$ such that $\mathrm{con}(\mathbf{u}_1) \subseteq \mathrm{con}(\mathbf{w}_1)$ (if the first letter of $\mathbf{a}_{i+1}$ does not occur in $\mathbf{w}_1$ then $\mathbf{u}_1 = \lambda$). Then $\mathbf{a}_{i+1} = \mathbf{u}_1\mathbf{b}$ for some possibly empty word $\mathbf{b}$. If $\mathbf{b} = \lambda$ then (4.18) holds with $k = 1$ and $\mathbf{u}_1' = \lambda$. Otherwise, let $\mathbf{u}_1'$ be the maximal prefix of $\mathbf{b}$ such that $\mathrm{con}(\mathbf{u}_1') \subseteq \mathrm{con}(\mathbf{w}_2)$. Then $\mathbf{a}_{i+1} = \mathbf{u}_1\mathbf{u}_1'\mathbf{c}$ for some possibly empty word $\mathbf{c}$. If $\mathbf{c} = \lambda$ then (4.18) holds with $k = 1$. Otherwise, let $\mathbf{u}_2$ be the maximal prefix of $\mathbf{c}$ such that $\mathrm{con}(\mathbf{u}_2) \subseteq \mathrm{con}(\mathbf{w}_1)$. Continuing this process, we obtain the equality (4.18).

Put

$$\mathbf{w}_1' = \begin{cases} t_0\mathbf{b}_1t_1\cdots t_{i-1}\mathbf{b}_i & \text{if } 0 < i \le m-1,\\ \lambda & \text{if } i = 0 \end{cases}$$

and

$$\mathbf{w}_2' = \begin{cases} \mathbf{b}_{i+2}t_{i+2}\cdots \mathbf{b}_mt_m & \text{if } 0 \le i < m-1,\\ \lambda & \text{if } i = m-1. \end{cases}$$

The same arguments as above show that $\mathbf{b}_{i+1} = \mathbf{v}_1\mathbf{v}_1'\mathbf{v}_2\mathbf{v}_2'\cdots\mathbf{v}_r\mathbf{v}_r'$ for some natural r, possibly empty words $\mathbf{v}_1, \mathbf{v}_r'$ and non-empty words $\mathbf{v}_1', \mathbf{v}_2, \mathbf{v}_2', \dots, \mathbf{v}_r$ such that $\mathrm{con}(\mathbf{v}_j) \subseteq \mathrm{con}(\mathbf{w}_1')$ and $\mathrm{con}(\mathbf{v}_j') \subseteq \mathrm{con}(\mathbf{w}_2')$ for all $j = 1, \dots, r$. Therefore,

$$\mathbf{v} = \mathbf{w}_1't_i\mathbf{v}_1\mathbf{v}_1'\mathbf{v}_2\mathbf{v}_2'\cdots\mathbf{v}_r\mathbf{v}_r't_{i+1}\mathbf{w}_2'.$$

Further, we may assume without loss of generality that $k \ge r$. We are going to verify that $k = r$, $\mathrm{con}(\mathbf{u}_j) = \mathrm{con}(\mathbf{v}_j)$ and $\mathrm{con}(\mathbf{u}_j') = \mathrm{con}(\mathbf{v}_j')$ for all $j = 1, \dots, r$.

Let $x \in \mathrm{con}(\mathbf{u}_1)$. As we have shown above, $\mathbf{u}(x, t_i) = xt_ix$. Therefore, $\mathbf{v}(x, t_i) = xt_ix$ too, whence $\mathrm{occ}_x(\mathbf{w}_1') = 1$. Note that $\mathbf{v}(x, t_{i+1}) \ne xt_{i+1}x$ because $\mathbf{u}(x, t_{i+1}) = x^2t_{i+1}$. Therefore, $x \notin \mathrm{con}(\mathbf{w}_2')$, whence $x \in \mathrm{con}(\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_r)$. If $x \notin \mathrm{con}(\mathbf{v}_1)$ then $x \in \mathrm{con}(\mathbf{v}_p)$ for some $p > 1$. Then there exists a letter $y \in \mathrm{con}(\mathbf{v}_{p-1}')$. Note that $\mathbf{u}(y, t_{i+1}) = \mathbf{v}(y, t_{i+1}) = yt_{i+1}y$. Therefore, $y \in \mathrm{con}(\mathbf{w}_2)$, whence $y \in \mathrm{con}(\mathbf{u}_j')$ for some $1 \le j \le k$. Then $\mathbf{u}(x, y, t_i, t_{i+1}) = xt_ixyt_{i+1}y$, while $\mathbf{v}(x, y, t_i, t_{i+1}) = xt_iyxt_{i+1}y$. This contradicts the fact that the word $xt_ixyt_{i+1}y$ is an isoterm for $\mathbf{Z}$. Thus, $x \in \mathrm{con}(\mathbf{v}_1)$, whence $\mathrm{con}(\mathbf{u}_1) \subseteq \mathrm{con}(\mathbf{v}_1)$. Analogously, $\mathrm{con}(\mathbf{v}_1) \subseteq \mathrm{con}(\mathbf{u}_1)$. Therefore, $\mathrm{con}(\mathbf{u}_1) = \mathrm{con}(\mathbf{v}_1)$.

Let $x \in \mathrm{con}(\mathbf{u}_1')$. As we have shown above, $\mathbf{u}(x, t_{i+1}) = xt_{i+1}x$. Therefore, $\mathbf{v}(x, t_{i+1}) = xt_{i+1}x$ too, whence $\mathrm{occ}_x(\mathbf{w}_2') = 1$. Note that $\mathbf{v}(x, t_i) \ne xt_ix$ because $\mathbf{u}(x, t_i) = t_ix^2$. Therefore, $x \notin \mathrm{con}(\mathbf{w}_1')$, whence $x \in \mathrm{con}(\mathbf{v}_1'\mathbf{v}_2'\cdots\mathbf{v}_r')$. If $x \notin \mathrm{con}(\mathbf{v}_1')$ then $x \in \mathrm{con}(\mathbf{v}_p')$ for some $p > 1$. Then there exists a letter $y \in \mathrm{con}(\mathbf{v}_p)$. Note that $\mathbf{u}(y, t_i) = \mathbf{v}(y, t_i) = yt_iy$. Therefore, $y \in \mathrm{con}(\mathbf{w}_1)$, whence $y \in \mathrm{con}(\mathbf{u}_j)$ for some $1 \le j \le k$. Note that $y \notin \mathrm{con}(\mathbf{u}_1)$. Indeed, if $y \in \mathrm{con}(\mathbf{u}_1)$ then $y \in \mathrm{con}(\mathbf{v}_1)$ because $\mathrm{con}(\mathbf{u}_1) = \mathrm{con}(\mathbf{v}_1)$. Hence $\mathrm{occ}_y(\mathbf{v}) \ge \mathrm{occ}_y(\mathbf{v}_1\mathbf{v}_p\mathbf{w}_2') \ge 3$, a contradiction. So, $y \in \mathrm{con}(\mathbf{u}_j)$ for some $2 \le j \le k$. Then $\mathbf{u}(x, y, t_i, t_{i+1}) = xt_ixyt_{i+1}y$, while $\mathbf{v}(x, y, t_i, t_{i+1}) = xt_iyxt_{i+1}y$. This contradicts the fact that the word $xt_ixyt_{i+1}y$ is an isoterm for $\mathbf{Z}$. Thus, $x \in \mathrm{con}(\mathbf{v}_1')$, whence $\mathrm{con}(\mathbf{u}_1') \subseteq \mathrm{con}(\mathbf{v}_1')$. Analogously, $\mathrm{con}(\mathbf{v}_1') \subseteq \mathrm{con}(\mathbf{u}_1')$. We prove that $\mathrm{con}(\mathbf{u}_1') = \mathrm{con}(\mathbf{v}_1')$.

Repeating with evident modifications arguments from the previous two paragraphs, we can check that $\mathrm{con}(\mathbf{u}_i) = \mathrm{con}(\mathbf{v}_i)$ and $\mathrm{con}(\mathbf{u}_i') = \mathrm{con}(\mathbf{v}_i')$ for $i = 2, \ldots, r$.

If $k > r$ then there is a letter $x \in \mathrm{con}(\mathbf{u}_{r+1})$. As we have shown above, $\mathbf{u}(x, t_i) = xt_ix$. Therefore, $\mathbf{v}(x, t_i) = xt_ix$ too, whence $\mathrm{occ}_x(\mathbf{w}_1') = 1$. Note also that $\mathbf{v}(x, t_{i+1}) = \mathbf{u}(x, t_{i+1}) = x^2t_{i+1}$. In particular, $\mathbf{v}(x, t_{i+1}) \ne xt_{i+1}x$. Therefore, $x \notin \mathrm{con}(\mathbf{w}_2')$, whence $x \in \mathrm{con}(\mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_r)$. Then $x \in \mathrm{con}(\mathbf{u}_1\mathbf{u}_2 \cdots \mathbf{u}_r)$ because $\mathrm{con}(\mathbf{u}_i) = \mathrm{con}(\mathbf{v}_i)$ for $i = 1, 2, \ldots, r$. Thus, $\mathrm{occ}_x(\mathbf{u}) \ge \mathrm{occ}_x(\mathbf{w}_1\mathbf{u}_1\mathbf{u}_2 \cdots \mathbf{u}_{r+1}) \ge 3$, a contradiction. Therefore, $k = r$.

We prove that $k = r$, $\mathrm{con}(\mathbf{u}_i) = \mathrm{con}(\mathbf{v}_i)$ and $\mathrm{con}(\mathbf{u}_i') = \mathrm{con}(\mathbf{v}_i')$ for all $i = 1, \ldots, k$. One can fix an index $s \in \{1, 2, \ldots, k\}$. Then $\mathbf{u}_s$ and $\mathbf{v}_s$ are linear words depending on the same letters. The same is true for the words $\mathbf{u}_s'$ and $\mathbf{v}_s'$. The identity σ_1 [respectively σ_2] allows us to swap the first [the second] occurrences of two multiple letters whenever these occurrences are each adjacent to other. Therefore, the identities σ_1 and σ_2 allow us to reorder letters within the words $\mathbf{u}_s$ and $\mathbf{u}_s'$ in an arbitrary way. Thus, if we replace $\mathbf{u}_s$ by $\mathbf{v}_s$ and $\mathbf{u}_s'$ by $\mathbf{v}_s'$ in $\mathbf{u}$ then the word we obtain should be equal to $\mathbf{u}$ in $\mathbf{L}$. This is true for all $s = 1, \ldots, k$. Hence $\mathbf{L}$ satisfies the identities

$$\begin{aligned}\mathbf{u} = \mathbf{w}_1 t_i \mathbf{a}_{i+1} t_{i+1} \mathbf{w}_2 &= \mathbf{w}_1 t_i \mathbf{u}_1 \mathbf{u}_1' \mathbf{u}_2 \mathbf{u}_2' \cdots \mathbf{u}_k \mathbf{u}_k' t_{i+1} \mathbf{w}_2 \\ &\approx \mathbf{w}_1 t_i \mathbf{v}_1 \mathbf{v}_1' \mathbf{v}_2 \mathbf{v}_2' \cdots \mathbf{v}_k \mathbf{v}_k' t_{i+1} \mathbf{w}_2 = \mathbf{w}_1 t_i \mathbf{b}_{i+1} t_{i+1} \mathbf{w}_2.\end{aligned}$$

Thus, if we replace $\mathbf{a}_{i+1}$ by $\mathbf{b}_{i+1}$ in $\mathbf{u}$ then the word we obtain should be equal to $\mathbf{u}$ in $\mathbf{L}$. This is true for all $i = 0, \ldots, m-1$. Therefore, $\mathbf{L}$ satisfies the identities

$$\mathbf{u} = t_0\mathbf{a}_1t_1\mathbf{a}_2t_2 \cdots t_{m-1}\mathbf{a}_mt_m \approx t_0\mathbf{b}_1t_1\mathbf{b}_2t_2 \cdots t_{m-1}\mathbf{b}_mt_m = \mathbf{v}.$$

The lemma is proved. □

Lemma 4.6 and [8, Lemma 5.10] imply that any proper subvariety of $\mathbf{L}$ is contained in $\mathrm{var}\, S(xyx)$. Lemmas 2.7 and 2.8 imply now the following

Corollary 4.7. *The lattice $L(\mathbf{L})$ is the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \mathbf{L}$.* □

A non-finitely based variety all whose proper subvarieties are finitely based is called *limit*. The variety $\mathrm{var}\, S(xzxyty)$ is limit by [8, Proposition 5.1]. Thus, Lemma 4.6 implies

Corollary 4.8. *The variety $\mathbf{L}$ is a limit variety. In particular, it does not have a finite basis of identities.* □

According to the result of [11] mentioned in Section 1, there are uncountably many periodic group varieties whose subvariety lattice is the 3-element chain. Let $\mathcal{G}$ be the class of all such varieties. Since the class of finitely based group varieties is countably infinite, the class $\mathcal{G}$ contains non-finitely based varieties. Group varieties whose subvariety lattice is the 2-element chain are varieties of Abelian groups of a prime exponent. They are finitely based. Thus, all non-finitely based varieties from the class $\mathcal{G}$ are limit varieties. But explicit examples of limit chain group varieties have not been published anywhere so far.

We denote by $\mathbf{M}$ the subvariety of $\mathbf{N}$ given within $\mathbf{N}$ by the following identity:

$$\alpha_1: \quad x_1y_1x_0x_1y_1 \approx y_1x_1x_0x_1y_1.$$

Note that α_1 belongs to a countably infinite series of identities α_k that will be defined in Subsection 6.1.

Lemma 4.9. *Let $\mathbf{X}$ be a monoid variety and $\mathbf{D}_2 \subseteq \mathbf{X}$.*

(i) *If $\mathbf{L} \nsubseteq \mathbf{X}$ then $\mathbf{X}$ satisfies the identity γ_1.*
(ii) *If $\mathbf{M} \nsubseteq \mathbf{X}$ then $\mathbf{X}$ satisfies the identity σ_1.*

Proof. (i) According to Lemmas 2.3 and 4.6, the variety $\mathbf{X}$ satisfies a non-trivial identity of the form $xzxyty \approx \mathbf{w}$. Note that the word xyx is an isoterm for $\mathbf{X}$ by Lemmas 2.3 and 2.7. Then Fact 4.1(i) of [20] implies that $\mathbf{w} = xzyxty$. Therefore, the identity γ_1 holds in $\mathbf{X}$.

(ii) According to Lemmas 2.3 and 2.7, the word xyx is an isoterm for $\mathbf{X}$. Further, the variety $\mathbf{M}$ is generated by the monoid $S(xyzxty)$ (this fact is dual to Proposition 1 in Erratum to [8]). Then $S(xyzxty) \notin \mathbf{X}$, whence $\mathbf{X}$ satisfies a non-trivial identity of the form $xyzxty \approx \mathbf{w}$ by Lemma 2.3. Fact 4.1(ii) of [20] implies that $\mathbf{w} = yxzxty$. Therefore, the identity σ_1 holds in $\mathbf{X}$. □

Return to an examination of a chain variety $\mathbf{V}$. In Subsection 4.1 we reduce considerations to the case when $\mathbf{D}_2 \subseteq \mathbf{V}$. Then $\mathbf{E} \nsubseteq \mathbf{V}$ because the varieties $\mathbf{D}_2$ and $\mathbf{E}$ are non-comparable. The variety $\mathbf{V}$ satisfies the identity (4.6) by Lemma 4.3. Similarly, the fact that $\overleftarrow{\mathbf{E}} \nsubseteq \mathbf{V}$ implies that $\mathbf{V}$ satisfies the identity (4.7) by the dual of Lemma 4.3. Hence the identity (4.15) holds in $\mathbf{V}$. If $\mathbf{V}$ does not contain $\mathbf{L}$, $\mathbf{M}$ and $\overleftarrow{\mathbf{M}}$ then Lemma 4.9 and the dual of its claim (ii) imply that $\mathbf{V}$ satisfies σ_1, σ_2 and γ_1, whence $\mathbf{V} \subseteq \mathbf{D}$.

It remains to consider the case when $\mathbf{V}$ contains one of the varieties $\mathbf{L}$, $\mathbf{M}$ or $\overleftarrow{\mathbf{M}}$. Then $\mathbf{V}$ does not contain the variety $\mathbf{D}_3$ because $\mathbf{L}$, $\mathbf{M}$ and $\overleftarrow{\mathbf{M}}$ are non-comparable with $\mathbf{D}_3$. Lemma 2.15 and the fact that $\mathbf{V}$ satisfies the identity (4.15) imply that the identity (4.14) holds in $\mathbf{V}$.

Let $\mathbf{M} \subseteq \mathbf{V}$. Then $\mathbf{V}$ does not contain $\mathbf{L}$ and $\overleftarrow{\mathbf{M}}$. Lemma 4.9(i) and the dual of Lemma 4.9(ii) imply that $\mathbf{V} \subseteq \mathbf{N}$. Dual arguments show that if $\overleftarrow{\mathbf{M}} \subseteq \mathbf{V}$ then $\mathbf{V} \subseteq \overleftarrow{\mathbf{N}}$.

We reduce our considerations to the case when $\mathbf{L} \subseteq \mathbf{V}$.

4.3. **The case when $\mathbf{L} \subseteq \mathbf{V}$.** Clearly, here $\mathbf{M}, \overleftarrow{\mathbf{M}} \nsubseteq \mathbf{V}$. Lemma 4.9(ii) and the dual of it imply that $\mathbf{V}$ satisfies the identities σ_1 and σ_2. As we have already seen above, $\mathbf{V}$ satisfies the identities (4.14) and (4.15) as well. Thus, $\mathbf{V}$ is contained in the variety

$$\mathbf{O} = \operatorname{var}\{x^2y \approx yx^2,\ xyxzx \approx x^2yz,\ \sigma_1,\ \sigma_2\}.$$

To complete the proof of the necessity in Theorem 1.1, it suffices to verify that $\mathbf{V} \subseteq \mathbf{L}$ in the case under consideration. To achieve this goal, it remains to check that $\mathbf{V}$ satisfies all identities of the form (4.12) where n is a natural number and $\pi, \tau \in S_n$. To do this, we need several auxiliary claims. Let $n \in \mathbb{N}$,

$0 \le k \le \ell \le n$ and $\pi, \tau \in S_n$. Put

$$\mathbf{w}_n^{k,\ell}(\pi,\tau) = \biggl(\prod_{i=1}^{n} z_i t_i\biggr)\biggl(\prod_{i=1}^{k} z_{\pi(i)} z_{n+\tau(i)}\biggr) x \biggl(\prod_{i=k+1}^{\ell} z_{\pi(i)} z_{n+\tau(i)}\biggr) x \cdot \biggl(\prod_{i=\ell+1}^{n} z_{\pi(i)} z_{n+\tau(i)}\biggr)\biggl(\prod_{i=n+1}^{2n} t_i z_i\biggr).$$

We note that $\mathbf{w}_n^{0,n}(\pi,\tau) = \mathbf{w}_n(\pi,\tau)$ and $\mathbf{w}_n^{0,0}(\pi,\tau) = \mathbf{w}_n'(\pi,\tau)$.

Lemma 4.10. *Let* $\mathbf{X}$ *be a monoid variety such that* $\mathbf{L} \subseteq \mathbf{X} \subseteq \mathbf{O}$*,* n *be a natural number and* $\pi, \tau \in S_n$*. If* $S\bigl(\mathbf{w}_n(\pi,\tau)\bigr) \notin \mathbf{X}$ *then* $\mathbf{X}$ *satisfies a non-trivial identity of the form*

$$\mathbf{w}_n(\pi,\tau) \approx \mathbf{w}_n^{k,\ell}(\pi,\tau) \tag{4.19}$$

for some $0 \le k \le \ell \le n$*.*

Proof. Suppose that $S\bigl(\mathbf{w}_n(\pi,\tau)\bigr) \notin \mathbf{X}$. Then Lemma 2.3 applies and we conclude that the variety $\mathbf{X}$ satisfies a non-trivial identity of the form

$$\mathbf{w}_n(\pi,\tau) = \biggl(\prod_{i=1}^{n} z_i t_i\biggr) x \biggl(\prod_{i=1}^{n} z_{\pi(i)} z_{n+\tau(i)}\biggr) x \biggl(\prod_{i=n+1}^{2n} t_i z_i\biggr) \approx \mathbf{w}. \tag{4.20}$$

Put $\mathbf{a} = z_{\pi(1)}$, $\mathbf{b} = t_{\pi(1)} z_{\pi(1)+1} t_{\pi(1)+1} \cdots z_n t_n x$, $\mathbf{c} = z_{n+\tau(1)}$ and

$$\mathbf{d} = z_{\pi(2)} z_{n+\tau(2)} \cdots z_{\pi(n)} z_{n+\tau(n)} x t_{n+1} z_{n+1} \cdots t_{n+\tau(1)-1} z_{n+\tau(1)-1} t_{n+\tau(1)}.$$

The word $\mathbf{w}_n(\pi,\tau)$ contains the subword $\mathbf{abacdc}$. Therefore, the submonoid of the monoid $S\bigl(\mathbf{w}_n(\pi,\tau)\bigr)$ generated by the elements $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ is isomorphic to $S(xzxyty)$. Now Lemmas 2.3 and 4.6 apply with the conclusion that the word $xzxyty$ is an isoterm for $\mathbf{X}$. Now we are going to verify that

$$\begin{gathered} \ell_2(\mathbf{w}, z_i) < \ell_1(\mathbf{w}, z_{n+j}) \text{ if and only if} \\ \ell_2\bigl(\mathbf{w}_n(\pi,\tau), z_i\bigr) < \ell_1\bigl(\mathbf{w}_n(\pi,\tau), z_{n+j}\bigr) \end{gathered} \tag{4.21}$$

for any $1 \le i, j \le n$. Indeed, let $1 \le i, j \le n$. The word xyx is an isoterm for $\mathbf{X}$. Since $\bigl[\mathbf{w}_n(\pi,\tau)\bigr](z_i, t_i) = z_i t_i z_i$, $\bigl[\mathbf{w}_n(\pi,\tau)\bigr](z_{n+j}, t_{n+j}) = z_{n+j} t_{n+j} z_{n+j}$ and (4.20) holds in $\mathbf{X}$, we have

$$\mathbf{w}(z_i, t_i) = z_i t_i z_i \text{ and } \mathbf{w}(z_{n+j}, t_{n+j}) = z_{n+j} t_{n+j} z_{n+j}.$$

The variety $\mathbf{X}$ is non-commutative, whence $\ell_1(\mathbf{w}, t_i) < \ell_1(\mathbf{w}, t_{n+j})$. Therefore,

$$\begin{aligned} \mathbf{w}(z_i, t_{n+j}) &= \bigl[\mathbf{w}_n(\pi,\tau)\bigr](z_i, t_{n+j}) = z_i^2 t_{n+j}, \\ \mathbf{w}(z_{n+j}, t_i) &= \bigl[\mathbf{w}_n(\pi,\tau)\bigr](z_{n+j}, t_i) = t_i z_{n+j}^2. \end{aligned}$$

Summarizing the above, we have

$$\begin{aligned} \mathbf{w}(z_i, t_i, t_{n+j}) &= \bigl[\mathbf{w}_n(\pi,\tau)\bigr](z_i, t_i, t_{n+j}) = z_i t_i z_i t_{n+j}, \\ \mathbf{w}(z_{n+j}, t_i, t_{n+j}) &= \bigl[\mathbf{w}_n(\pi,\tau)\bigr](z_{n+j}, t_i, t_{n+j}) = t_i z_{n+j} t_{n+j} z_{n+j}. \end{aligned}$$

Suppose that $\ell_2(\mathbf{w}, z_i) < \ell_1(\mathbf{w}, z_{n+j})$. Then observations given in the previous paragraph imply that $\mathbf{w}(z_i, z_{n+j}, t_i, t_{n+j}) = z_i t_i z_i z_{n+j} t_{n+j} z_{n+j}$. Since the word $xzxyty$ is an isoterm for $\mathbf{X}$, we have

$$\big[\mathbf{w}_n(\pi,\tau)\big](z_i, z_{n+j}, t_i, t_{n+j}) = z_i t_i z_i z_{n+j} t_{n+j} z_{n+j} = \mathbf{w}(z_i, z_{n+j}, t_i, t_{n+j}),$$

whence $\ell_2(\mathbf{w}_n(\pi,\tau), z_i) < \ell_1(\mathbf{w}_n(\pi,\tau), z_{n+j})$.

Suppose now that $\ell_2\big(\mathbf{w}_n(\pi,\tau), z_i\big) < \ell_1\big(\mathbf{w}_n(\pi,\tau), z_{n+j}\big)$. Then

$$\big[\mathbf{w}_n(\pi,\tau)\big](z_i, z_{n+j}, t_i, t_{n+j}) = z_i t_i z_i z_{n+j} t_{n+j} z_{n+j}.$$

Now we apply the fact that $xzxyty$ is an isoterm for $\mathbf{X}$ again and obtain

$$\mathbf{w}(z_i, z_{n+j}, t_i, t_{n+j}) = \big[\mathbf{w}_n(\pi,\tau)\big](z_i, z_{n+j}, t_i, t_{n+j}) = z_i t_i z_i z_{n+j} t_{n+j} z_{n+j},$$

whence $\ell_2(\mathbf{w}, z_i) < \ell_1(\mathbf{w}, z_{n+j})$.

The claim (4.21) is proved. Then

$$\mathbf{w}_x = \bigg(\prod_{i=1}^{n} z_i t_i\bigg)\bigg(\prod_{i=1}^{n} z_{\pi(i)} z_{n+\tau(i)}\bigg)\bigg(\prod_{i=n+1}^{2n} t_i z_i\bigg).$$

Being a subvariety of $\mathbf{O}$, the variety $\mathbf{X}$ satisfies the identities $xyxzx \approx x^2yz \approx yzx^2$. Therefore, we can assume that $\mathrm{occ}_x(\mathbf{w}) = 2$ for any $x \in \mathrm{con}(\mathbf{w})$. So,

$$\mathbf{w} = \bigg(\prod_{i=1}^{n} \mathbf{p}_{2i-1} z_i \mathbf{p}_{2i} t_i\bigg)\mathbf{q}_0\bigg(\prod_{i=1}^{n} z_{\pi(i)} \mathbf{q}_{2i-1} z_{n+\tau(i)} \mathbf{q}_{2i}\bigg)\bigg(\prod_{i=n+1}^{2n} t_i \mathbf{r}_{2i-2n-1} z_i \mathbf{r}_{2i-2n}\bigg)$$

where

$$\bigg(\prod_{i=1}^{2n} \mathbf{p}_i\bigg)\bigg(\prod_{i=0}^{2n} \mathbf{q}_i\bigg)\bigg(\prod_{i=1}^{2n} \mathbf{r}_i\bigg) = x^2.$$

Suppose at first that $x \in \mathrm{con}(\mathbf{p}_{2j-1}\mathbf{p}_{2j})$ for some $1 \le j \le n$ and j is the least number with this property. If $\mathbf{p}_{2j-1}\mathbf{p}_{2j} = x$ then

$$\bigg(\prod_{i=2j+1}^{2n} \mathbf{p}_i\bigg)\bigg(\prod_{i=0}^{2n} \mathbf{q}_i\bigg)\bigg(\prod_{i=1}^{2n} \mathbf{r}_i\bigg) = x.$$

It can be easily verified directly that substituting 1 for all letters except x and t_j in the identity (4.20) we obtain here the identity $t_j x^2 \approx x t_j x$. But this is impossible because xzx is an isoterm for $\mathbf{X}$. Therefore, $\mathbf{p}_{2j-1}\mathbf{p}_{2j} = x^2$, i.e., either $\mathbf{p}_{2j-1} = \mathbf{p}_{2j} = x$ or $\mathbf{p}_{2j-1} = x^2$ or $\mathbf{p}_{2j} = x^2$. If $\mathbf{p}_{2j-1} = \mathbf{p}_{2j} = x$ then $\mathbf{X}$ satisfies the identities

$$\begin{aligned}
\mathbf{w}_n(\pi,\tau) \approx \mathbf{w} = {} & \bigg(\prod_{i=1}^{j-1} z_i t_i\bigg) x z_j x t_j \bigg(\prod_{i=j+1}^{n} z_i t_i\bigg)\bigg(\prod_{i=1}^{n} z_{\pi(i)} z_{n+\tau(i)}\bigg)\bigg(\prod_{i=n+1}^{2n} t_i z_i\bigg) \\
\stackrel{\sigma_1}{\approx} {} & \bigg(\prod_{i=1}^{j-1} z_i t_i\bigg) z_j x^2 t_j \bigg(\prod_{i=j+1}^{n} z_i t_i\bigg)\bigg(\prod_{i=1}^{n} z_{\pi(i)} z_{n+\tau(i)}\bigg)\bigg(\prod_{i=n+1}^{2n} t_i z_i\bigg) \\
\stackrel{(4.15)}{\approx} {} & \bigg(\prod_{i=1}^{n} z_i t_i\bigg) x^2 \bigg(\prod_{i=1}^{n} z_{\pi(i)} z_{n+\tau(i)}\bigg)\bigg(\prod_{i=n+1}^{2n} t_i z_i\bigg) \\
= {} & \mathbf{w}'_n(\pi,\tau) = \mathbf{w}_n^{0,0}(\pi,\tau),
\end{aligned}$$

and we are done. If $\mathbf{p}_{2j-1} = x^2$ or $\mathbf{p}_{2j} = x^2$ then we can apply the identity (4.15) and obtain the required conclusion. So, we can assume that $\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_{2n} = \lambda$.

The case when $x \in \operatorname{con}(\mathbf{r}_{2j-1}\mathbf{r}_{2j})$ for some $1 \le j \le n$ can be considered quite analogously to the previous case with the use of the identity σ_2 rather than σ_1.

Finally, let $x \notin \operatorname{con}(\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_{2n})$ and $x \notin \operatorname{con}(\mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_{2n})$. Then $\mathbf{q}_0\mathbf{q}_1\cdots\mathbf{q}_{2n} = x^2$. Note that either $x \notin \operatorname{con}(\mathbf{q}_0)$ or $x \notin \operatorname{con}(\mathbf{q}_{2n})$ because otherwise the identity (4.20) is trivial. Assume without loss of generality that $x \notin \operatorname{con}(\mathbf{q}_0)$, whence $\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_{2n} = x^2$. Let $x \in \operatorname{con}(\mathbf{q}_k)$ and k is the least number with this property. If $\mathbf{q}_k = x^2$ then we can apply the identity (4.15) and obtain the required conclusion. Suppose now that $x \in \operatorname{con}(\mathbf{q}_{\ell+1})$ for some $k \le \ell \le 2n-1$.

Each occurrence of x in $\mathbf{w}$ lies either in a subword like $z_{\pi(i)}xz_{n+\tau(i)}$ or in a subword like $z_{\pi(i)}z_{n+\tau(i)}xz_{\pi(i+1)}z_{n+\tau(i+1)}$. We need to verify that $\mathbf{w}$ is equal in $\mathbf{X}$ to some word which has the same structure as $\mathbf{w}$ but contains only occurrences of x of the second type. If both occurrences of x in $\mathbf{w}$ are of the second type then we are done. Suppose that both occurrences are of the first type. Then the variety $\mathbf{X}$ satisfies the identities

$$\begin{aligned}
\mathbf{w}_n(\pi,\tau) \approx \mathbf{w} = {} & \biggl(\prod_{i=1}^{n} z_it_i\biggr)\biggl(\prod_{i=1}^{k-1} z_{\pi(i)}z_{n+\tau(i)}\biggr)z_{\pi(k)}xz_{n+\tau(k)}\biggl(\prod_{i=k+1}^{\ell} z_{\pi(i)}z_{n+\tau(i)}\biggr)\\
& \cdot\, \underline{z_{\pi(\ell+1)}x}\, z_{n+\tau(\ell+1)}\biggl(\prod_{i=\ell+2}^{n} z_{\pi(i)}z_{n+\tau(i)}\biggr)\biggl(\prod_{i=n+1}^{2n} t_iz_i\biggr)\\
\stackrel{\sigma_2}{\approx} {} & \biggl(\prod_{i=1}^{n} z_it_i\biggr)\biggl(\prod_{i=1}^{k-1} z_{\pi(i)}z_{n+\tau(i)}\biggr)z_{\pi(k)}\,\underline{xz_{n+\tau(k)}}\biggl(\prod_{i=k+1}^{\ell} z_{\pi(i)}z_{n+\tau(i)}\biggr)\\
& \cdot\, x\biggl(\prod_{i=\ell+1}^{n} z_{\pi(i)}z_{n+\tau(i)}\biggr)\biggl(\prod_{i=n+1}^{2n} t_iz_i\biggr)\\
\stackrel{\sigma_1}{\approx} {} & \biggl(\prod_{i=1}^{n} z_it_i\biggr)\biggl(\prod_{i=1}^{k} z_{\pi(i)}z_{n+\tau(i)}\biggr)x\biggl(\prod_{i=k+1}^{\ell} z_{\pi(i)}z_{n+\tau(i)}\biggr)x\\
& \cdot\, \biggl(\prod_{i=\ell+1}^{n} z_{\pi(i)}z_{n+\tau(i)}\biggr)\biggl(\prod_{i=n+1}^{2n} t_iz_i\biggr)\\
= {} & \mathbf{w}_n^{k,\ell}(\pi,\tau)
\end{aligned}$$

(for reader convenience, we underline here pairs of adjacent letters that are transposed by one of the identities σ_1 or σ_2). Finally, if two occurrences of x in $\mathbf{w}$ are of different types, then we can use analogous but simpler arguments. If occurrence of the first type lies in $\mathbf{q}_k$ [in $\mathbf{q}_{\ell+1}$] then it suffices to apply the identity σ_1 [respectively σ_2] only. Thus, in all cases an identity of the form (4.19) holds in $\mathbf{X}$. □

Lemma 4.11. *Let m be a natural number, $0 \le k < \ell < m$, $q = \ell - k$ and $\pi, \tau \in S_m$. Then there are permutations $\rho, \sigma \in S_q$ such that the identity $\mathbf{w}_q(\rho,\sigma) \approx \mathbf{w}'_q(\rho,\sigma)$ implies the identity $\mathbf{w}_m^{k,\ell}(\pi,\tau) \approx \mathbf{w}_m^{k,k}(\pi,\tau)$.*

Proof. For convenience, we put

$$\{z_{\pi(k+1)}, z_{\pi(k+2)}, \dots, z_{\pi(\ell)}\} = \{z_{p_1}, z_{p_2}, \dots, z_{p_q}\}$$

and

$$\{z_{m+\tau(k+1)}, z_{m+\tau(k+2)}, \dots, z_{m+\tau(\ell)}\} = \{z_{r_1}, z_{r_2}, \dots, z_{r_q}\}$$

where $1 \le p_1 < p_2 < \cdots < p_q \le m < r_1 < r_2 < \cdots < r_q \le 2m$. The word $\mathbf{w}_m^{k,\ell}(\pi,\tau)$ has the form

$$\mathbf{u}_0 z_{p_1} \mathbf{u}_1 \cdots z_{p_q} \mathbf{u}_q x z_{\pi(k+1)} z_{m+\tau(k+1)} \cdots z_{\pi(\ell)} z_{m+\tau(\ell)} x \mathbf{u}_{q+1} z_{r_1} \cdots \mathbf{u}_{2q} z_{r_q} \mathbf{u}_{2q+1}$$

where

$$\begin{aligned}
\mathbf{u}_0 &= \prod_{i=1}^{p_1-1} z_i t_i,\\
\mathbf{u}_s &= t_{p_s}\Biggl(\prod_{i=p_s+1}^{p_{s+1}-1} z_i t_i\Biggr) \text{ for all } 1 \le s < q,\\
\mathbf{u}_q &= t_{p_q}\Biggl(\prod_{i=p_q+1}^{m} z_i t_i\Biggr)\Biggl(\prod_{i=1}^{k} z_{\pi(i)} z_{m+\tau(i)}\Biggr),\\
\mathbf{u}_{q+1} &= \Biggl(\prod_{i=\ell+1}^{m} z_{\pi(i)} z_{m+\tau(i)}\Biggr)\Biggl(\prod_{i=m+1}^{r_1-1} t_i z_i\Biggr) t_{r_1},\\
\mathbf{u}_{q+1+s} &= \Biggl(\prod_{i=r_{s-1}+1}^{r_s-1} t_i z_i\Biggr) t_{r_s} \text{ for all } 1 \le s < q,\\
\mathbf{u}_{2q+1} &= \prod_{i=r_q+1}^{2m} t_i z_i.
\end{aligned}$$

We are going to rename all letters except x in the word $\mathbf{w}_m^{k,\ell}(\pi,\tau)$. First, we rename all letters from the set

$$\mathrm{con}\bigl(\mathbf{w}_m^{k,\ell}(\pi,\tau)\bigr) \setminus \{x, z_{p_1}, z_{p_2} \dots, z_{p_q}, z_{r_1}, z_{r_2}, \dots, z_{r_q}\}$$

by some pairwise different letters that do not occur in $\mathbf{w}_m^{k,\ell}(\pi,\tau)$. Further, we perform the substitution

$$(z_{p_1}, z_{p_2}, \dots, z_{p_q}, z_{r_1}, z_{r_2}, \dots, z_{r_q}) \mapsto (z_1, z_2, \dots, z_q, z_{q+1}, z_{q+2}, \dots, z_{2q}).$$

As a result, we get the word

$$\mathbf{u}' = \mathbf{u}_0' z_1 \mathbf{u}_1' \cdots z_q \mathbf{u}_q' x z_{\rho(1)} z_{q+\sigma(1)} \cdots z_{\rho(q)} z_{q+\sigma(q)} x \mathbf{u}_{q+1}' z_{q+1} \cdots \mathbf{u}_{2q}' z_{2q} \mathbf{u}_{2q+1}'$$

for appropriate permutations $\rho, \sigma \in S_q$ and some words $\mathbf{u}_0', \mathbf{u}_1', \dots, \mathbf{u}_{2q+1}'$.

Now we can perform the substitution $(t_1, t_2, \dots, t_{2q}) \mapsto (\mathbf{u}_1', \mathbf{u}_2', \dots, \mathbf{u}_{2q}')$ in the identity $\mathbf{w}_q(\rho,\sigma) \approx \mathbf{w}_q'(\rho,\sigma)$. We get the identity

$$\begin{aligned}
&z_1 \mathbf{u}_1' \cdots z_q \mathbf{u}_q' x z_{\rho(1)} z_{q+\sigma(1)} \cdots z_{\rho(q)} z_{q+\sigma(q)} x \mathbf{u}_{q+1}' z_{q+1} \cdots \mathbf{u}_{2q}' z_{2q}\\
&\approx z_1 \mathbf{u}_1' \cdots z_q \mathbf{u}_q' x^2 z_{\rho(1)} z_{q+\sigma(1)} \cdots z_{\rho(q)} z_{q+\sigma(q)} \mathbf{u}_{q+1}' z_{q+1} \cdots \mathbf{u}_{2q}' z_{2q}.
\end{aligned}$$

We apply this identity to the word $\mathbf{u}'$ and obtain the identity

$$\mathbf{u}' \approx \mathbf{u}_0' z_1 \mathbf{u}_1' \cdots z_q \mathbf{u}_q' x^2 z_{\rho(1)} z_{q+\sigma(1)} \cdots z_{\rho(q)} z_{q+\sigma(q)} \mathbf{u}_{q+1}' z_{q+1} \cdots \mathbf{u}_{2q}' z_{2q} \mathbf{u}_{2q+1}'.$$

Now we implement in this identity the renaming of letters, the reverse of the one described above. Then we obtain the identity

$$\mathbf{w}_m^{k,\ell}(\pi,\tau) \approx \mathbf{u}_0 z_{p_1}\mathbf{u}_1 \cdots z_{p_q}\mathbf{u}_q x^2 z_{\pi(k+1)} z_{m+\tau(k+1)} \cdots z_{\pi(\ell)} z_{m+\tau(\ell)} x \cdot \mathbf{u}_{q+1} z_{r_1} \cdots \mathbf{u}_{2q} z_{r_q}\mathbf{u}_{2q+1} = \mathbf{w}_m^{k,k}(\pi,\tau).$$

The lemma is proved. □

Now we are well prepared to complete the proof of necessity of Theorem 1.1. Recall that we reduce our considerations to the case when $\mathbf{L} \subseteq \mathbf{V} \subseteq \mathbf{O}$. We denote by $\mathcal{K}$ the class of all varieties of the form $\operatorname{var} S\big(\mathbf{w}_n(\pi,\tau)\big)$ where $n \in \mathbb{N}$ and $\pi,\tau \in S_n$. It is clear that $\mathbf{L} \subseteq \mathbf{X}$ whenever $\mathbf{X} \in \mathcal{K}$. We use this fact below without references. Let $\mathbf{X} \in \mathcal{K}$. We are going to verify that then the variety $\mathbf{X}$ contains at least two incomparable subvarieties from the class $\mathcal{K}$.

For an arbitrary permutation $\xi \in S_n$, we define the following two permutations from S_{n+2}:

$$\xi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n+2 \\ \xi(1)+2 & 1 & 2 & \xi(2)+2 & \xi(3)+2 & \dots & \xi(n)+2 \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n+2 \\ \xi(1)+2 & 2 & 1 & \xi(2)+2 & \xi(3)+2 & \dots & \xi(n)+2 \end{pmatrix}.$$

We have $\mathbf{X} = \operatorname{var} S\big(\mathbf{w}_n(\pi,\tau)\big)$ for some n, π and τ. Let $T_1 = S_{n+2}(\pi_1,\tau_1)$ and $T_2 = S_{n+2}(\pi_2,\tau_1)$. If $T_1 \notin \mathbf{X}$ then Lemma 4.10 allows us to assume without loss of generality that $\mathbf{X}$ satisfies a non-trivial identity of the form $\mathbf{w}_{n+2}(\pi_1,\tau_1) \approx \mathbf{w}_{n+2}^{k,\ell}(\pi_1,\tau_1)$ for some $1 \le k \le \ell \le n+2$. Then we substitute

- 1 for z_1, z_2, z_{n+3}, z_{n+4}, t_1, t_2, t_{n+3} and t_{n+4},
- z_{i-2} for z_i whenever $3 \le i \le n+2$ and z_{i-4} for z_i whenever $n+5 \le i \le 2n+4$,
- t_{i-2} for t_i whenever $3 \le i \le n+2$ and t_{i-4} for t_i whenever $n+5 \le i \le 2n+4$

in $\mathbf{w}_{n+2}(\pi_1,\tau_1) \approx \mathbf{w}_{n+2}^{k,\ell}(\pi_1,\tau_1)$. Then we obtain $\mathbf{X}$ satisfies the identity $\mathbf{w}_n(\pi,\tau) \approx \mathbf{w}_n^{s,t}(\pi,\tau)$ where $s = 1$ whenever $k \le 3$ and $s = k-2$ whenever $k > 3$, while $t = 1$ whenever $\ell \le 3$ and $t = \ell - 2$ whenever $\ell > 3$. Since $s \ge 1$, this identity is non-trivial. We obtain a contradiction with the fact that $\mathbf{X} = \operatorname{var} S\big(\mathbf{w}_n(\pi,\tau)\big)$ and Lemma 2.3. Thus, we have proved that $T_1 \in \mathbf{X}$. Analogously, $T_2 \in \mathbf{X}$.

Suppose that $T_1 \in \operatorname{var} T_2$. Then Lemma 2.3 applies and we conclude that the word $\mathbf{w}_{n+2}(\pi_1,\tau_1)$ is an isoterm for $\operatorname{var} T_2$. At the same time, it is easy to verify that $\operatorname{var} T_2$ satisfies $\mathbf{w}_{n+2}(\pi_1,\tau_1) \approx \mathbf{w}'_{n+2}(\pi_1,\tau_1)$. Therefore, $\operatorname{var} T_1 \nsubseteq \operatorname{var} T_2$. Analogously, $\operatorname{var} T_2 \nsubseteq \operatorname{var} T_1$. We see that the varieties $\operatorname{var} T_1$ and $\operatorname{var} T_2$ are incomparable. Besides that, it is evident that these two varieties lie in $\mathcal{K}$.

Thus, if $\mathbf{X} = \operatorname{var} S\big(\mathbf{w}_n(\pi,\tau)\big)$ for some n, π and τ then the variety $\mathbf{X}$ is not chain. Therefore, $S\big(\mathbf{w}_n(\pi,\tau)\big) \notin \mathbf{V}$ for all n, π and τ. For an arbitrary n, we denote the trivial permutation from S_n by ε. Then $S\big(\mathbf{w}_1(\varepsilon,\varepsilon)\big) \notin \mathbf{V}$. According to Lemma 4.10, $\mathbf{V}$ satisfies a non-trivial identity of the form $\mathbf{w}_1(\varepsilon,\varepsilon) \approx \mathbf{w}_1^{k,\ell}(\varepsilon,\varepsilon)$ where $0 \le k \le \ell \le 1$. Since $\mathbf{w}_1^{0,0}(\varepsilon,\varepsilon) = \mathbf{w}'_1(\varepsilon,\varepsilon)$, $\mathbf{w}_1^{0,1}(\varepsilon,\varepsilon) = \mathbf{w}_1(\varepsilon,\varepsilon)$ and the identity $\mathbf{w}_1(\varepsilon,\varepsilon) \approx \mathbf{w}_1^{k,\ell}(\varepsilon,\varepsilon)$ is non-trivial, we have $\mathbf{V}$ satisfies one of the identities $\mathbf{w}_1(\varepsilon,\varepsilon) \approx \mathbf{w}_1'(\varepsilon,\varepsilon)$ or $\mathbf{w}_1(\varepsilon,\varepsilon) \approx \mathbf{w}_1^{1,1}(\varepsilon,\varepsilon)$. Clearly, the latter identity together with (4.15) implies the former one. Thus, $\mathbf{V}$ satisfies the identity $\mathbf{w}_1(\varepsilon,\varepsilon) \approx \mathbf{w}_1'(\varepsilon,\varepsilon)$.

Thus, there is a number n such that $\mathbf{V}$ satisfies the identities of the form (4.12) for all $\pi,\tau \in S_n$ (for instance, $n = 1$). We are going to verify that an arbitrary n possesses this property. Arguing by contradiction, we suppose that the above-mentioned claim is true for $1, 2, \dots, n$ but is false for $n + 1$. Let $\pi_1,\tau_1 \in S_{n+1}$. Since $S\big(\mathbf{w}_{n+1}(\pi_1,\tau_1)\big) \notin \mathbf{V}$, Lemma 4.10 implies that $\mathbf{V}$ satisfies an identity of the form $\mathbf{w}_{n+1}(\pi_1,\tau_1) \approx \mathbf{w}_{n+1}^{k,\ell}(\pi_1,\tau_1)$ for some $0 \le k \le \ell < n + 1$. Suppose that $k < \ell$. Then Lemma 4.11 with $m = n + 1$, $\pi = \pi_1$ and $\tau = \tau_1$ applies and we conclude that there exist permutations $\rho,\sigma \in S_{\ell-k}$ such that the identity $\mathbf{w}_{\ell-k}(\rho,\sigma) \approx \mathbf{w}_{\ell-k}'(\rho,\sigma)$ implies the identity $\mathbf{w}_{n+1}^{k,\ell}(\pi_1,\tau_1) \approx \mathbf{w}_{n+1}^{k,k}(\pi_1,\tau_1)$. The first of these identities holds in $\mathbf{V}$ because $\ell - k \le n$. Thus, in any case $\mathbf{V}$ satisfies the identity $\mathbf{w}_{n+1}(\pi_1,\tau_1) \approx \mathbf{w}_{n+1}^{k,k}(\pi_1,\tau_1)$. Note that $\mathbf{w}_{n+1}^{k,k}(\pi_1,\tau_1) \stackrel{(4.15)}{\approx} \mathbf{w}_{n+1}^{0,0}(\pi_1,\tau_1) = \mathbf{w}_{n+1}'(\pi_1,\tau_1)$. Therefore, the identity $\mathbf{w}_{n+1}(\pi_1,\tau_1) \approx \mathbf{w}_{n+1}'(\pi_1,\tau_1)$ holds in $\mathbf{V}$. This is true for any $\pi_1,\tau_1 \in S_{n+1}$. This contradicts the choice of n. So, the variety $\mathbf{V}$ satisfies the identities of the form (4.12) for all n and $\pi,\tau \in S_n$, whence $\mathbf{V} = \mathbf{L}$.

We have thus completed the proof of the "only if" part of Theorem 1.1.

5. The proof of the "if" part: all varieties except $\mathbf{K}$

In this and the following sections we are going to prove that if $\mathbf{X}$ is a subvariety of one of the varieties listed in Theorem 1.1 then $\mathbf{X}$ is a chain variety. Since the property of being a chain variety is inherited by subvarieties, we can assume that $\mathbf{X}$ coincides with one of the varieties listed in Theorem 1.1. By symmetry, we can exclude from considerations the varieties $\overleftarrow{\mathbf{K}}$ and $\overleftarrow{\mathbf{N}}$. Thus, it suffices to verify that $\mathbf{C}_n$, $\mathbf{D}$, $\mathbf{K}$, $\mathbf{L}$, $\mathbf{LRB}$, $\mathbf{N}$ and $\mathbf{RRB}$ are chain varieties. Here we consider all these varieties except the variety $\mathbf{K}$, the last variety will be examined in the following section.

Lemmas 2.8 and 2.9(ii) and Corollary 4.7 immediately imply that the varieties $\mathbf{D}$, $\mathbf{L}$, $\mathbf{LRB}$ and $\mathbf{RRB}$ are chain varieties.

Proposition 5.1. *The lattice $L(\mathbf{C}_n)$ is the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{C}_3 \subset \cdots \subset \mathbf{C}_n$.*

Proof. Let $\mathbf{V} \subseteq \mathbf{C}_n$. Then $\mathbf{V}$ is commutative and aperiodic. If $\mathbf{C}_2 \nsubseteq \mathbf{V}$ then $\mathbf{V}$ is completely regular by Corollary 2.6. Then $\mathbf{V} \subseteq \mathbf{SL}$, whence $\mathbf{V}$ coincides with either $\mathbf{T}$ or $\mathbf{SL}$. It remains to verify that if $\mathbf{C}_2 \subseteq \mathbf{V} \subseteq \mathbf{C}_n$ then $\mathbf{V} = \mathbf{C}_s$ for some $2 \le s \le n$. We will use induction on n. If $n = 2$ then the assertion is obvious. Let now $n > 2$. Suppose that $\mathbf{V} \ne \mathbf{C}_n$. Then Lemma 2.5 implies that $\mathbf{V}$ satisfies the identity $x^{n-1} \approx x^n$, whence $\mathbf{V} \subseteq \mathbf{C}_{n-1}$. By the induction assumption, $\mathbf{V} = \mathbf{C}_s$ for some $2 \le s \le n - 1$. □

It remains to consider the variety $\mathbf{N}$.

Proposition 5.2. *The lattice* $L(\mathbf{N})$ *is the chain* $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \mathbf{M} \subset \mathbf{N}$.

Proof. First of all, we are going to check that the variety $\mathbf{N}$ satisfies identities of the form (4.11) for all n, m and $\theta \in S_{n+m}$. Indeed, $\mathbf{w}_{n,m}(\theta) = \mathbf{p}x\mathbf{q}x\mathbf{r}$ where $\mathbf{p} = z_1t_1 \cdots z_nt_n$, $\mathbf{q} = z_{\theta(1)} \cdots z_{\theta(n+m)}$ and $\mathbf{r} = t_{n+1}z_{n+1} \cdots t_{n+m}z_{n+m}$. Suppose at first that $\theta(n+m) \le n$. Then

$$\mathbf{w}_{n,m}(\theta) = z_1t_1 \cdots \overset{(1)}{z_{\theta(n+m)}}\, t_{\theta(n+m)} \cdots z_nt_n \overset{(1)}{x}\, z_{\theta(1)} \cdots \overset{(2)}{z_{\theta(n+m)}} \overset{(2)}{x}\, \mathbf{r}.$$

We see that the second occurrences of the letters $z_{\theta(n+m)}$ and x in $\mathbf{w}_{n,m}(\theta)$ are each adjacent to other. The identity σ_2 allows us to swap these occurrences. In other words,

$$\mathbf{w}_{n,m}(\theta) \overset{\sigma_2}{\approx} \mathbf{p}xz_{\theta(1)} \cdots z_{\theta(n+m-1)}xz_{\theta(n+m)}\mathbf{r}.$$

Suppose now that $\theta(n+m) > n$. Then

$$\mathbf{w}_{n,m}(\theta) = \mathbf{p} \overset{(1)}{x}\, z_{\theta(1)} \cdots \overset{(1)}{z_{\theta(n+m)}} \overset{(2)}{x}\, t_{n+1}z_{n+1} \cdots t_{\theta(n+m)} \overset{(2)}{z_{\theta(n+m)}} \cdots t_{n+m}z_{n+m}.$$

We see that first occurrence of $z_{\theta(n+m)}$ and second occurrence x in $\mathbf{w}_{n,m}(\theta)$ are each adjacent to other. The identity γ_1 allows us to transpose these occurrences. In other words,

$$\mathbf{w}_{n,m}(\theta) \overset{\gamma_1}{\approx} \mathbf{p}xz_{\theta(1)} \cdots z_{\theta(n+m-1)}xz_{\theta(n+m)}\mathbf{r}.$$

We see that in any case the identity

$$\mathbf{w}_{n,m}(\theta) \approx \mathbf{p}xz_{\theta(1)} \cdots z_{\theta(n+m-1)}xz_{\theta(n+m)}\mathbf{r}$$

holds in $\mathbf{N}$. Analogous arguments show that we can successively swap second occurrence of x with $z_{\theta(n+m-1)}$, $z_{\theta(n+m-2)}$, ..., $z_{\theta(1)}$ and obtain $\mathbf{N}$ satisfies the identities

$$\mathbf{w}_{n,m}(\theta) \approx \mathbf{p}x^2z_{\theta(1)} \cdots z_{\theta(n+m)}\mathbf{r} = \mathbf{p}x^2\mathbf{q}\mathbf{r} = \mathbf{w}'_{n,m}(\theta).$$

Therefore, we can apply Lemma 4.5 below.

Suppose that $\mathbf{V} \subseteq \mathbf{N}$. If $\mathbf{M} \nsubseteq \mathbf{V}$ then $\mathbf{V} \subseteq \mathbf{D}$ by Lemma 4.9(ii). Therefore, in view of Lemma 2.8, it suffices to consider the case when $\mathbf{M} \subseteq \mathbf{V}$. We need to verify that $\mathbf{V}$ coincides with one of the varieties $\mathbf{M}$ or $\mathbf{N}$. Let $\mathbf{u} \approx \mathbf{v}$ be an arbitrary identity that holds in the variety $\mathbf{V}$. Our aim is to verify that $\mathbf{u} \approx \mathbf{v}$ either implies the identity α_1 or holds in the variety $\mathbf{N}$. Proposition 2.2 implies that $\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v})$. Let $\mathrm{sim}(\mathbf{u}) = \{t_0, t_1, \dots, t_m\}$. As in the proof of Lemma 4.6, we can assume that

$$\mathbf{u} = t_0\mathbf{a}_1t_1\mathbf{a}_2t_2 \cdots t_{m-1}\mathbf{a}_mt_m \text{ and } \mathbf{v} = t_0\mathbf{b}_1t_1\mathbf{b}_2t_2 \cdots t_{m-1}\mathbf{b}_mt_m$$

for some possibly empty words $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ and $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$.

Let x be a letter multiple in $\mathbf{u}$ and $\mathbf{u}(x, y) \ne xyx$ for any letter y. By Lemma 4.5, the variety $\mathbf{V}$ satisfies the identity (4.16). Obviously, $\mathbf{C}_2 \subseteq \mathbf{M} \subseteq \mathbf{V}$, whence $x \in \mathrm{mul}(\mathbf{v})$ by Proposition 2.2. Since $\mathbf{D}_2 \subseteq \mathbf{M} \subseteq \mathbf{V}$, we apply Lemmas 2.3 and 2.7 and conclude that the word xyx is an isoterm for $\mathbf{V}$. Therefore, $\mathbf{v}(x, y) \ne xyx$ for any letter y. We apply Lemma 4.5 again and conclude that the identity $\mathbf{v} \approx x^2\mathbf{v}_x$ holds in $\mathbf{V}$. Thus, the identity $\mathbf{u} \approx \mathbf{v}$ follows from the identities (4.16), $\mathbf{v} \approx x^2\mathbf{v}_x$ and $\mathbf{u}_x \approx \mathbf{v}_x$. So, we can remove from $\mathbf{u} \approx \mathbf{v}$ all multiple letters x such that $\mathbf{u}(x, y) \neq xyx$ for any y. In other words, we may assume without loss of generality that for any letter $x \in \mathrm{mul}(\mathbf{u})$ there is a letter y such that $\mathbf{u}(x, y) = xyx = \mathbf{v}(x, y)$. In particular, $\mathrm{occ}_x(\mathbf{u}), \mathrm{occ}_x(\mathbf{v}) \leq 2$ for any letter x.

Let $0 \leq i \leq m-1$. Then $\mathbf{u} = \mathbf{w}_1 t_i \mathbf{a}_{i+1} t_{i+1} \mathbf{w}_2$ where

$$\mathbf{w}_1 = \begin{cases} t_0 \mathbf{a}_1 t_1 \cdots t_{i-1} \mathbf{a}_i & \text{if } 0 < i \leq m-1, \\ \lambda & \text{if } i = 0 \end{cases}$$

and

$$\mathbf{w}_2 = \begin{cases} \mathbf{a}_{i+2} t_{i+2} \cdots \mathbf{a}_m t_m & \text{if } 0 \leq i < m-1, \\ \lambda & \text{if } i = m-1. \end{cases}$$

Analogously, $\mathbf{v} = \mathbf{w}_1' t_i \mathbf{b}_{i+1} t_{i+1} \mathbf{w}_2'$ where

$$\mathbf{w}_1' = \begin{cases} t_0 \mathbf{b}_1 t_1 \cdots t_{i-1} \mathbf{b}_i & \text{if } 0 < i \leq m-1, \\ \lambda & \text{if } i = 0 \end{cases}$$

and

$$\mathbf{w}_2' = \begin{cases} \mathbf{b}_{i+2} t_{i+2} \cdots \mathbf{b}_m t_m & \text{if } 0 \leq i < m-1, \\ \lambda & \text{if } i = m-1. \end{cases}$$

Suppose that the word $\mathbf{a}_{i+1}$ contains the subword $\mathbf{d} = x_i x_j$ where $x_i \in \mathrm{con}(\mathbf{w}_1)$ and $x_j \in \mathrm{con}(\mathbf{w}_2)$. The occurrence of the letter x_i in the word $\mathbf{d}$ is second occurrence of x_i in $\mathbf{u}$, while the occurrence of the letter x_j in the word $\mathbf{d}$ is first occurrence of x_j in $\mathbf{u}$. The identity γ_1 allows us to swap these two occurrences. Therefore, the variety $\mathbf{N}$ satisfies the identity $\mathbf{u} \approx \mathbf{w}_1 t_i \mathbf{p}_1 \mathbf{q}_1 t_{i+1} \mathbf{w}_2$ where $\mathrm{con}(\mathbf{p}_1) \subseteq \mathrm{con}(\mathbf{w}_2)$ and $\mathrm{con}(\mathbf{q}_1) \subseteq \mathrm{con}(\mathbf{w}_1)$. Analogously, we can prove that $\mathbf{N}$ satisfies $\mathbf{v} \approx \mathbf{w}_1' t_i \mathbf{p}_2 \mathbf{q}_2 t_{i+1} \mathbf{w}_2'$ where $\mathrm{con}(\mathbf{p}_2) \subseteq \mathrm{con}(\mathbf{w}_2')$ and $\mathrm{con}(\mathbf{q}_2) \subseteq \mathrm{con}(\mathbf{w}_1')$.

We are going to verify that $\mathrm{con}(\mathbf{p}_1) = \mathrm{con}(\mathbf{p}_2)$ and $\mathrm{con}(\mathbf{q}_1) = \mathrm{con}(\mathbf{q}_2)$. Let $x \in \mathrm{con}(\mathbf{p}_1)$. Then $\mathbf{u}(x, t_{i+1}) = x t_{i+1} x$. Therefore, $\mathbf{v}(x, t_{i+1}) = x t_{i+1} x$. This means that $x \in \mathrm{con}(\mathbf{w}_1' \mathbf{p}_2 \mathbf{q}_2)$ and $x \in \mathrm{con}(\mathbf{w}_2')$. If $x \in \mathrm{con}(\mathbf{q}_2)$ then $x \in \mathrm{con}(\mathbf{w}_1')$ as well, whence $\mathrm{occ}_x(\mathbf{v}) \geq 3$. Therefore, $x \notin \mathrm{con}(\mathbf{q}_2)$. Note that $\mathbf{u}(x, t_i) = t_i x^2$. Therefore, $\mathbf{v}(x, t_i) \neq x t_i x$, whence $x \notin \mathrm{con}(\mathbf{w}_1')$. We see that $x \in \mathrm{con}(\mathbf{p}_2)$. We have just proved that $\mathrm{con}(\mathbf{p}_1) \subseteq \mathrm{con}(\mathbf{p}_2)$. By symmetry, $\mathrm{con}(\mathbf{p}_2) \subseteq \mathrm{con}(\mathbf{p}_1)$, whence $\mathrm{con}(\mathbf{p}_1) = \mathrm{con}(\mathbf{p}_2)$. Analogous arguments imply that $\mathrm{con}(\mathbf{q}_1) = \mathrm{con}(\mathbf{q}_2)$.

Therefore, $\mathbf{p}_1 = x_1 x_2 \cdots x_k$ and $\mathbf{p}_2 = x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(k)}$ for some letters $x_1, x_2, \ldots, x_k \in \mathrm{con}(\mathbf{w}_2) \cap \mathrm{con}(\mathbf{w}_2')$ and some permutation $\pi \in S_k$, whence $\mathbf{N}$ satisfies the identities

$$\mathbf{u} \approx \mathbf{w}_1 t_i x_1 x_2 \cdots x_k \mathbf{q}_1 t_{i+1} \mathbf{w}_2 \text{ and } \mathbf{v} \approx \mathbf{w}_1' t_i x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(k)} \mathbf{q}_2 t_{i+1} \mathbf{w}_2'.$$

Then the identity

$$\mathbf{w}_1 t_i x_1 x_2 \cdots x_k \mathbf{q}_1 t_{i+1} \mathbf{w}_2 \approx \mathbf{w}_1' t_i x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(k)} t_{i+1} \mathbf{q}_2 \mathbf{w}_2' \tag{5.1}$$

holds in $\mathbf{V}$.

Suppose that the permutation π is non-trivial. Then there are j and ℓ such that $j < \ell$ but $\pi(j) > \pi(\ell)$. Substituting 1 for all letters occurring in the identity (5.1) except x_j, x_ℓ and t_{i+1}, we obtain the identity $x_j x_\ell t_{i+1} \mathbf{s} \approx x_\ell x_j t_{i+1} \mathbf{s}'$

where $\mathbf{s}, \mathbf{s}' \in \{x_j x_\ell, x_\ell x_j\}$. Now we apply the identity σ_2 and get $x_j x_\ell t_{i+1} x_j x_\ell \approx x_\ell x_j t_{i+1} x_j x_\ell$. The last identity is nothing but α_1 (up to renaming of letters). So, if the permutation π is non-trivial then $\mathbf{V}$ satisfies α_1. This means that $\mathbf{V} \subseteq \mathbf{M}$, whence $\mathbf{V} = \mathbf{M}$. In other words, if $\mathbf{p}_1 \ne \mathbf{p}_2$ then $\mathbf{V} = \mathbf{M}$.

Let now $\mathbf{p}_1 = \mathbf{p}_2$. The words $\mathbf{q}_1$ and $\mathbf{q}_2$ are linear and $\mathrm{con}(\mathbf{q}_1) = \mathrm{con}(\mathbf{q}_2) \subseteq \mathrm{con}(\mathbf{w}_1) \cap \mathrm{con}(\mathbf{w}_1')$. Thus, if some letter z occurs in $\mathrm{con}(\mathbf{q}_1)$ then this occurrence is the second occurrence of z in the word $\mathbf{u}$. Hence the identity σ_2 allows us to reorder letters in $\mathbf{q}_1$ in an arbitrary way. Therefore, we can replace $\mathbf{q}_1$ by $\mathbf{q}_2$ in $\mathbf{u}$, and the word we obtain should be equal to $\mathbf{u}$ in $\mathbf{N}$. Thus, $\mathbf{N}$ satisfies the identities

$$\mathbf{u} = \mathbf{w}_1 t_i \mathbf{a}_{i+1} t_{i+1} \mathbf{w}_2 \approx \mathbf{w}_1 t_i \mathbf{p}_1 \mathbf{q}_1 t_{i+1} \mathbf{w}_2 \approx \mathbf{w}_1 t_i \mathbf{p}_2 \mathbf{q}_2 t_{i+1} \mathbf{w}_2 \approx \mathbf{w}_1 t_i \mathbf{b}_{i+1} t_{i+1} \mathbf{w}_2.$$

This is true for all $i = 0, 1, \ldots, m-1$. Therefore, $\mathbf{N}$ satisfies the identities

$$\mathbf{u} = t_0 \mathbf{a}_1 t_1 \mathbf{a}_2 t_2 \cdots t_{m-1} \mathbf{a}_m t_m \approx t_0 \mathbf{b}_1 t_1 \mathbf{b}_2 t_2 \cdots t_{m-1} \mathbf{b}_m t_m = \mathbf{v}.$$

The proposition is proved. □

6. The proof of the "if" part: the variety K

Here we are going to verify that $\mathbf{K}$ is a chain variety. This case is much more complex than all the ones discussed in the previous section, taken together, and its consideration will be many times longer. For reader convenience, we divide this section into four subsections.

6.1. **Reduction to the interval $[\mathbf{E}, \mathbf{K}]$.** We fix notation for the following identity system:

$$\Phi = \{xyx \approx xyx^2,\ x^2y^2 \approx y^2x^2,\ x^2y \approx x^2yx\}.$$

Note that $\mathbf{K} = \mathrm{var}\,\Phi$. For any $s \in \mathbb{N}$ and $1 \le q \le s$, we put

$$\mathbf{b}_{s,q} = x_{s-1} x_s x_{s-2} x_{s-1} \cdots x_{q-1} x_q.$$

For brevity, we will write $\mathbf{b}_s$ rather than $\mathbf{b}_{s,1}$. We put also $\mathbf{b}_0 = \lambda$ for convenience. We introduce the following four countably infinite series of identities:

$$\alpha_k:\ x_k y_k x_{k-1} x_k y_k \mathbf{b}_{k-1} \approx y_k x_k x_{k-1} x_k y_k \mathbf{b}_{k-1},$$
$$\beta_k:\ x x_k x \mathbf{b}_k \approx x_k x^2 \mathbf{b}_k,$$
$$\gamma_k:\ y_1 y_0 x_k y_1 \mathbf{b}_k \approx y_1 y_0 y_1 x_k \mathbf{b}_k,$$
$$\delta_k^m:\ y_{m+1} y_m x_k y_{m+1} \mathbf{b}_{k,m} y_m \mathbf{b}_{m-1} \approx y_{m+1} y_m y_{m+1} x_k \mathbf{b}_{k,m} y_m \mathbf{b}_{m-1}$$

where $k \in \mathbb{N}$ and $1 \le m \le k$. Note that the identities α_1 and γ_1 have already appeared above. We define the following four countably infinite series of varieties:

$$\mathbf{F}_k = \mathrm{var}\{\Phi, \alpha_k\},\ \mathbf{H}_k = \mathrm{var}\{\Phi, \beta_k\},\ \mathbf{I}_k = \mathrm{var}\{\Phi, \gamma_k\},\ \mathbf{J}_k^m = \mathrm{var}\{\Phi, \delta_k^m\}.$$

In this section we are going to verify the following

Proposition 6.1. 1) *The lattice $L(\mathbf{K})$ is the set-theoretical union of the lattice $L(\mathbf{E})$ and the interval $[\mathbf{E}, \mathbf{K}]$.*
2) *The lattice $L(\mathbf{E})$ is the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{E}$.*

3) *If* $\mathbf{X}$ *is a monoid variety such that* $\mathbf{E} \subset \mathbf{X} \subset \mathbf{K}$ *then* $\mathbf{X}$ *belongs to the interval* $[\mathbf{F}_k, \mathbf{F}_{k+1}]$ *for some* k*.*
4) *The interval* $[\mathbf{F}_k, \mathbf{F}_{k+1}]$ *is the chain*

$$\mathbf{F}_k \subset \mathbf{H}_k \subset \mathbf{I}_k \subset \mathbf{J}_k^1 \subset \mathbf{J}_k^2 \subset \cdots \subset \mathbf{J}_k^k \subset \mathbf{F}_{k+1}. \tag{6.1}$$

This proposition immediately implies that the lattice $L(\mathbf{K})$ is the following chain:

$$\begin{aligned}\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}_1 \subset \mathbf{E} &\subset \mathbf{F}_1 \subset \mathbf{H}_1 \subset \mathbf{I}_1 \subset \mathbf{J}_1^1\\ &\subset \mathbf{F}_2 \subset \mathbf{H}_2 \subset \mathbf{I}_2 \subset \mathbf{J}_2^1 \subset \mathbf{J}_2^2\\ &\vdots\\ &\subset \mathbf{F}_k \subset \mathbf{H}_k \subset \mathbf{I}_k \subset \mathbf{J}_k^1 \subset \mathbf{J}_k^2 \subset \cdots \subset \mathbf{J}_k^k\\ &\vdots\\ &\subset \mathbf{K}.\end{aligned}$$

In the remainder of this subsection we verify the claim 1) of Proposition 6.1. The claim 2) follows from Lemma 2.10(ii). The claims 3) and 4) are proved in Subsections 6.3 and 6.4 respectively. Subsection 6.2 contains auxiliary assertions.

Let $\mathbf{X}$ be a monoid variety with $\mathbf{X} \subseteq \mathbf{K}$. We need to verify that either $\mathbf{E} \subseteq \mathbf{X}$ or $\mathbf{X} \subseteq \mathbf{E}$. Substituting 1 for y in the identity (4.9), we obtain $\mathbf{X}$ satisfies the identity (4.5). If $\mathbf{X}$ is commutative then $\mathbf{X} \subseteq \mathbf{C}_2 \subseteq \mathbf{E}$, and we are done. Thus, we can assume that $\mathbf{X}$ is non-commutative. The variety $\mathbf{X}$ is aperiodic because it satisfies the identity (4.5). Suppose that the variety $\mathbf{X}$ is completely regular. Every aperiodic completely regular variety is a variety of band monoids and every band satisfying the identity (4.4) is commutative. Thus, if $\mathbf{X}$ is completely regular then it is commutative, a contradiction. Hence we can assume that $\mathbf{X}$ is non-completely regular. Then $\mathbf{D}_1 \subseteq \mathbf{X}$ by Lemma 2.14.

Suppose that $\mathbf{E} \nsubseteq \mathbf{X}$. Then $\mathbf{X}$ satisfies the identity (4.6) by Lemma 4.3. Further, $\mathbf{X}$ satisfies the identity (4.10) as well because $\mathbf{X} \subseteq \mathbf{K}$. Hence $x^2y \stackrel{(4.10)}{\approx} x^2yx^2 \stackrel{(4.6)}{\approx} yx^2$. We see that the identity (4.15) holds in $\mathbf{X}$. Besides that,

$$xyx \stackrel{(4.9)}{\approx} xyx^2 \stackrel{(4.6)}{\approx} x^3yx^2 \stackrel{(4.5)}{\approx} x^2yx^2 \stackrel{(4.6)}{\approx} yx^2 \stackrel{(4.15)}{\approx} x^2y,$$

whence the identity

$$xyx \approx x^2y \tag{6.2}$$

holds in $\mathbf{X}$. So, $\mathbf{X}$ satisfies the identities $yx^2 \stackrel{(4.15)}{\approx} x^2y \stackrel{(6.2)}{\approx} xyx$. The identities (4.15) (4.4) and (6.2) evidently imply σ_1, σ_2 and γ_1. Thus, $\mathbf{X} \subseteq \mathbf{D}_1 \subseteq \mathbf{E}$. We proved that if $\mathbf{E} \nsubseteq \mathbf{X}$ then $\mathbf{X} \subseteq \mathbf{E}$. Hence the claim 1) of Proposition 6.1 is proved.

6.2. **Several auxiliary results.** Here we prove several lemmas that will be used many times below. This subsection is divided into three subsubsections.

6.2.1. *Some properties of the varieties* $\mathbf{F}_k$, $\mathbf{H}_k$, $\mathbf{I}_k$, $\mathbf{J}_k^m$, $\mathbf{K}$ *and their identities.*

Lemma 6.2. *The variety* $\mathbf{K}$ *satisfies:*

(i) *the identity* σ_2;

(ii) *the identity*

$$xyxzx \approx xyxz; \tag{6.3}$$

(iii) *any identity* $\mathbf{u} \approx \mathbf{v}$ *such that* $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ *and* $\mathrm{occ}_x(\mathbf{u}), \mathrm{occ}_x(\mathbf{v}) \geq 2$ *for any letter* $x \in \mathrm{con}(\mathbf{u})$.

Proof. (i) We have $xzytxy \stackrel{(4.9)}{\approx} xzytx^2y^2 \stackrel{(4.4)}{\approx} xzyty^2x^2 \stackrel{(4.9)}{\approx} xzytyx$.

(ii) Here we have $xyxzx \stackrel{(4.9)}{\approx} xyx^2zx \stackrel{(4.10)}{\approx} xyx^2z \stackrel{(4.9)}{\approx} xyxz$.

(iii) According to the claim (ii), $\mathbf{V}$ satisfies the identity (6.3). This allows us to assume that $\mathrm{occ}_x(\mathbf{u}) = \mathrm{occ}_x(\mathbf{v}) = 2$ for any letter $x \in \mathrm{con}(\mathbf{u})$. Let $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v}) = \{x_1, x_2, \dots, x_k\}$. We are going to verify that $\mathbf{u} \approx x_1^2x_2^2 \cdots x_k^2$ holds in $\mathbf{K}$. We will use induction on k.

Induction base. Suppose that $k = 1$. Here the identity $\mathbf{u} \approx \mathbf{v}$ has the form $x_1^2 \approx x_1^2$, whence it trivially holds in $\mathbf{K}$.

Induction step. Let now $k > 1$. We may assume without loss of generality that $\ell_1(\mathbf{u}, x_i) < \ell_1(\mathbf{u}, x_k)$ for any $1 \leq i < k$. Then

$$\mathbf{u} = \mathbf{u}' x_k x_{j_1} x_{j_2} \cdots x_{j_s} x_k x_{j_{s+1}} x_{j_{s+2}} \cdots x_{j_{s+t}}$$

where $x_{j_r} \in \mathrm{con}(\mathbf{u}')$ for any $1 \leq r \leq s+t$. Then the identities

$$\begin{aligned}
\mathbf{u} &\stackrel{(4.9)}{\approx} \mathbf{u}' x_k x_{j_1}^2 x_{j_2}^2 \cdots x_{j_s}^2 x_k^2 x_{j_{s+1}}^2 x_{j_{s+2}}^2 \cdots x_{j_{s+t}}^2 \\
&\stackrel{(4.4)}{\approx} \mathbf{u}' x_k^3 x_{j_1}^2 x_{j_2}^2 \cdots x_{j_{s+t}}^2 \\
&\stackrel{(4.5)}{\approx} \mathbf{u}' x_k^2 x_{j_1}^2 x_{j_2}^2 \cdots x_{j_{s+t}}^2 \\
&\stackrel{(4.4)}{\approx} \mathbf{u}' x_{j_1}^2 x_{j_2}^2 \cdots x_{j_{s+t}}^2 x_k^2 \\
&\stackrel{(6.3)}{\approx} \mathbf{u}' x_{j_1} x_{j_2} \cdots x_{j_{s+t}} x_k^2 \\
&= \mathbf{u}_{x_k} x_k^2
\end{aligned}$$

hold in $\mathbf{K}$. The word $\mathbf{u}_{x_k}$ contains exactly $k-1$ letters. By the induction assumption, the identity $\mathbf{u}_{x_k} \approx x_1^2x_2^2 \cdots x_{k-1}^2$ holds in $\mathbf{K}$, whence this variety satisfies the identities $\mathbf{u} \approx \mathbf{u}_{x_k} x_k^2 \approx x_1^2x_2^2 \cdots x_k^2$. Similarly, $\mathbf{v} \approx x_1^2x_2^2 \cdots x_k^2$ holds in $\mathbf{K}$, whence $\mathbf{K}$ satisfies $\mathbf{u} \approx \mathbf{v}$. □

Lemma 6.3. *The identity system* Φ *together with the identity*

$$xx_kx\mathbf{b}_k \approx x^2x_k\mathbf{b}_k \tag{6.4}$$

forms an identity basis of the variety $\mathbf{J}_k^k$.

Proof. First of all, we note that the identity (6.4) holds in the variety $\mathbf{J}_k^k$. To check this fact, it suffices to perform the substitution $(y_k, y_{k+1}) \mapsto (1, x)$ in the identity δ_k^k and use the equality $\mathbf{b}_k = x_{k-1}x_k\mathbf{b}_{k-1}$. So, we need to verify that

δ_k^k follows from Φ and (6.4). In view of Lemma 6.2, we can use the identities σ_2 and (6.3). Here is the required deduction (letters in the right column refer to comments after the deduction):

$$\begin{aligned}
y_{k+1}y_kx_ky_{k+1}\mathbf{b}_{k,k}y_k\mathbf{b}_{k-1} &= y_{k+1}y_kx_ky_{k+1}x_{k-1}x_ky_k\mathbf{b}_{k-1} && \text{(a)}\\
&\approx y_{k+1}y_kx_ky_{k+1}x_{k-1}y_kx_k\mathbf{b}_{k-1} && \text{(b)}\\
&\approx y_{k+1}^2y_kx_kx_{k-1}y_kx_k\mathbf{b}_{k-1} && \text{(c)}\\
&\approx y_{k+1}^2y_kx_kx_{k-1}x_ky_k\mathbf{b}_{k-1} && \text{(d)}\\
&= y_{k+1}^2y_kx_kx_{k-1}x_ky_kx_{k-2}x_{k-1}\mathbf{b}_{k-2} && \text{(e)}\\
&\approx y_{k+1}^2y_kx_kx_{k-1}x_ky_kx_{k-2}x_kx_{k-1}x_k\mathbf{b}_{k-2} && \text{(f)}\\
&\approx y_{k+1}y_ky_{k+1}x_kx_{k-1}x_ky_kx_{k-2}x_kx_{k-1}x_k\mathbf{b}_{k-2} && \text{(g)}\\
&\approx y_{k+1}y_ky_{k+1}x_kx_{k-1}x_ky_kx_{k-2}x_{k-1}\mathbf{b}_{k-2} && \text{(h)}\\
&= y_{k+1}y_ky_{k+1}x_k\mathbf{b}_{k,k}y_k\mathbf{b}_{k-1}. && \text{(i)}
\end{aligned}$$

(a) Here we use the equality $\mathbf{b}_{k,k} = x_{k-1}x_k$.

(b) Here we modify the subword $y_kx_ky_{k+1}x_{k-1}x_ky_k$ by performing the substitution $(x, t, y, z) \mapsto (y_k, 1, x_k, y_{k+1}x_{k-1})$ in σ_2.

(c) Here we perform the substitution $(x, x_k) \mapsto (y_{k+1}, y_kx_k)$ in (6.4) and use the equality $\mathbf{b}_k = x_{k-1}x_k\mathbf{b}_{k-1}$.

(d) Here we modify the subword $y_kx_kx_{k-1}y_kx_k$ by performing the substitution $(x, t, y, z) \mapsto (y_k, 1, x_k, x_{k-1})$ in σ_2.

(e) Here we use the equality $\mathbf{b}_{k-1} = x_{k-2}x_{k-1}\mathbf{b}_{k-2}$.

(f) Here (6.3) allows us to add two new occurrences of the letter x_k after its second occurrence in the word $y_{k+1}^2y_k \overset{(1)}{x_k} x_{k-1} \overset{(2)}{x_k} y_kx_{k-2}x_{k-1}\mathbf{b}_{k-2}$.

(g) Here we perform the substitution $(x, x_k, x_{k-1}) \mapsto (y_{k+1}, y_k, x_kx_{k-1}x_k)$ in (6.4) and use the equality $\mathbf{b}_k = x_{k-1}x_kx_{k-2}x_{k-1}\mathbf{b}_{k-2}$.

(h) Here (6.3) allows us to delete the third and the fourth occurrences of the letter x_k in the word $y_{k+1}y_ky_{k+1} \overset{(1)}{x_k} x_{k-1} \overset{(2)}{x_k} y_kx_{k-2} \overset{(3)}{x_k} x_{k-1} \overset{(4)}{x_k} \mathbf{b}_{k-2}$.

(i) Here we use the equalities $\mathbf{b}_{k,k} = x_{k-1}x_k$ and $\mathbf{b}_{k-1} = x_{k-2}x_{k-1}\mathbf{b}_{k-2}$. □

Lemma 6.4. *The inclusions*

$$\mathbf{F}_k \subseteq \mathbf{H}_k \subseteq \mathbf{I}_k \subseteq \mathbf{J}_k^1 \subseteq \mathbf{J}_k^2 \subseteq \cdots \subseteq \mathbf{J}_k^k \subseteq \mathbf{F}_{k+1} \tag{6.5}$$

are valid.

Proof. Since all varieties that appear in the inclusions (6.5) are contained in $\mathbf{K}$, we can apply Lemma 6.2. In particular, this allows us to use below the identities σ_2 and (6.3).

1°. *The inclusion* $\mathbf{F}_k \subseteq \mathbf{H}_k$. We need to verify that β_k follows from Φ and α_k. Here is the required deduction:

$$\begin{aligned}
xx_kx\mathbf{b}_k &= xx_kxx_{k-1}x_k\mathbf{b}_{k-1} && \text{because } \mathbf{b}_k = x_{k-1}x_k\mathbf{b}_{k-1}\\
&\approx xx_kxx_{k-1}x_kx^2\mathbf{b}_{k-1} && \text{by (6.3)}\\
&\approx x_kx^2x_{k-1}x_kx^2\mathbf{b}_{k-1} && \text{we perform the substitution}\\
& && (x_k, y_k) \mapsto (x_kx, x) \text{ in } \alpha_k
\end{aligned}$$

$$\approx x_k x^2 x_{k-1} x_k \mathbf{b}_{k-1} \qquad \text{by (6.3)}$$
$$= x_k x^2 \mathbf{b}_k \qquad \text{because } \mathbf{b}_k = x_{k-1} x_k \mathbf{b}_{k-1}.$$

2°. *The inclusion* $\mathbf{H}_k \subseteq \mathbf{I}_k$. Here we need to verify that γ_k follows from Φ and β_k. Indeed,

$$y_1 y_0 x_k y_1 \mathbf{b}_k \approx y_1 y_0 x_k y_1^2 \mathbf{b}_k \qquad \text{by (6.3)}$$
$$\approx y_1 y_0 y_1 x_k y_1 \mathbf{b}_k \qquad \text{we modify the subword } x_k y_1^2 \mathbf{b}_k \text{ by substitution } y_1 \text{ for } x \text{ in } \beta_k$$
$$\approx y_1 y_0 y_1 x_k \mathbf{b}_k \qquad \text{by (6.3).}$$

3°. *The inclusion* $\mathbf{I}_k \subseteq \mathbf{J}_k^1$. It suffices to verify that δ_k^1 follows from γ_k. Since $\mathbf{b}_{k,1} = \mathbf{b}_k$ and $\mathbf{b}_0 = \lambda$, the identity δ_k^1 has the form

$$y_2 y_1 x_k y_2 \mathbf{b}_k y_1 \approx y_2 y_1 y_2 x_k \mathbf{b}_k y_1.$$

To deduce this identity from γ_k, it suffices to modify the subword $y_2 y_1 x_k y_2 \mathbf{b}_k$ by performing the substitution $(y_0, y_1) \mapsto (y_1, y_2)$ in γ_k.

4°. *The inclusion* $\mathbf{J}_k^m \subseteq \mathbf{J}_k^{m+1}$ *where* $1 \le m < k$. It suffices to verify that δ_k^{m+1} follows from δ_k^m. Indeed, we get δ_k^{m+1} if we multiply δ_k^m by $x_{-1} x_0$ on the left and then increase by 1 the index of each letter in the identity we obtain.

5°. *The inclusion* $\mathbf{J}_k^k \subseteq \mathbf{F}_{k+1}$. In view of Lemma 6.3, it suffices to verify that α_{k+1} follows from Φ and (6.4). We have:

$$x_{k+1} y_{k+1} x_k x_{k+1} y_{k+1} \mathbf{b}_k \approx (x_{k+1} y_{k+1})^2 x_k \mathbf{b}_k \qquad \text{(a)}$$
$$\approx (y_{k+1} x_{k+1})^2 x_k \mathbf{b}_k \qquad \text{(b)}$$
$$\approx y_{k+1} x_{k+1} x_k y_{k+1} x_{k+1} \mathbf{b}_k \qquad \text{(c)}$$
$$\approx y_{k+1} x_{k+1} x_k x_{k+1} y_{k+1} \mathbf{b}_k. \qquad \text{(d)}$$

(a) Here we substitute $x_k y_k$ for x in (6.4).
(b) Here we apply the identity $(xy)^2 \approx (yx)^2$ that holds in $\mathbf{K}$ according to Lemma 6.2(iii).
(c) Here we substitute $y_k x_k$ for x in (6.4).
(d) Here we perform the substitution $(x, t, y, z) \mapsto (y_{k+1}, 1, x_{k+1}, x_k)$ in the identity σ_2. □

Below we often use the inclusions (6.5) without references to Lemma 6.4. Note that in fact strict inclusions (6.1) are valid. We will prove these inclusions in Subsubsection 6.4.6.

6.2.2. *k-decompositions of sides of the identities α_k, β_k, γ_k and δ_k^m.*

Lemma 6.5. *Let* $\mathbf{u}$ *be a left-hand or right-hand side of one of the identities* α_k, β_k, γ_k *or* δ_k^m. *Then:*

1) *If* $x_i, y_j \in \operatorname{con}(\mathbf{u})$ *then* $D(\mathbf{u}, x_i) = i$ *and* $D(\mathbf{u}, y_j) = j$. *The depth of the letter* x *in the left-hand [right-hand] side of the identity* β_k *equals* $k+1$ *[respectively* ∞*].*
2) *The* k*-decomposition of the word* $\mathbf{u}$ *has the form indicated in Table* 6.1.

TABLE 6.1. k-decompositions of some words

The identity	The k-decomposition of the left-hand side	The k-decomposition of the right-hand side
α_k	$\lambda \cdot \underline{\lambda} \cdot x_k \cdot \underline{\lambda} \cdot y_k \cdot \underline{\lambda} \cdot x_{k-1} \cdot \underline{x_k y_k}$ $\cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{\lambda} \cdot y_k \cdot \underline{\lambda} \cdot x_k \cdot \underline{\lambda} \cdot x_{k-1} \cdot \underline{x_k y_k}$ $\cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$
β_k	$\lambda \cdot \underline{x} \cdot x_k \cdot \underline{x} \cdot x_{k-1} \cdot \underline{x_k} \cdot x_{k-2}$ $\cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{\lambda} \cdot x_k \cdot \underline{x^2} \cdot x_{k-1} \cdot \underline{x_k} \cdot x_{k-2}$ $\cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$
γ_k	$\lambda \cdot \underline{\lambda} \cdot y_1 \cdot \underline{\lambda} \cdot y_0 \cdot \underline{\lambda} \cdot x_k \cdot \underline{y_1} \cdot x_{k-1}$ $\cdot \underline{x_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{\lambda} \cdot y_1 \cdot \underline{\lambda} \cdot y_0 \cdot \underline{y_1} \cdot x_k \cdot \underline{\lambda} \cdot x_{k-1}$ $\cdot \underline{x_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$
δ_k^m with $m < k$	$\lambda \cdot \underline{\lambda} \cdot y_{m+1} \cdot \underline{\lambda} \cdot y_m \cdot \underline{\lambda} \cdot x_k \cdot \underline{y_{m+1}}$ $\cdot x_{k-1} \cdot \underline{x_k} \cdots x_{m-1} \cdot \underline{x_m y_m} \cdot x_{m-2}$ $\cdot \underline{x_{m-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$	$\lambda \cdot \underline{\lambda} \cdot y_{m+1} \cdot \underline{\lambda} \cdot y_m \cdot \underline{y_{m+1}} \cdot x_k \cdot \underline{\lambda}$ $\cdot x_{k-1} \cdot \underline{x_k} \cdots x_{m-1} \cdot \underline{x_m y_m} \cdot x_{m-2}$ $\cdot \underline{x_{m-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0 \cdot \underline{x_1}$
δ_k^k	$\lambda \cdot \underline{y_{k+1}} \cdot y_k \cdot \underline{\lambda} \cdot x_k \cdot \underline{y_{k+1}} \cdot x_{k-1}$ $\cdot \underline{x_k y_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0$ $\cdot \underline{x_1}$	$\lambda \cdot \underline{y_{k+1}} \cdot y_k \cdot \underline{y_{k+1}} \cdot x_k \cdot \underline{\lambda} \cdot x_{k-1}$ $\cdot \underline{x_k y_k} \cdot x_{k-2} \cdot \underline{x_{k-1}} \cdots x_1 \cdot \underline{x_2} \cdot x_0$ $\cdot \underline{x_1}$

As in Example 3.3, we underline k-blocks of words in Table 6.1 to distinguish them from k-dividers.

Proof of Lemma 6.5. We allow ourselves to verify both the claims for the left-hand side of the identity α_k only. In all other cases the proof is given by quite similar considerations. We denote the left-hand side of the identity α_k by $\mathbf{u}_k$. So,

$$\mathbf{u}_k = x_k y_k x_{k-1} x_k y_k x_{k-2} x_{k-1} x_{k-3} x_{k-2} \cdots x_1 x_2 x_0 x_1.$$

1) The letter x_0 is simple in $\mathbf{u}_k$, whence $D(\mathbf{u}_k, x_0) = 0$. All other letters from $\operatorname{con}(\mathbf{u}_k)$ occur in $\mathbf{u}_k$ exactly twice. In particular, they are multiple in $\mathbf{u}_k$, and therefore their depth in $\mathbf{u}_k$ grater than 0. First occurrence of x_1 in $\mathbf{u}_k$ is not preceded by any simple letter. Therefore, $h_1^0(\mathbf{u}_k, x_1) = \lambda$. Further, only x_0 is the simple in $\mathbf{u}_k$ letter that precedes second occurrence of x_1 in $\mathbf{u}_k$. Hence $h_2^0(\mathbf{u}_k, x_1) = x_0$. We see that $h_1^0(\mathbf{u}_k, x_1) \ne h_2^0(\mathbf{u}_k, x_1)$, whence $D(\mathbf{u}_k, x_1) = 1$.

Both the first and the second occurrences of x_2 in $\mathbf{u}_k$ are not preceded by any simple in $\mathbf{u}_k$ letter. This means that $h_1^0(\mathbf{u}_k, x_2) = h_2^0(\mathbf{u}_k, x_2) = \lambda$, whence $D(\mathbf{u}_k, x_2) > 1$. Second occurrence of x_2 in $\mathbf{u}_k$ is preceded by exactly one occurrence of x_1 and there are no any letters between these occurrences of x_1 and x_2. Besides that, $h_1^0(\mathbf{u}_k, x_1) \ne h_2^0(\mathbf{u}_k, x_1)$. Therefore, $h_2^1(\mathbf{u}_k, x_2) = x_1$. On the other hand, $h_1^1(\mathbf{u}_k, x_2) \ne x_1$ because x_1 does not occur before first occurrence of x_2 in $\mathbf{u}_k$. Thus, $h_1^1(\mathbf{u}_k, x_2) \ne h_2^1(\mathbf{u}_k, x_2)$, whence $D(\mathbf{u}_k, x_2) = 2$.

We introduce some new notation to facilitate further considerations. For a letter $a \in \operatorname{mul}(\mathbf{u}_k)$, we denote by $\mathbf{u}_k[a; 1, 2]$ the subword of $\mathbf{u}_k$ located between the first and the second occurrences of a in $\mathbf{u}_k$. For instance, $\mathbf{u}_k[x_k; 1, 2] = y_k x_{k-1}$, $\mathbf{u}_k[y_k; 1, 2] = x_{k-1} x_k$, while $\mathbf{u}_k[x_1; 1, 2] = x_2 x_0$. Let now $2 < r < k$. Suppose that we prove the equality $D(\mathbf{u}_k, x_i) = i$ for all $i = 0, 1, \dots, r-1$. We are going to check that $D(\mathbf{u}_k, x_r) = r$. Suppose that $D(\mathbf{u}_k, x_r) = s < r$. This means that $h_1^{s-1}(\mathbf{u}_k, x_r) \ne h_2^{s-1}(\mathbf{u}_k, x_r)$. Therefore, there is a letter z such that first occurrence of z in $\mathbf{u}_k$ lies in $\mathbf{u}_k[x_r; 1, 2]$ and $h_1^{s-2}(\mathbf{u}_k, z) \ne h_2^{s-2}(\mathbf{u}_k, z)$.

But $\mathbf{u}_k[x_r;1,2] = x_{r+1}x_{r-1}$ whenever $r < k-1$ and $\mathbf{u}_k[x_{k-1};1,2] = x_k y_k x_{k-2}$. In any case, a unique letter whose first occurrence in $\mathbf{u}_k$ lies in $\mathbf{u}_k[x_r;1,2]$ is x_{r-1}. In view of our assumption, $D(\mathbf{u}_k, x_{r-1}) = r-1$. Since $s-2 < r-2$ the latest equality implies that $h_1^{s-2}(\mathbf{u}_k, x_{r-1}) = h_2^{s-2}(\mathbf{u}_k, x_{r-1})$. Thus, there are no letters z with the above-mentioned properties. Therefore, $D(\mathbf{u}_k, x_r) \ge r$. Suppose now that $D(\mathbf{u}_k, x_r) = t > r$. Then $h_1^{r-1}(\mathbf{u}_k, x_r) = h_2^{r-1}(\mathbf{u}_k, x_r)$. Therefore, there are no letters z such that first occurrence of z in $\mathbf{u}_k$ lies in $\mathbf{u}_k[x_r;1,2]$ and $D(\mathbf{u}_k, z) = r-1$. But our assumption implies that the letter x_{r-1} have these properties. Thus, $D(\mathbf{u}_k, x_r) = r$.

The arguments quite analogous to ones from the previous paragraph permit establish that $D(\mathbf{u}_k, y_k) = k$. It is necessary to take into account the equality $D(\mathbf{u}_k, x_{k-1}) = k-1$ proved above and the fact that a unique letter whose first occurrence in $\mathbf{u}_k$ lies in $\mathbf{u}[y_k;1,2]$ is x_{k-1}.

It remains to verify that $D(\mathbf{u}_k, x_k) = k$. We note that both the first and the second occurrences of x_k in $\mathbf{u}_k$ are not preceded by any simple letter, whence $h_1^0(\mathbf{u}_k, x_k) = h_2^0(\mathbf{u}_k, x_k) = \lambda$. Suppose that $h_1^i(\mathbf{u}_k, x_k) \ne h_2^i(\mathbf{u}_k, x_k)$ for some $0 < i < k-1$. Then there is a letter z such that first occurrence of z in $\mathbf{u}_k$ lies in $\mathbf{u}[x_k;1,2]$ and $h_1^{i-1}(\mathbf{u}_k, z) = h_2^{i-1}(\mathbf{u}_k, z)$. The latest equality means that $D(\mathbf{u}_k, z) \le i < k-1$. Further, $\mathbf{u}[x_k;1,2] = y_k x_{k-1}$ and occurrences of both y_k and x_{k-1} are the first occurrences of these letters in $\mathbf{u}_k$. As we have seen above, $D(\mathbf{u}_k, y_k), D(\mathbf{u}_k, x_{k-1}) \ge k-1$. Thus, $h_1^i(\mathbf{u}_k, x_k) = h_2^i(\mathbf{u}_k, x_k)$ for all $0 \le i < k-1$. Now we check that $h_1^k(\mathbf{u}_k, x_k) \ne h_2^k(\mathbf{u}_k, x_k)$. Indeed, we have seen above that $D(\mathbf{u}_k, x_{k-1}) = k-1$ and $D(\mathbf{u}_k, y_k) = k$. Therefore, $h_1^{k-2}(\mathbf{u}_k, x_{k-1}) \ne h_2^{k-2}(\mathbf{u}_k, x_{k-1})$ and $h_1^{k-2}(\mathbf{u}_k, y_k) = h_2^{k-2}(\mathbf{u}_k, y_k)$. This implies that $h_2^{k-1}(\mathbf{u}_k, x_k) = x_{k-1}$. On the other hand, first occurrence of x_k in $\mathbf{u}_k$ is not preceded by any letter, whence $h_1^{k-1}(\mathbf{u}_k, x_k) = \lambda$. We see that $h_1^k(\mathbf{u}_k, x_k) \ne h_2^k(\mathbf{u}_k, x_k)$. In view of the above, this means that $D(\mathbf{u}_k, x_k) = k$.

2) By Lemma 3.7, k-dividers of a word $\mathbf{w}$ are exactly the first occurrences of letters $x \in \mathrm{con}(\mathbf{w})$ with $D(\mathbf{w}, x) \le k$ and the empty word at the beginning of the word $\mathbf{w}$. As we have proved above, $D(\mathbf{u}_k, x) \le k$ for any letter $x \in \mathrm{con}(\mathbf{u}_k)$. Thus, k-dividers of $\mathbf{u}_k$ are just the first occurrences of all letters from $\mathrm{con}(\mathbf{u}_k)$ and the empty word at the beginning of $\mathbf{u}_k$. All subwords of $\mathbf{u}_k$ between these k-dividers and only they are k-blocks of $\mathbf{u}_k$. Thus, the k-decomposition of the word $\mathbf{u}_k$ has the form indicated in Table 6.1. □

Note that the claim 1) of Lemma 6.5 explains the choice of indexes of letters in the identities α_k, β_k, γ_k and δ_k^m.

6.2.3. *Swapping letters within k-blocks.* In this subsubsection we verify only one statement. It can be called the "core" of the whole proof of Theorem 1.1. Its proof is very long and based on a quite hard technique. At the same time, it is the basis for the rest of the proof of Theorem 1.1 and plays a key role there.

Lemma 6.6. *Let $\mathbf{V}$ be a monoid variety such that $\mathbf{V} \subseteq \mathbf{K}$, $\mathbf{u}$ be a word and k be a natural number. Further, let $\mathbf{u} = \mathbf{u}'ab\mathbf{u}''$ where $\mathbf{u}'$ and $\mathbf{u}''$ are possibly empty words, while ab is a subword of some $(k-1)$-block of $\mathbf{u}$. Suppose that one of the following holds:*

(i) $\mathbf{V}$ *satisfies* δ_k^m*,* $a \in \mathrm{con}(\mathbf{u}')$ *and* $D(\mathbf{u}, a) > m$*;*
(ii) $\mathbf{V}$ *satisfies* γ_k *and* $a \in \mathrm{con}(\mathbf{u}')$*;*
(iii) $\mathbf{V}$ *satisfies* β_k *and* $D(\mathbf{u}, a) \ne D(\mathbf{u}, b)$*;*
(iv) $\mathbf{V}$ *satisfies* α_k*.*

Then $\mathbf{V}$ *satisfies the identity* $\mathbf{u} \approx \mathbf{u}'ba\mathbf{u}''$*.*

Proof. We will prove the assertions (i)–(iv) simultaneously. Suppose that the variety $\mathbf{V}$ satisfies the hypothesis of one of these four claims. In particular, $\mathbf{V}$ satisfies δ_k^k in any case. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$ and ab is a subword of $\mathbf{u}_i$ for some $0 \le i \le m$. Then $\mathbf{u}_i = \mathbf{u}_i' ab \mathbf{u}_i''$ for some possibly empty words $\mathbf{u}_i'$ and $\mathbf{u}_i''$. Clearly, $\mathbf{u}' = t_0\mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_i \mathbf{u}_i'$ and $\mathbf{u}'' = \mathbf{u}_i'' t_{i+1}\mathbf{u}_{i+1} \cdots t_m \mathbf{u}_m$.

If $a, b \in \mathrm{con}(\mathbf{u}')$ then

$$\mathbf{u} = \mathbf{u}'ab\mathbf{u}'' \stackrel{(4.9)}{\approx} \mathbf{u}'a^2b^2\mathbf{u}'' \stackrel{(4.4)}{\approx} \mathbf{u}'b^2a^2\mathbf{u}'' \stackrel{(4.9)}{\approx} \mathbf{u}'ba\mathbf{u}'',$$

and we are done. Thus, we can assume without loss of generality that

$$b \notin \mathrm{con}(\mathbf{u}'). \tag{6.6}$$

If $D(\mathbf{u}, b) \le k-1$ then b is a $(k-1)$-divider of $\mathbf{u}$ by Lemma 3.7. But this is not the case because first occurrence of b in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$. Therefore, $D(\mathbf{u}, b) \ge k$. Further, if $a \in \mathrm{mul}(\mathbf{u}')$ then Lemma 6.2(ii) implies that the identities $\mathbf{u}'ab\mathbf{u}'' \approx \mathbf{u}'b\mathbf{u}'' \approx \mathbf{u}'ba\mathbf{u}''$ hold in $\mathbf{V}$. Thus, we can assume that

$$\text{if } a \in \mathrm{con}(\mathbf{u}') \text{ then } a \in \mathrm{sim}(\mathbf{u}'). \tag{6.7}$$

Further considerations are divided into three cases depending on the depth of b in $\mathbf{u}$: $D(\mathbf{u}, b) = k$, $k < D(\mathbf{u}, b) < \infty$ and $D(\mathbf{u}, b) = \infty$. Each of these cases is divided into subcases corresponding to the claims (i)–(iv). Thus, the proof of each of the assertions (i)–(iv) will be completed after considering the corresponding subcase of Case 3.

Case 1: $D(\mathbf{u}, b) = k$. This case is the most difficult from the technical point of view and the longest. By examining two other cases, we will repeatedly refer to properties that will be verified here. Let $\mathbf{p} \approx \mathbf{q}$ be one of the identities α_k, β_k, γ_k or δ_k^m. In a sense, the identity $\mathbf{p} \approx \mathbf{q}$ "looks like" to $\mathbf{u}'ab\mathbf{u}'' \approx \mathbf{u}'ba\mathbf{u}''$. We have in mind that the words $\mathbf{p}$ and $\mathbf{q}$ start with the same prefix (which is empty for α_k and β_k) and end with the same suffix, and the subword between these prefix and suffix is the product of two letters in $\mathbf{p}$ and the product of the same two letters in the reverse order in $\mathbf{q}$. This makes it possible in principle to apply the identity $\mathbf{p} \approx \mathbf{q}$ to one of the sides of the identity $\mathbf{u}'ab\mathbf{u}'' \approx \mathbf{u}'ba\mathbf{u}''$ in order to obtain the other side of it. To realize this possibility, we need, with the use of the identities that hold in $\mathbf{K}$, to reduce, say, the right-hand side of the identity $\mathbf{u}'ab\mathbf{u}'' \approx \mathbf{u}'ba\mathbf{u}''$ to a form to which the identity $\mathbf{p} \approx \mathbf{q}$ can be applied. To do this, we first need to find "inside of" the word $\mathbf{u}$ the letters $x_0, x_1, \dots, x_k$ which would appear in the same order as the letters with the same names in one of the sides of the identity $\mathbf{p} \approx \mathbf{q}$.

Put $x_k = b$. Let X_{k-1} be the set of $(k-1)$-dividers z of $\mathbf{u}$ such that

$$\ell_1(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_k).$$

The fact that $D(\mathbf{u}, x_k) = k$ implies that $h_1^{k-1}(\mathbf{u}, x_k) \neq h_2^{k-1}(\mathbf{u}, x_k)$, whence $h_2^{k-1}(\mathbf{u}, x_k) \in X_{k-1}$. Therefore, the set X_{k-1} is non-empty. Further, Lemma 3.9(ii) implies that $D(\mathbf{u}, z) = k-1$ and $\ell_2(\mathbf{u}, x_k) < \ell_2(\mathbf{u}, z)$ for any $z \in X_{k-1}$. Now we consider the letter $x_{k-1} \in X_{k-1}$ such that $\ell_2(\mathbf{u}, z) \leq \ell_2(\mathbf{u}, x_{k-1})$ for any $z \in X_{k-1}$.

Let X_{k-2} be the set of $(k-2)$-dividers z of $\mathbf{u}$ such that $\ell_1(\mathbf{u}, x_{k-1}) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{k-1})$. Then the fact that $D(\mathbf{u}, x_{k-1}) = k-1$ implies that $h_1^{k-2}(\mathbf{u}, x_{k-1}) \neq h_2^{k-2}(\mathbf{u}, x_{k-1})$, whence $h_2^{k-2}(\mathbf{u}, x_{k-1}) \in X_{k-2}$. Therefore, the set X_{k-2} is non-empty. Further, Lemma 3.9(ii) implies that $D(\mathbf{u}, z) = k-2$ and $\ell_2(\mathbf{u}, x_{k-1}) < \ell_2(\mathbf{u}, z)$ for any $z \in X_{k-2}$. Now we consider the letter $x_{k-2} \in X_{k-2}$ such that $\ell_2(\mathbf{u}, z) \leq \ell_2(\mathbf{u}, x_{k-2})$ for any $z \in X_{k-2}$. Since $\ell_1(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, x_{k-1}) < \ell_1(\mathbf{u}, x_{k-2})$, Lemma 3.13 implies that $\ell_2(\mathbf{u}, x_k) < \ell_1(\mathbf{u}, x_{k-2})$.

Further, for $s = k-3, k-4, \dots, 1$ we denote one by one the set X_s and the letter x_s in the following way: X_s is the set of all s-dividers z of $\mathbf{u}$ such that $\ell_1(\mathbf{u}, x_{s+1}) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{s+1})$, and x_s is a letter such that $x_s \in X_s$ and $\ell_2(\mathbf{u}, z) \leq \ell_2(\mathbf{u}, x_s)$ for any $z \in X_s$. Arguments similar to those from the previous two paragraphs allow us to verify that the set X_s is non-empty, $D(\mathbf{u}, x_s) = s$, $\ell_j(\mathbf{u}, x_{s+1}) < \ell_j(\mathbf{u}, x_s)$ for any $j = 1, 2$ and $\ell_2(\mathbf{u}, x_{s+2}) < \ell_1(\mathbf{u}, x_s)$.

Finally, put $x_0 = h_2^0(\mathbf{u}, x_1)$. In view of Lemma 3.9, $D(\mathbf{u}, x_0) = 0$ and $\ell_1(\mathbf{u}, x_1) < \ell_1(\mathbf{u}, x_0)$. Since $\ell_1(\mathbf{u}, x_2) < \ell_1(\mathbf{u}, x_1)$, Lemma 3.13 implies that $\ell_2(\mathbf{u}, x_2) < \ell_1(\mathbf{u}, x_0)$. Then

$$\mathbf{u} = \mathbf{u}' ab\mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0 \tag{6.8}$$

for some possibly empty words $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2k}$. One can verify that if $2 \leq s \leq k$ then

$$\ell_2(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{s-1}) \text{ for any } z \in \mathrm{con}(\mathbf{v}_{2s}\mathbf{v}_{2s-1}). \tag{6.9}$$

Put

$$\mathbf{w}_s = \mathbf{u}' ab\mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_{2s+2} x_s \mathbf{v}_{2s+1} x_{s+1}.$$

The word $\mathbf{w}_s$ is the prefix of $\mathbf{u}$ that immediately precedes the word $\mathbf{v}_{2s}$, while the word $\mathbf{v}_{2s-1}$ precedes second occurrence of x_{s-1} in $\mathbf{u}$. This implies the required conclusion whenever $z \in \mathrm{con}(\mathbf{w}_s)$. Suppose now that $z \notin \mathrm{con}(\mathbf{w}_s)$. Then $\ell_1(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_s)$. If z is an $(s-1)$-divider of $\mathbf{u}$ then $z \in X_{s-1}$, whence $\ell_2(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{s-1})$ by the choice of the letter x_{s-1}. Otherwise $D(\mathbf{u}, z) > s-1$ by Lemma 3.7. Then since $\ell_1(\mathbf{u}, z) < \ell_1(\mathbf{u}, x_{s-2})$, Lemma 3.13 implies that $\ell_2(\mathbf{u}, z) < \ell_1(\mathbf{u}, x_{s-2})$, whence $\ell_2(\mathbf{u}, z) < \ell_2(\mathbf{u}, x_{s-1})$.

Further realization of the plan outlined at the beginning of Case 1 depends on the identity that plays the role of $\mathbf{p} \approx \mathbf{q}$. Therefore, further considerations are divided into four subcases.

Subcase 1.1: $\mathbf{V}$ satisfies the hypothesis of the claim (i), i.e., δ_k^m holds in $\mathbf{V}$, $a \in \mathrm{con}(\mathbf{u}')$ and $D(\mathbf{u}, a) > m$. The claim (6.7) allows us to assume that $a \in \mathrm{sim}(\mathbf{u}')$. Then $\mathbf{u}' = \mathbf{w}a\mathbf{v}$ for some possibly empty words $\mathbf{v}$ and $\mathbf{w}$. This implies that

$$\begin{aligned} \mathbf{u} = {} & \mathbf{w}a\mathbf{v}ab\mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \\ & \cdot \mathbf{v}_1 x_1 \mathbf{v}_0. \end{aligned} \tag{6.10}$$

Put $D(\mathbf{u},a)=r$. Further considerations are divided into two parts corresponding to the cases when $r\le k+1$ and $r>k+1$.

A) $r\le k+1$. Here we need to define two more letters, namely y_{r-1} and y_{r-2} and clarify the location of these letters within $\mathbf{u}$. Let Y_{r-1} be the set of $(r-1)$-dividers z of $\mathbf{u}$ such that $\ell_1(\mathbf{u},a)<\ell_1(\mathbf{u},z)<\ell_2(\mathbf{u},a)$. The fact that $D(\mathbf{u},a)=r$ implies that $h_1^{r-1}(\mathbf{u},a)\ne h_2^{r-1}(\mathbf{u},a)$, whence $h_2^{r-1}(\mathbf{u},a)\in Y_{r-1}$. Therefore, the set Y_{r-1} is non-empty. Lemma 3.9(ii) implies that $D(\mathbf{u},z)=r-1$ and $\ell_2(\mathbf{u},a)<\ell_2(\mathbf{u},z)$ for any $z\in Y_{r-1}$. Then $\ell_1(\mathbf{u},b)<\ell_2(\mathbf{u},z)$ for any $z\in Y_{r-1}$. Now we consider the letter $y_{r-1}\in Y_{r-1}$ such that $\ell_2(\mathbf{u},z)\le\ell_2(\mathbf{u},y_{r-1})$ for any $z\in Y_{r-1}$.

Now we check some additional properties of the letter x_r, which are fulfilled under certain restrictions to r. Suppose that $r<k+1$. Then the letter x_r is defined. Our aim is to prove that

$$\ell_2(\mathbf{u},x_r)<\ell_2(\mathbf{u},y_{r-1}). \tag{6.11}$$

Put $y_{r-2}=h_2^{r-2}(\mathbf{u},y_{r-1})$. Since $D(\mathbf{u},y_{r-1})=r-1$, Lemma 3.9 implies that $D(\mathbf{u},y_{r-2})=r-2$ and $\ell_1(\mathbf{u},y_{r-1})<\ell_1(\mathbf{u},y_{r-2})$. Recall that $\ell_1(\mathbf{u},a)<\ell_1(\mathbf{u},y_{r-1})$, whence $\ell_1(\mathbf{u},a)<\ell_1(\mathbf{u},y_{r-2})$. Since $D(\mathbf{u},a)=r$, we can apply Lemma 3.13 and conclude that $\ell_2(\mathbf{u},a)<\ell_1(\mathbf{u},y_{r-2})$. Second occurrence of a in $\mathbf{u}$ immediately precedes first occurrence of $b=x_k$, whence $\ell_1(\mathbf{u},x_k)<\ell_1(\mathbf{u},y_{r-2})$. Then Lemma 3.13 implies that $\ell_2(\mathbf{u},x_k)<\ell_1(\mathbf{u},y_{r-2})$. This implies that $\ell_1(\mathbf{u},x_{k-1})<\ell_2(\mathbf{u},x_k)<\ell_1(\mathbf{u},y_{r-2})$. If $k-1\ge r$ then Lemma 3.13 applies with the conclusion that $\ell_2(\mathbf{u},x_{k-1})<\ell_1(\mathbf{u},y_{r-2})$. Continuing this process, we eventually obtain $\ell_2(\mathbf{u},x_r)<\ell_1(\mathbf{u},y_{r-2})$. The choice of y_{r-2} implies that first occurrence of y_{r-2} in $\mathbf{u}$ precedes second occurrence of y_{r-1}. Therefore, $\ell_2(\mathbf{u},x_r)<\ell_2(\mathbf{u},y_{r-1})$. So, we have proved that if $r<k+1$ then the claim (6.11) is true.

Let now $r>2$. Note that

$$\ell_1(\mathbf{u},y_{r-1})<\ell_2(\mathbf{u},a)<\ell_1(\mathbf{u},b)=\ell_1(\mathbf{u},x_k)<\ell_1(\mathbf{u},x_{k-1})<\cdots<\ell_1(\mathbf{u},x_{r-3}).$$

If $\ell_1(\mathbf{u},x_{r-3})<\ell_2(\mathbf{u},y_{r-1})$ then the letter x_{r-3} lies between the first and the second occurrences of y_{r-1} in $\mathbf{u}$. Since x_{r-3} is an $(r-3)$-divider of $\mathbf{u}$, we obtain a contradiction with the equality $D(\mathbf{u},y_{r-1})=r-1$. Thus,

$$\ell_2(\mathbf{u},y_{r-1})<\ell_1(\mathbf{u},x_{r-3}) \tag{6.12}$$

whenever $r>2$.

One can return to arbitrary $r\le k+1$. This restriction on r guarantees that the letters x_{r-2} and x_{r-1} are defined. There are three possibilities for second occurrence of the letter y_{r-1} in $\mathbf{u}$:

$$\ell_1(\mathbf{u},x_{r-2})<\ell_2(\mathbf{u},y_{r-1})<\ell_2(\mathbf{u},x_{r-1}); \tag{6.13}$$

$$\ell_2(\mathbf{u},y_{r-1})<\ell_1(\mathbf{u},x_{r-2}); \tag{6.14}$$

$$\ell_2(\mathbf{u},x_{r-1})<\ell_2(\mathbf{u},y_{r-1}). \tag{6.15}$$

The equality (6.10) may be rewritten in the form

$$\begin{aligned}\mathbf{u} = {} & \mathbf{w}a\mathbf{v}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r}\overset{(1)}{x_{r-1}}\\ & \cdot\mathbf{v}_{2r-1}\overset{(2)}{x_r}\mathbf{v}_{2r-2}\overset{(1)}{x_{r-2}}\mathbf{v}_{2r-3}\overset{(2)}{x_{r-1}}\mathbf{v}_{2r-4}\overset{(1)}{x_{r-3}}\mathbf{v}_{2r-5}\overset{(2)}{x_{r-2}}\cdots\\ & \cdot\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned}\tag{6.16}$$

Suppose that the claim (6.13) holds. Then second occurrence of y_{r-1} in $\mathbf{u}$ belongs to the word $\mathbf{v}_{2r-3}$, whence $\mathbf{v}_{2r-3} = \mathbf{v}'_{2r-3}y_{r-1}\mathbf{v}''_{2r-3}$ for possibly empty words $\mathbf{v}'_{2r-3}$ and $\mathbf{v}''_{2r-3}$. Further, since $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, y_{r-1}) < \ell_2(\mathbf{u}, a)$, first occurrence of y_{r-1} belongs to $\mathbf{v}$. Therefore, $\mathbf{v} = \mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}$ for possibly empty words $\mathbf{v}_{2k+2}$ and $\mathbf{v}_{2k+1}$.

Combining all we above, we can clarify the presentation (6.10) of the word $\mathbf{u}$ and write this word in the form

$$\begin{aligned}\mathbf{u} = {} & \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\\ & \cdot\ \mathbf{v}_{2r}x_{r-1}\mathbf{v}_{2r-1}x_r\mathbf{v}_{2r-2}x_{r-2}\mathbf{v}'_{2r-3}y_{r-1}\mathbf{v}''_{2r-3}x_{r-1}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}x_{r-2}\cdots\\ & \cdot\ \mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned}$$

Note that $\mathbf{u}' = \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}$ and

$$\begin{aligned}\mathbf{u}'' = {} & \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r}x_{r-1}\mathbf{v}_{2r-1}x_r\mathbf{v}_{2r-2}x_{r-2}\\ & \cdot\ \mathbf{v}'_{2r-3}y_{r-1}\mathbf{v}''_{2r-3}x_{r-1}\mathbf{v}_{2r-4}x_{r-3}\mathbf{v}_{2r-5}x_{r-2}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned}$$

Similarly to the proof of (6.9), we can verify that if $z \in \mathrm{con}(\mathbf{v}_{2k+2}\mathbf{v}_{2k+1})$ then $\ell_2(\mathbf{u}, z) \le \ell_2(\mathbf{u}, y_{r-1})$.

Now we are ready to begin the process of modifying the word $\mathbf{u}$ to get the word $\mathbf{u}'ba\mathbf{u}''$. But first, we will outline the general scheme of further considerations, since arguments of that type will be repeated many times below. We are based on the fact that the identity (6.3) is satisfied by the variety $\mathbf{K}$. This allows us to add any letter that is multiple in a given word to any place after second occurrence of this letter in the word. Using this, we will add different missing letters or even words in different places in $\mathbf{u}$ (or in a word which equals $\mathbf{u}$ in $\mathbf{V}$) in order to make it possible to apply that word to the identity that is fulfilled in $\mathbf{V}$ at the moment (now such identity is δ_k^m). After that, we will apply this identity, and then "reverse the process", i.e., based on (6.3), remove unnecessary letters or even words from the resulting word to obtain the word $\mathbf{u}'ba\mathbf{u}''$.

Let us proceed with the implementation of this plan. First, we apply the identity (6.3) to the word $\mathbf{u}$ and insert the letter y_{r-1} after second occurrence of x_{r-1} in $\mathbf{u}$. We obtain the identity

$$\begin{aligned}\mathbf{u} \approx {} & \mathbf{w}a\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\\ & \cdot\mathbf{v}_{2r}\overset{(1)}{x_{r-1}}\mathbf{v}_{2r-1}x_r\mathbf{v}_{2r-2}x_{r-2}\mathbf{v}_{2r-3}\overset{(2)}{x_{r-1}}y_{r-1}\mathbf{v}_{2r-4}x_{r-3}\cdots\\ & \cdot\mathbf{v}_{2r-5}x_{r-2}\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned}\tag{6.17}$$

Further, we apply the identity (6.3) sufficiently many times to the right-hand side of the identity (6.17) and replace there third occurrence of y_{r-1} with $\mathbf{v}_{2k+2}y_{r-1}\mathbf{v}_{2k+1}$ and second occurrence of x_{s-1} with $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ for any $2 \le$

$s \le k$. We have $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w} a \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1} ab \mathbf{p} \mathbf{v}_0 \tag{6.18}$$

where

$$\begin{aligned}\mathbf{p} = {} & \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} \mathbf{v}_{2k-4} \cdots \mathbf{v}_{2r-2} x_{r-2} \mathbf{v}_{2r-3} \\ & \cdot \mathbf{v}_{2r} x_{r-1} \mathbf{v}_{2r-1} \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1} \mathbf{v}_{2r-4} x_{r-3} \mathbf{v}_{2r-5} \mathbf{v}_{2r-2} x_{r-2} \mathbf{v}_{2r-3} \mathbf{v}_{2r-6} \cdots \\ & \cdot \mathbf{v}_4 x_1 \mathbf{v}_3 \mathbf{v}_6 x_2 \mathbf{v}_5 \mathbf{v}_2 x_0 \mathbf{v}_1 \mathbf{v}_4 x_1 \mathbf{v}_3.\end{aligned}$$

By the hypothesis, $r = D(\mathbf{u}, a) > m$. Then by Lemma 6.4, δ_k^{r-1} holds in $\mathbf{V}$. Now we perform the substitution

$$(x_0, \dots, x_{k-1}, x_k, y_{r-1}, y_r) \mapsto (\mathbf{v}_2 x_0 \mathbf{v}_1, \dots, \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1}, b, \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1}, a)$$

in this identity. Then we obtain the identity

$$a \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1} ab \mathbf{p} \approx a \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1} ba \mathbf{p}.$$

This identity together with (6.18) implies that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w} a \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1} ba \mathbf{p} \mathbf{v}_0.$$

Now we apply the identity (6.3) to the right-hand side of the last identity "in the opposite direction" and replace the subword $\mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1}$ with y_{r-1} and the subword $\mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain the identity

$$\begin{aligned}\mathbf{u} \approx {} & \mathbf{w} a \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1} ba \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_{2r-2} x_{r-2} \\ & \cdot \mathbf{v}_{2r-3} x_{r-1} y_{r-1} \mathbf{v}_{2r-4} x_{r-3} \mathbf{v}_{2r-5} x_{r-2} \mathbf{v}_{2r-6} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0.\end{aligned}$$

Finally, we apply the identity (6.3) to the right-hand side of the last identity and delete third occurrence y_{r-1}. We obtain the identity

$$\begin{aligned}\mathbf{u} \approx {} & \mathbf{w} a \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1} ba \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_{2r-2} x_{r-2} \\ & \cdot \mathbf{v}'_{2r-3} y_{r-1} \mathbf{v}''_{2r-3} x_{r-1} \mathbf{v}_{2r-4} x_{r-3} \mathbf{v}_{2r-5} x_{r-2} \mathbf{v}_{2r-6} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 \\ & \cdot x_0 \mathbf{v}_1 x_1 \mathbf{v}_0 \\ = {} & \mathbf{u}' ba \mathbf{u}''.\end{aligned}$$

It remains to consider the case when either (6.14) or (6.15) holds. We are going to verify that in both the cases the identity (6.17) holds. It suffices to our aim because then we can complete considerations in the same arguments as above. If the claim (6.14) holds then (6.11) and (6.16) imply that the word $\mathbf{u}$ has the form

$$\begin{aligned}\mathbf{u} = {} & \mathbf{w} a \mathbf{v}_{2k+2} y_{r-1} \mathbf{v}_{2k+1} ab \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1} b \mathbf{v}_{2k-2} x_{k-2} \mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_{2r} \overset{(1)}{x_{r-1}} \mathbf{v}_{2r-1} \\ & \cdot x_r \mathbf{v}'_{2r-2} y_{r-1} \mathbf{v}''_{2r-2} x_{r-2} \mathbf{v}_{2r-3} \overset{(2)}{x_{r-1}} \mathbf{v}_{2r-4} x_{r-3} \mathbf{v}_{2r-5} x_{r-2} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \\ & \cdot \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{v}'_{2r-2}, \mathbf{v}''_{2r-2}$ such that $\mathbf{v}_{2r-2} = \mathbf{v}'_{2r-2} y_{r-1} \mathbf{v}''_{2r-2}$. Here we add one more occurrence of the letter y_{r-1} immediately after second occurrence of x_{r-1}. As a result, we obtain the identity (6.17). Finally, if (6.15) is the case then we use (6.12). The word $\mathbf{u}$ here has the form

$$\begin{aligned}\mathbf{u} = \mathbf{w}a\mathbf{v}_{2k+2}\, \overset{(1)}{y_{r-1}}\, \mathbf{v}_{2k+1}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r}x_{r-1}\mathbf{v}_{2r-1}\\ \cdot\, x_r\mathbf{v}_{2r-2}x_{r-2}\mathbf{v}_{2r-3}x_{r-1}\mathbf{v}'_{2r-4}\, \overset{(2)}{y_{r-1}}\, \mathbf{v}''_{2r-4}\, \overset{(1)}{x_{r-3}}\, \mathbf{v}_{2r-5}x_{r-2}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\\ \cdot\, \mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{v}'_{2r-4}, \mathbf{v}''_{2r-4}$ such that $\mathbf{v}_{2r-4} = \mathbf{v}'_{2r-4}y_{r-1}\mathbf{v}''_{2r-4}$. Then we can add third occurrence of the letter x_{r-1} immediately before second occurrence of y_{r-1} and obtain the identity

$$\begin{aligned}\mathbf{u} \approx \mathbf{w}a\mathbf{v}_{2k+2}\, \overset{(1)}{y_{r-1}}\, \mathbf{v}_{2k+1}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_{2r}\, \overset{(1)}{x_{r-1}}\\ \cdot\, \mathbf{v}_{2r-1}x_r\mathbf{v}_{2r-2}x_{r-2}\mathbf{v}_{2r-3}\, \overset{(2)}{x_{r-1}}\, \mathbf{v}'_{2r-4}\, \overset{(3)}{x_{r-1}}\overset{(2)}{y_{r-1}}\, \mathbf{v}''_{2r-4}\, \overset{(1)}{x_{r-3}}\, \mathbf{v}_{2r-5}x_{r-2}\cdots\\ \cdot\, \mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.\end{aligned}$$

The last identity is nothing but (6.17) (up to renaming of $\mathbf{v}_{2r-3}x_{r-1}\mathbf{v}'_{2r-4}$ to $\mathbf{v}_{2r-3}$, and $\mathbf{v}''_{2r-4}$ to $\mathbf{v}_{2r-4}$).

B) $r > k+1$. Recall that the equality (6.10) is true. Suppose that the word $\mathbf{v}$ is non-empty. Let $y \in \mathrm{con}(\mathbf{v})$. Suppose that $\ell_1(\mathbf{u}, x_{k-1}) < \ell_2(\mathbf{u}, y)$. This implies that $h_1^{k-1}(\mathbf{u}, y) \neq h_2^{k-1}(\mathbf{u}, y)$ because x_{k-1} is a $(k-1)$-divider of $\mathbf{u}$. Then y is a k-divider of $\mathbf{u}$. Since $\mathbf{v}$ (and, in particular, y) is located between the first and the second occurrences of a in $\mathbf{u}$, this contradicts the fact that $D(\mathbf{u}, a) = r > k+1$. So, $\ell_2(\mathbf{u}, y) \le \ell_1(\mathbf{u}, x_{k-1})$ for any $y \in \mathrm{con}(\mathbf{v})$. Then we apply the identity (6.3) sufficiently many times to the right-hand side of the identity (6.10), namely, we insert the word $\mathbf{v}$ after second occurrence of b there. Clearly, we can formally insert the word $\mathbf{v}$ after second occurrence of b whenever $\mathbf{v} = \lambda$ too. Further, in view of (6.9), we can replace second occurrence of x_{s-1} in the right-hand side of (6.10) with the word $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We have $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w}a\mathbf{v}ab\mathbf{p}\mathbf{v}_0 \tag{6.19}$$

where

$$\begin{aligned}\mathbf{p} = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\\ \cdot\, \mathbf{v}_4x_1\mathbf{v}_3.\end{aligned}$$

In view of Lemma 6.4, $\mathbf{V}$ satisfies the identity δ_k^k. Now we perform the substitution

$$(x_0, \dots, x_{k-1}, x_k, y_k, y_{k+1}) \mapsto (\mathbf{v}_2x_0\mathbf{v}_1, \dots, \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}, b, \mathbf{v}, a)$$

in this identity. Then we obtain the identity

$$a\mathbf{v}ab\mathbf{p} \approx a\mathbf{v}ba\mathbf{p}.$$

This identity together with (6.19) implies that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w}a\mathbf{v}ba\mathbf{p}\mathbf{v}_0.$$

Now we apply the identity (6.3) to the right-hand side of the last identity "in the opposite direction", namely delete the word $\mathbf{v}$ after second occurrence of

b and replace the subword $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain the identity

$$\begin{aligned}\mathbf{u} &\approx \mathbf{w}a\mathbf{v}ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0\\ &= \mathbf{u}'ba\mathbf{u}''.\end{aligned}$$

Subcase 1.2: $\mathbf{V}$ satisfies the hypothesis of the claim (ii), i.e., γ_k holds in $\mathbf{V}$ and $a \in \mathrm{con}(\mathbf{u}')$. Recall that the equality (6.8) is true. The claim (6.7) allows us to assume that $a \in \mathrm{sim}(\mathbf{u}')$. Then, as well as in Subcase 1.1, the word $\mathbf{u}$ has the form (6.10). Note that $\mathbf{u}' = \mathbf{w}a\mathbf{v}$ and

$$\mathbf{u}'' = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Recall that the claim (6.9) is true for any $2 \le s \le k$. Now we can apply the identity (6.3) sufficiently many times to the right-hand side of the identity (6.10) and replace second occurrence of x_{s-1} with the word $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We have $\mathbf{V}$ satisfies the identity

$$\begin{aligned}\mathbf{u} \approx{}& \mathbf{w}a\mathbf{v}ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots\\ &\cdot\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_0.\end{aligned}$$

Put $\mathbf{p}_1 = a\mathbf{v}$ and

$$\begin{aligned}\mathbf{p}_2 ={}& \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\\ &\cdot\mathbf{v}_4x_1\mathbf{v}_3.\end{aligned}$$

Then the last identity has the form

$$\mathbf{u} \approx \mathbf{w}\mathbf{p}_1ab\mathbf{p}_2\mathbf{v}_0. \tag{6.20}$$

By the hypothesis, $\mathbf{V}$ satisfies the identity γ_k. Now we perform the substitution

$$(x_0, x_1, \dots, x_{k-1}, x_k, y_0, y_1) \mapsto (\mathbf{v}_2x_0\mathbf{v}_1, \mathbf{v}_4x_1\mathbf{v}_3, \dots, \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}, b, \mathbf{v}, a)$$

in this identity. Then we obtain the identity $\mathbf{p}_1ba\mathbf{p}_2 \approx \mathbf{p}_1ab\mathbf{p}_2$. This identity together with (6.20) implies that $\mathbf{V}$ satisfies the identity $\mathbf{u} \approx \mathbf{w}\mathbf{p}_1ba\mathbf{p}_2\mathbf{v}_0$, i.e., the identity

$$\begin{aligned}\mathbf{u} \approx{}& \mathbf{w}a\mathbf{v}ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots\\ &\cdot\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_0.\end{aligned}$$

Now we apply the identity (6.3) to the right-hand side of the last identity "in the opposite direction" and replace the subword $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain the identity

$$\begin{aligned}\mathbf{u} &\approx \mathbf{w}a\mathbf{v}ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0\\ &= \mathbf{u}'ba\mathbf{u}''.\end{aligned}$$

Subcase 1.3: $\mathbf{V}$ satisfies the hypothesis of the claim (iii), i.e., β_k holds in $\mathbf{V}$ and $D(\mathbf{u}, a) \ne D(\mathbf{u}, b)$. Subcase 1.2 allows us to assume that $a \notin \mathrm{con}(\mathbf{u}')$. This fact and (6.6) immediately imply that $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, b)$. If $D(\mathbf{u}, a) \le k-1$ then a is a $(k-1)$-divider of $\mathbf{u}$ by Lemma 3.7. But this is not the case. Therefore, $D(\mathbf{u}, a) \ge k$. Since $D(\mathbf{u}, b) \ne D(\mathbf{u}, a)$ and $D(\mathbf{u}, b) = k$, we obtain $D(\mathbf{u}, a) > k$.

Note that $\ell_2(\mathbf{u},a) < \ell_1(\mathbf{u},x_{k-1})$ because $h_1^{k-1}(\mathbf{u},a) = h_2^{k-1}(\mathbf{u},a)$ and x_{k-1} is a $(k-1)$-divider. Recall that the equality (6.8) is true. Then $\mathbf{v}_{2k} = \mathbf{v}'_{2k} a \mathbf{v}''_{2k}$ for some possibly empty words $\mathbf{v}'_{2k}, \mathbf{v}''_{2k}$. Thus,

$$\mathbf{u} = \mathbf{u}' ab\mathbf{v}'_{2k} a\mathbf{v}''_{2k} x_{k-1}\mathbf{v}_{2k-1} b\mathbf{v}_{2k-2} x_{k-2}\mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0.$$

Now we are going to verify that the identity

$$\mathbf{u} \approx \mathbf{u}' aba\mathbf{v}_{2k} x_{k-1}\mathbf{v}_{2k-1} b\mathbf{v}_{2k-2} x_{k-2}\mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0 \tag{6.21}$$

holds in $\mathbf{V}$. This statement is evident whenever $\mathbf{v}'_{2k} = \lambda$. Suppose now that $\mathbf{v}'_{2k} = \mathbf{v}^* d$ for some possibly empty word $\mathbf{v}^*$ and some letter d. Then $\mathbf{u}$ may be rewritten in the form

$$\mathbf{u} = \mathbf{u}' \overset{(1)}{a}\, b\mathbf{v}^* d \,\overset{(2)}{a}\, \mathbf{v}''_{2k} x_{k-1}\mathbf{v}_{2k-1} b\mathbf{v}_{2k-2} x_{k-2}\mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0.$$

Note that the subword da located between $\mathbf{v}^*$ and $\mathbf{v}''_{2k}$ lies in some $(k-1)$-block of $\mathbf{u}$. Indeed, the occurrence of d in this subword is not a $(k-1)$-divider of $\mathbf{u}$ because otherwise the first and the second occurrences of a in $\mathbf{u}$ lies in different $(k-1)$-blocks, contradicting the inequality $D(\mathbf{u},a) > k$, while the occurrence of a in this subword is not a $(k-1)$-divider of $\mathbf{u}$ because this is not first occurrence of a in $\mathbf{u}$.

According to Lemma 6.4, the variety $\mathbf{V}$ satisfies the identity γ_k. In view of the statement that was proved in Subcase 1.2, $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{u}' ab\mathbf{v}^* ad\mathbf{v}''_{2k} x_{k-1}\mathbf{v}_{2k-1} b\mathbf{v}_{2k-2} x_{k-2}\mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0.$$

Acting in this way, we can successively swap the letter a with all letters of the word $\mathbf{v}'_{2k}$ and obtain

$$\mathbf{u} \approx \mathbf{u}' aba\mathbf{v}'_{2k}\mathbf{v}''_{2k} x_{k-1}\mathbf{v}_{2k-1} b\mathbf{v}_{2k-2} x_{k-2}\mathbf{v}_{2k-3} x_{k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0$$

holds in $\mathbf{V}$. Now we apply the identity (6.3) to the right-hand side of the last identity and insert the letter a after the word $\mathbf{v}'_{2k}$. We obtain the identity (6.21).

Recall that (6.9) is true for any $2 \le s \le k$. Now we can apply the identity (6.3) sufficiently many times to the right-hand side of the identity (6.21) and replace second occurrence of x_{s-1} with the word $\mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We have $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{u}' aba\mathbf{p}\mathbf{v}_0 \tag{6.22}$$

where

$$\mathbf{p} = \mathbf{v}_{2k} x_{k-1}\mathbf{v}_{2k-1} b\mathbf{v}_{2k-2} x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k} x_{k-1}\mathbf{v}_{2k-1} \cdots \mathbf{v}_4 x_1 \mathbf{v}_3 \mathbf{v}_6 x_2 \mathbf{v}_5 \mathbf{v}_2 x_0 \mathbf{v}_1 \cdot \mathbf{v}_4 x_1 \mathbf{v}_3.$$

Now we perform the substitution

$$(x_0, x_1, \dots, x_{k-1}, x_k, x) \mapsto (\mathbf{v}_2 x_0 \mathbf{v}_1, \mathbf{v}_4 x_1 \mathbf{v}_3, \dots, \mathbf{v}_{2k} x_{k-1} \mathbf{v}_{2k-1}, b, a)$$

in the identity β_k. Then we obtain the identity $aba\mathbf{p} \approx ba^2\mathbf{p}$. One can apply this identity to the identity (6.22). We get that the identity

$$\mathbf{u} \approx \mathbf{u}' ba^2\mathbf{p}\mathbf{v}_0 = \mathbf{u}' ba^2 \mathbf{v}_{2k} x_{k-1}\mathbf{v}_{2k-1} b\mathbf{v}_{2k-2} x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k} x_{k-1}\mathbf{v}_{2k-1} \cdots \cdot \mathbf{v}_4 x_1 \mathbf{v}_3 \mathbf{v}_6 x_2 \mathbf{v}_5 \mathbf{v}_2 x_0 \mathbf{v}_1 \mathbf{v}_4 x_1 \mathbf{v}_3 \mathbf{v}_0$$

holds in $\mathbf{V}$. Now we apply the identity (6.3) to the right-hand side of the last identity "in the opposite direction" and replace the subword $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain the identity

$$\mathbf{u} \approx \mathbf{u}'ba^2\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Repeating arguments used above in the deduction of the identity (6.21), we obtain $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{u}'ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0 = \mathbf{u}'ba\mathbf{u}''$$
$$= \mathbf{u}'ba\mathbf{u}''.$$

Subcase 1.4: $\mathbf{V}$ satisfies the hypothesis of the claim (iv), i.e., α_k holds in $\mathbf{V}$. By Subcases 1.2 and 1.3 and the claim (6.6), we can assume that $a, b \notin \operatorname{con}(\mathbf{u}')$ and $D(\mathbf{u}, b) = D(\mathbf{u}, a)$. Recall that the equality (6.8) is true.

Note that $\ell_2(\mathbf{u}, a) < \ell_1(\mathbf{u}, x_{k-2})$ because $h_1^{k-2}(\mathbf{u}, a) = h_2^{k-2}(\mathbf{u}, a)$ and x_{k-2} is a $(k-2)$-divider. Therefore, there are possibly empty words $\mathbf{v}'$ and $\mathbf{v}''$ such that one of the following equalities holds:

$$\mathbf{v}_{2k} = \mathbf{v}'a\mathbf{v}'',\ \mathbf{v}_{2k-1} = \mathbf{v}'a\mathbf{v}'' \text{ or } \mathbf{v}_{2k-2} = \mathbf{v}'a\mathbf{v}''.$$

Then one of the following equalities holds:

$$\mathbf{u} = \mathbf{u}'ab\mathbf{v}'a\mathbf{v}''x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0,$$
$$\mathbf{u} = \mathbf{u}'ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}'a\mathbf{v}''b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0,$$
$$\mathbf{u} = \mathbf{u}'ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}'a\mathbf{v}''x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

We consider only the first case. Two other cases can be considered similarly. Since the variety $\mathbf{V}$ satisfies the identity (6.3), we obtain the identity

$$\mathbf{u} \approx \mathbf{u}'ab\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}ab\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0 \tag{6.23}$$

holds in this variety.

Recall that the claim (6.9) is true for any $2 \le s \le k$. Now we can apply the identity (6.3) sufficiently many times to the right-hand side of the identity (6.23) and replace second occurrence of x_{s-1} with the word $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ for any $2 \le s \le k$. We have $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{u}'ab\mathbf{p}\mathbf{v}_0 \tag{6.24}$$

where

$$\mathbf{p} = \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}ab\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3\mathbf{v}_6x_2\mathbf{v}_5\mathbf{v}_2x_0\mathbf{v}_1$$
$$\cdot\ \mathbf{v}_4x_1\mathbf{v}_3.$$

Now we perform the substitution

$$(x_0, x_1, \dots, x_{k-1}, x_k, y_k) \mapsto (\mathbf{v}_2x_0\mathbf{v}_1, \mathbf{v}_4x_1\mathbf{v}_3, \dots, \mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}, a, b)$$

in the identity α_k. Then we obtain the identity $ab\mathbf{p} \approx ba\mathbf{p}$. Let us apply this identity to the identity (6.24). We get that the identity $\mathbf{u} \approx \mathbf{u}'ba\mathbf{p}\mathbf{v}_0$ holds in $\mathbf{V}$. Now we apply the identity (6.3) to the right-hand side of the last identity "in the opposite direction" and replace the subword $\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}$ with x_{s-1} for any $2 \le s \le k$. As a result, we obtain the identity

$$\mathbf{u} \approx \mathbf{u}'ba\mathbf{v}_{2k}x_{k-1}\mathbf{v}_{2k-1}ab\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0.$$

Now we apply the identity (6.3) again and delete the occurrence of a located between $\mathbf{v}_{2k-1}$ and second occurrence of b in the right-hand side of the last identity. We obtain $\mathbf{V}$ satisfies the identity

$$\begin{aligned}\mathbf{u} &\approx \mathbf{u}'ba\mathbf{v}'a\mathbf{v}''x_{k-1}\mathbf{v}_{2k-1}b\mathbf{v}_{2k-2}x_{k-2}\mathbf{v}_{2k-3}x_{k-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0\\ &= \mathbf{u}'ba\mathbf{u}''.\end{aligned}$$

Case 2: $k < D(\mathbf{u}, b) < \infty$. As we will see below, this case reduces to the previous one by relatively simple arguments. Put $D(\mathbf{u}, b) = r$. Further considerations are divided into three subcases.

Subcase 2.1: $\mathbf{V}$ satisfies the hypothesis of one of the claims (i) and (ii). Here $a \in \operatorname{con}(\mathbf{u}')$. Hence the occurrence of the letter a in the subword ab of the word $\mathbf{u}$ mentioned in the formulation of the lemma is not first occurrence of a in $\mathbf{u}$. Therefore, this occurrence of a in $\mathbf{u}$ is not an $(r-1)$-divider of $\mathbf{u}$. Lemma 3.7 together with the fact that $D(\mathbf{u}, b) = r$ implies that the occurrence of the letter b in the same subword ab of the word $\mathbf{u}$ also is not an $(r-1)$-divider of $\mathbf{u}$. Therefore, the above-mentioned subword ab of the word $\mathbf{u}$ lies in some $(r-1)$-block of $\mathbf{u}$.

Let

$$s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_n\mathbf{w}_n \tag{6.25}$$

be the $(r-1)$-decomposition of $\mathbf{u}$. Then there exists a number $0 \le j \le n$ such that $\mathbf{w}_j = \mathbf{w}_j'ab\mathbf{w}_j''$, whence

$$\mathbf{u}' = s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j' \text{ and } \mathbf{u}'' = \mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n.$$

Since $\mathbf{J}_k^m \subseteq \mathbf{J}_r^m$ and $\mathbf{I}_k \subseteq \mathbf{I}_r$ by Lemma 6.4, we apply the statements that proved in Subcases 1.1 and 1.2 and obtain the required conclusion that the identity $\mathbf{u} \approx \mathbf{u}'ba\mathbf{u}''$ holds in $\mathbf{V}$.

Subcase 2.2: $\mathbf{V}$ satisfies the hypothesis of the claim (iii), i.e., β_k holds in $\mathbf{V}$ and $D(\mathbf{u}, a) \ne D(\mathbf{u}, b)$. Subcase 2.1 allows us to assume that $a \notin \operatorname{con}(\mathbf{u}')$.

Suppose that $D(\mathbf{u}, a) = s < r$. If $s \le k-1$ then a is a $(k-1)$-divider of $\mathbf{u}$ by Lemma 3.7. But this is not the case because first occurrence of a in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$. Therefore, $s \ge k$. Let (6.25) be the $(s-1)$-decomposition of $\mathbf{u}$. Then there exists a number $0 \le j \le n$ such that $\mathbf{w}_j = \mathbf{w}_j'ab\mathbf{w}_j''$, $\mathbf{u}' = s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j'$ and $\mathbf{u}'' = \mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. Put $\mathbf{u}^* = \mathbf{u}'ba\mathbf{u}''$. Since $a, b \notin \{s_1, s_2, \dots, s_n\}$, the $(s-1)$-decomposition of $\mathbf{u}^*$ has the form

$$s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j^*\cdots s_n\mathbf{w}_n$$

where $\mathbf{w}_j^* = \mathbf{w}_j'ba\mathbf{w}_j''$. Then the claims (2.1) and (3.6) with $\mathbf{v} = \mathbf{u}^*$ and $\ell = s$ are true. Now Lemma 3.12 applies with the conclusion that $D(\mathbf{u}^*, a) = s$. Since $\mathbf{V}$ satisfies the identity β_s by Lemma 6.4, we apply the statement that proved in Subcase 1.3 and obtain the identity $\mathbf{u}^* = \mathbf{u}'ba\mathbf{u}'' \approx \mathbf{u}'ab\mathbf{u}'' = \mathbf{u}$ holds in $\mathbf{V}$.

Suppose now that $D(\mathbf{u}, a) > r$. Let now (6.25) be the $(r-1)$-decomposition of $\mathbf{u}$. Then there exists a number $0 \le j \le n$ such that $\mathbf{w}_j = \mathbf{w}_j'ab\mathbf{w}_j''$, whence $\mathbf{u}' = s_0\mathbf{w}_0s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j'$ and $\mathbf{u}'' = \mathbf{w}_j''s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. Since $\mathbf{H}_k \subseteq \mathbf{H}_r$ by Lemma 6.4, we apply the statement that proved in Subcase 1.3 and obtain the identity $\mathbf{u} \approx \mathbf{u}'ba\mathbf{u}''$ holds in $\mathbf{V}$.

Subcase 2.3: $\mathbf{V}$ satisfies the hypothesis of the claim (iv), i.e., α_k holds in $\mathbf{V}$. Subcase 2.2 allows us to assume that $D(\mathbf{u},a) = D(\mathbf{u},b)$. Put $D(\mathbf{u},a) = r$. Then the subword ab of the word $\mathbf{u}$ above-mentioned in the formulation of the lemma lies in some $(r-1)$-block of $\mathbf{u}$. Let (6.25) be the $(r-1)$-decomposition of $\mathbf{u}$. Then there exists a number $0 \le j \le n$ such that $\mathbf{w}_j = \mathbf{w}_j' ab \mathbf{w}_j''$, whence $\mathbf{u}' = s_0\mathbf{w}_0 s_1 \mathbf{w}_1 \cdots s_j \mathbf{w}_j'$ and $\mathbf{u}'' = \mathbf{w}_j'' s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. Since $\mathbf{F}_k \subseteq \mathbf{F}_r$ by Lemma 6.4, we apply the statement that proved in Subcase 1.4 and obtain the identity $\mathbf{u} \approx \mathbf{u}' ba \mathbf{u}''$ holds in $\mathbf{V}$.

Case 3: $D(\mathbf{u},b) = \infty$. This case, as well as the previous one, is divided into three subcases.

Subcase 3.1: $\mathbf{V}$ satisfies the hypothesis of one of the claims (i) and (ii). Let s be a non-negative integer. Repeating literally arguments from Subcase 2.1, we obtain the subword ab of the word $\mathbf{u}$ above-mentioned in the formulation of the lemma lies in some s-block of $\mathbf{u}$. By Remark 3.2, there is a number $r \ge k$ such that (6.25) is the ℓ-decomposition of $\mathbf{u}$ for any $\ell \ge r$. Then ab is a subword of $\mathbf{w}_j$ for some $0 \le j \le n$. We have $\mathbf{w}_j = \mathbf{w}_j' ab \mathbf{w}_j''$ for some possibly empty words $\mathbf{w}_j'$ and $\mathbf{w}_j''$. Then $\mathbf{u}' = s_0\mathbf{w}_0 s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j'$ and $\mathbf{u}'' = \mathbf{w}_j'' s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$. One can prove that

$$\mathrm{occ}_z(\mathbf{w}_j) \ge 2 \tag{6.26}$$

for any letter $z \in \mathrm{con}(\mathbf{w}_j)$. Suppose at first that $s_j = h_1^r(\mathbf{u},z)$ and $\mathrm{occ}_z(\mathbf{w}_j) = 1$. If $\mathrm{occ}_z(\mathbf{u}) = 1$ then z is a 0-divider of $\mathbf{u}$. Lemma 3.5(i) implies that then $z \in \{s_1, s_2, \dots, s_n\}$, a contradiction. Therefore, $\mathrm{occ}_z(\mathbf{u}) \ge 2$. Since $\mathrm{occ}_z(\mathbf{w}_j) = 1$, we have $s_j \ne h_2^r(\mathbf{u},z)$. This means that $D(\mathbf{u},z) \le r+1$. According to Lemma 3.7, z is an $(r+1)$-divider of $\mathbf{u}$. We obtain a contradiction with the fact that (6.25) is the $(r+1)$-decomposition of $\mathbf{u}$. So, the claim (6.26) is true whenever $s_j = h_1^r(\mathbf{u},z)$. Suppose now that $s_j \ne h_1^r(\mathbf{u},z)$. Then the $(1,r)$-restrictor of z in $\mathbf{u}$ is s_p for some $p < j$. This means that $z \in \mathrm{con}(s_0\mathbf{w}_0 s_1\mathbf{w}_1\cdots s_{j-1}\mathbf{w}_{j-1})$. Then

$$\mathbf{u} = \mathbf{f} z \mathbf{g} \mathbf{w}_{j1} z \mathbf{w}_{j2} s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n$$

for some possibly empty words $\mathbf{f}, \mathbf{g}, \mathbf{w}_{j1}$ and $\mathbf{w}_{j2}$ with

$$\mathbf{f} z \mathbf{g} = s_0\mathbf{w}_0 s_1\mathbf{w}_1\cdots s_{j-1}\mathbf{w}_{j-1}\mathbf{s}_j$$

and $\mathbf{w}_j = \mathbf{w}_{j1} z \mathbf{w}_{j2}$. Then the identity (4.9) applies with the conclusion that $\mathbf{V}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{f} z \mathbf{g} \mathbf{w}_{j1} z^2 \mathbf{w}_{j2} s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n.$$

Therefore, we can assume that (6.26) is true again. Thus, this claim holds for any letter $z \in \mathrm{con}(\mathbf{w}_j)$. Then Lemma 6.2(iii) implies that the variety $\mathbf{V}$ satisfies $\mathbf{w}_j \approx \mathbf{w}_j' ba \mathbf{w}_j''$, whence

$$\begin{aligned}\mathbf{u} &= s_0\mathbf{w}_0 s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n\\ &\approx s_0\mathbf{w}_0 s_1\mathbf{w}_1\cdots s_j\mathbf{w}_j' ba \mathbf{w}_j'' s_{j+1}\mathbf{w}_{j+1}\cdots s_n\mathbf{w}_n\\ &= \mathbf{u}' ba \mathbf{u}''\end{aligned}$$

hold in this variety.

We have completed the proof of the statements (i) and (ii).

Subcase 3.2: $\mathbf{V}$ satisfies the hypothesis of the claim (iii), i.e., β_k holds in $\mathbf{V}$ and $D(\mathbf{u}, a) \ne D(\mathbf{u}, b)$. Then $D(\mathbf{u}, a) < \infty$. Put $D(\mathbf{u}, a) = r$. Repeating arguments from Subcase 1.3, we have $a \notin \mathrm{con}(\mathbf{u}')$ and $r \ge k$. Let (6.25) be the $(r-1)$-decomposition of $\mathbf{u}$. Then there exists a number $0 \le j \le n$ such that $\mathbf{w}_j = \mathbf{w}_j' ab \mathbf{w}_j''$, $\mathbf{u}' = s_0\mathbf{w}_0 s_1 \mathbf{w}_1 \cdots s_j \mathbf{w}_j'$ and $\mathbf{u}'' = \mathbf{w}_j'' s_{j+1}\mathbf{w}_{j+1} \cdots s_n \mathbf{w}_n$. Put $\mathbf{u}^* = \mathbf{u}' ba \mathbf{u}''$. Since $a, b \notin \{s_1, s_2, \dots, s_n\}$, the $(r-1)$-decomposition of $\mathbf{u}^*$ has the form $s_0\mathbf{w}_0 s_1 \mathbf{w}_1 \cdots s_j \mathbf{w}_j^* \cdots s_n \mathbf{w}_n$ where $\mathbf{w}_j^* = \mathbf{w}_j' ba \mathbf{w}_j''$. Then the claims (2.1) and (3.6) with $\mathbf{v} = \mathbf{u}^*$ and $\ell = r$ are true. Now Lemma 3.12 applies with the conclusion that $D(\mathbf{u}^*, a) = r$, whence a is an r-divider of $\mathbf{u}^*$ by Lemma 3.7. Then $h_1^r(\mathbf{u}^*, b) \ne h_2^r(\mathbf{u}^*, b)$. This implies that $D(\mathbf{u}^*, b) > r$. Since $\mathbf{V}$ satisfies the identity β_r by Lemma 6.4, we apply the statement that proved in Subcase 1.3 and obtain the identities $\mathbf{u}^* = \mathbf{u}' ba \mathbf{u}'' \approx \mathbf{u}' ab \mathbf{u}'' = \mathbf{u}$ hold in $\mathbf{V}$.

We have completed the proof of the statement (iii).

Subcase 3.3: $\mathbf{V}$ satisfies the hypothesis of the claim (iv), i.e., α_k holds in $\mathbf{V}$. Subcase 3.2 allows us to assume that $D(\mathbf{u}, a) = D(\mathbf{u}, b) = \infty$. This fact together with Lemma 3.7 implies that the subword ab of the word $\mathbf{u}$ abovementioned in the formulation of the lemma lies in some s-block of $\mathbf{u}$ for any s. Now we repeat literally arguments used in Subcase 3.1 and prove that the identity $\mathbf{u} \approx \mathbf{u}' ba \mathbf{u}''$ holds in $\mathbf{V}$.

We have completed the proof of the statement (iv) and the lemma as a whole. □

6.3. **Reduction to intervals of the form $[\mathbf{F}_k, \mathbf{F}_{k+1}]$.** Here we prove the claim 3) of Proposition 6.1. We need several auxiliary results.

Lemma 6.7. *Let $\mathbf{V}$ be a monoid variety such that $\mathbf{V} \subseteq \mathbf{K}$ and $\mathbf{V}$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$, and s be a natural number. Suppose that the claims* (2.1) *and* (3.6) *with $\ell = s$ are true and there are letters x and x_s such that $D(\mathbf{u}, x_s) = s$, $\ell_i(\mathbf{u}, x) < \ell_1(\mathbf{u}, x_s)$ and $\ell_1(\mathbf{v}, x_s) < \ell_i(\mathbf{v}, x)$ for some $i \in \{1, 2\}$.*

(i) *If $i = 1$ then $\mathbf{V} \subseteq \mathbf{H}_s$.*
(ii) *If $i = 2$ then $\mathbf{V} \subseteq \mathbf{J}_s^s$.*

Proof. Lemma 6.2(iii) allows us to assume that $\mathrm{occ}_y(\mathbf{u}), \mathrm{occ}_y(\mathbf{v}) \le 2$ for any letter y. Now Lemma 3.14 applies with the conclusion that there are letters $x_0, x_1, \dots, x_{s-1}$ such that $D(\mathbf{u}, x_r) = D(\mathbf{v}, x_r) = r$ for any $0 \le r < s$ and the identity $\mathbf{u} \approx \mathbf{v}$ has the form (3.7) for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2s+1}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2s+1}$.

Suppose that $i = 1$. Then $\ell_1(\mathbf{u}, x) < \ell_1(\mathbf{u}, x_s)$ and $\ell_1(\mathbf{v}, x_s) < \ell_1(\mathbf{v}, x)$. Suppose that $\ell_1(\mathbf{u}, x_s) < \ell_2(\mathbf{u}, x)$. In view of the above, we have

- first occurrence of x in $\mathbf{u}$ lies in $\mathbf{u}_{2s+1}$,
- second occurrence of x in $\mathbf{u}$ lies in $\mathbf{u}_{2s}\mathbf{u}_{2s-1} \cdots \mathbf{u}_0$,
- the first and the second occurrences of x in $\mathbf{v}$ lie in $\mathbf{v}_{2s}\mathbf{v}_{2s-1} \cdots \mathbf{v}_0$.

Now we substitute $x_s x^2$ for x_s in the identity $\mathbf{u} \approx \mathbf{v}$ and obtain the identity

$$\begin{aligned} &\mathbf{u}_{2s+1} x_s x^2 \mathbf{u}_{2s} x_{s-1} \mathbf{u}_{2s-1} x_s x^2 \mathbf{u}_{2s-2} x_{s-2} \mathbf{u}_{2s-3} x_{s-1} \cdots \\ &\cdot \mathbf{u}_4 x_1 \mathbf{u}_3 x_2 \mathbf{u}_2 x_0 \mathbf{u}_1 x_1 \mathbf{u}_0 \\ \approx\ &\mathbf{v}_{2s+1} x_s x^2 \mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1} x_s x^2 \mathbf{v}_{2s-2} x_{s-2} \mathbf{v}_{2s-3} x_{s-1} \cdots \\ &\cdot \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0. \end{aligned} \tag{6.27}$$

Further, we apply the identity (6.3) and delete the third and subsequent occurrences of x in both the sides of the identity (6.27). As a result, we obtain the identity

$$\mathbf{u}_{2s+1}x_sx(\mathbf{u}_{2s}x_{s-1}\mathbf{u}_{2s-1}x_s\mathbf{u}_{2(s-1)}x_{s-2}\mathbf{u}_{2s-1}x_{s-1}\cdots\mathbf{u}_4x_1\mathbf{u}_3x_2\mathbf{u}_2x_0\mathbf{u}_1x_1\mathbf{u}_0)_x$$
$$\approx \mathbf{v}_{2s+1}x_sx^2(\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}x_s\mathbf{v}_{2(s-1)}x_{s-2}\mathbf{v}_{2s-1}x_{s-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0)_x.$$

Now substitute 1 for all letters occurring in the last identity except $x, x_0, x_1, \dots, x_s$. We get the identity

$$xx_sxx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1 \approx x_sx^2x_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1,$$

i.e., the identity β_s. Therefore, $\mathbf{V}\subseteq\mathbf{H}_s$.

Suppose now that $\ell_2(\mathbf{u},x)<\ell_1(\mathbf{u},x_s)$. In view of the above, we have

- the first and the second occurrences of x in $\mathbf{u}$ lie in $\mathbf{u}_{2s+1}$,
- the first and the second occurrences of x in $\mathbf{v}$ lie in $\mathbf{v}_{2s}\mathbf{v}_{2s-1}\cdots\mathbf{v}_0$.

Now we substitute x_sx^2 for x_s in the identity $\mathbf{u}\approx\mathbf{v}$ and obtain the identity (6.27). The identity (6.3) allows us to delete the third and subsequent occurrences of x in both the sides of the identity (6.27). As a result, we obtain the identity

$$\mathbf{u}_{2s+1}x_s\mathbf{u}_{2s}x_{s-1}\mathbf{u}_{2s-1}x_s\mathbf{u}_{2s-2}x_{s-2}\mathbf{u}_{2s-3}x_{s-1}\cdots\mathbf{u}_4x_1\mathbf{u}_3x_2\mathbf{u}_2x_0\mathbf{u}_1x_1\mathbf{u}_0$$
$$\approx \mathbf{v}_{2s+1}x_sx^2(\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}x_s\mathbf{v}_{2(s-1)}x_{s-2}\mathbf{v}_{2s-1}x_{s-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0)_x.$$

Now substitute 1 for all letters occurring in the last identity except $x, x_0, x_1, \dots, x_s$. We get the identity

$$x^2x_sx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1 \approx x_sx^2x_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1. \tag{6.28}$$

Then $\mathbf{V}$ satisfies the identities

$$\begin{aligned}x_sx^2x_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1 &\stackrel{(6.28)}{\approx} x^2x_sx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1\\ &\stackrel{(4.5)}{\approx} x^3x_sx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1\\ &\stackrel{(6.28)}{\approx} xx_sx^2x_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1\\ &\stackrel{(6.3)}{\approx} xx_sxx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1,\end{aligned}$$

whence the identity β_s holds in $\mathbf{V}$. Therefore, $\mathbf{V}\subseteq\mathbf{H}_s$. The claim (i) is proved.

Suppose now that $i=2$. Then $\ell_1(\mathbf{u},x)<\ell_2(\mathbf{u},x)<\ell_1(\mathbf{u},x_s)$. If $\ell_1(\mathbf{v},x_s)<\ell_1(\mathbf{v},x)$ then we return to the already proved claim (i). Then $\mathbf{V}\subseteq\mathbf{H}_s\subseteq\mathbf{J}_s^s$ by Lemma 6.4. So, we can assume that $\ell_1(\mathbf{v},x)<\ell_1(\mathbf{v},x_s)$. In view of the above, we have

- the first and the second occurrences of x in $\mathbf{u}$ lie in $\mathbf{u}_{2s+1}$,
- first occurrence of x in $\mathbf{v}$ lies in $\mathbf{v}_{2s+1}$,
- second occurrence of x in $\mathbf{v}$ lies in $\mathbf{v}_{2s}\mathbf{v}_{2s-1}\cdots\mathbf{v}_0$.

Now we substitute x_sx^2 for x_s in the identity $\mathbf{u}\approx\mathbf{v}$ and obtain the identity (6.27). The identity (6.3) allows us to delete the third and subsequent occurrences of x in both the sides of the identity (6.27). As a result, we obtain the identity

$$\mathbf{u}_{2s+1}x_s\mathbf{u}_{2s}x_{s-1}\mathbf{u}_{2s-1}x_s\mathbf{u}_{2s-2}x_{s-2}\mathbf{u}_{2s-3}x_{s-1}\cdots\mathbf{u}_4x_1\mathbf{u}_3x_2\mathbf{u}_2x_0\mathbf{u}_1x_1\mathbf{u}_0$$
$$\approx \mathbf{v}_{2s+1}x_sx(\mathbf{v}_{2s}x_{s-1}\mathbf{v}_{2s-1}x_s\mathbf{v}_{2(s-1)}x_{s-2}\mathbf{v}_{2s-1}x_{s-1}\cdots\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0)_x.$$

Now substitute 1 for all letters occurring in the last identity except $x, x_0, x_1, \dots,$ x_s. We get the identity

$$x^2x_sx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1 \approx xx_sxx_{s-1}x_sx_{s-2}x_{s-1}\cdots x_1x_2x_0x_1,$$

i.e., the identity (6.4) with $k=s$. Lemma 6.3 implies now that $\mathbf{V}\subseteq\mathbf{J}_s^s$. The claim (ii) is proved. □

Lemma 6.8. *Let* $\mathbf{V}$ *be a monoid variety such that* $\mathbf{V}\subseteq\mathbf{K}$ *and* $\mathbf{V}$ *satisfies an identity* $\mathbf{u}\approx\mathbf{v}$*. If the claim* (2.1) *is true, while the claim* (3.6) *is false for some* $\ell>1$ *then* $\mathbf{V}\subseteq\mathbf{J}_{\ell-1}^{\ell-1}$.

Proof. Suppose that the claim (2.1) is true, while the claim (3.6) is false for some $\ell=k>1$ and k is the least number with this property. Then there exists a letter x such that $h_i^{k-1}(\mathbf{u},x)\ne h_i^{k-1}(\mathbf{v},x)$ where either $i=1$ or $i=2$. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$. In particular, the set of $(k-1)$-dividers of $\mathbf{u}$ is $\{t_0,t_1,\dots,t_m\}$. Since the claim (3.6) with $\ell=k-1$ is true, Lemma 3.10 applies with the conclusion that $\mathbf{v}$ has the same $(k-1)$-dividers as $\mathbf{u}$ (but the order of the first occurrences of these letters in the words $\mathbf{u}$ and $\mathbf{v}$ may be different). Put $t_p=h_i^{k-1}(\mathbf{u},x)$ and $t_q=h_i^{k-1}(\mathbf{v},x)$. Clearly, $p\ne q$.

Suppose at first that $\ell_i(\mathbf{u},x)<\ell_1(\mathbf{u},t_q)$. The choice of t_p and t_q guarantees that $\ell_1(\mathbf{u},t_p)<\ell_i(\mathbf{u},x)$ and $\ell_1(\mathbf{v},t_q)<\ell_i(\mathbf{v},x)$. Therefore, $\ell_1(\mathbf{u},t_p)<\ell_1(\mathbf{u},t_q)$, whence $p<q$ in the case we consider. If t_q is simple in $\mathbf{u}$ then the claim (2.1) implies that t_q is simple in $\mathbf{v}$ too. Therefore, the letter t_q is a 0-divider of the words $\mathbf{u}$ and $\mathbf{v}$. Since $t_q=h_i^{k-1}(\mathbf{v},x)$, we have $t_q=h_i^0(\mathbf{v},x)$. The claim (3.6) with $\ell=1$ implies that $t_q=h_i^0(\mathbf{u},x)$. But this contradicts the fact that $p<q$. So, t_q is multiple in $\mathbf{u}$, whence t_q is multiple in $\mathbf{v}$ as well by the claim (2.1). Therefore, $D(\mathbf{v},t_q)>0$. Besides that, $D(\mathbf{v},t_q)\le k-1$ by Lemma 3.7 because t_q is a $(k-1)$-divider of $\mathbf{v}$. Put $r=D(\mathbf{v},t_q)$. If $i=1$ then Lemma 6.7(i) with $s=r$ and $x_s=t_q$ applies with the conclusion that $\mathbf{V}\subseteq\mathbf{H}_r\subseteq\mathbf{J}_{k-1}^{k-1}$. If $i=2$ then $\mathbf{V}\subseteq\mathbf{J}_r^r\subseteq\mathbf{J}_{k-1}^{k-1}$ by Lemma 6.7(ii) with $s=r$ and $x_s=t_q$.

If $\ell_i(\mathbf{v},x)<\ell_1(\mathbf{v},t_p)$ then we can obtain the required conclusion using arguments similar to ones from the previous paragraph.

Finally, suppose that $\ell_1(\mathbf{u},t_q)<\ell_i(\mathbf{u},x)$ and $\ell_1(\mathbf{v},t_p)<\ell_i(\mathbf{v},x)$. The first of these inequalities implies that first occurrence of t_q in $\mathbf{u}$ precedes ith occurrence of x in $\mathbf{u}$. But t_p is the right-most $(k-1)$-divider of the word $\mathbf{u}$ precedes ith occurrence of x. Therefore, $\ell_1(\mathbf{u},t_q)<\ell_1(\mathbf{u},t_p)$. Analogously, it follows from $\ell_1(\mathbf{v},t_p)<\ell_i(\mathbf{v},x)$ and $t_q=h_i^{k-1}(\mathbf{v},x)$ that $\ell_1(\mathbf{v},t_p)<\ell_1(\mathbf{v},t_q)$. Suppose that t_p is simple in $\mathbf{u}$. Then the claim (2.1) implies that t_p is simple in $\mathbf{v}$ too. Then the letter t_p is a 0-divider of the words $\mathbf{u}$ and $\mathbf{v}$. Since $t_p=h_i^{k-1}(\mathbf{u},x)$, we have $t_p=h_i^0(\mathbf{u},x)$. The claim (3.6) with $\ell=1$ implies that $t_p=h_i^0(\mathbf{v},x)$. Note that $\ell_1(\mathbf{v},t_p)<\ell_1(\mathbf{v},t_q)<\ell_i(\mathbf{v},x)$. Being the right-most simple in $\mathbf{v}$ letter that is located to the left of x, the letter t_p turns out to be also the right-most simple in $\mathbf{v}$ letter that is located to the left of t_q. In other words, $t_p = h_1^0(\mathbf{v}, t_q)$. The claim (3.6) with $\ell = 1$ implies that $t_p = h_1^0(\mathbf{u}, t_q)$. But this contradicts the fact that $\ell_1(\mathbf{u}, t_q) < \ell_1(\mathbf{u}, t_p)$. So, t_p is multiple in $\mathbf{u}$. Therefore, $D(\mathbf{u}, t_p) > 0$. Besides that, $D(\mathbf{u}, t_p) \le k-1$ by Lemma 3.7 because t_p is a $(k-1)$-divider of $\mathbf{u}$. Put $r = D(\mathbf{u}, t_p)$. Then the hypothesis of Lemma 6.7 with $i = 1$, $s = r$, $x = t_q$ and $x_s = t_p$ holds. Therefore, Lemma 6.7(i) implies that $\mathbf{V} \subseteq \mathbf{H}_r \subseteq \mathbf{J}_{k-1}^{k-1}$. $\square$

The following statement opens a series of one-type assertions, which also includes Propositions 6.12, 6.14 and 6.17. These results provide solutions of the word problem in the varieties $\mathbf{F}_k$, $\mathbf{H}_k$, $\mathbf{I}_k$, $\mathbf{J}_k^m$ and $\mathbf{K}$. All of them are proved in the same scheme. For the "only if" part, this scheme almost does not changed from proposition to proposition. As to the "if" part, the mentioned scheme is generally outlined in the proof of Proposition 6.9(i) but technically its implementation will be more and more complicated each time.

Proposition 6.9. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds:*

(i) *in the variety* $\mathbf{F}_k$ *if and only if the claims* (2.1) *and* (3.6) *with* $\ell = k$ *are true;*

(ii) *in the variety* $\mathbf{K}$ *if and only if the claims* (2.1) *and* (3.6) *for all* ℓ *are true.*

Proof. (i) *Necessity.* Suppose that a non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{F}_k$. Proposition 2.2 and the inclusion $\mathbf{C}_2 \subseteq \mathbf{F}_k$ imply that the claim (2.1) is true. Since $\mathbf{F}_k$ satisfies $\mathbf{u} \approx \mathbf{v}$, there is a sequence of words $\mathbf{u} = \mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_n = \mathbf{v}$ such that, for any $i = 0, 1, \dots, n-1$, there are words $\mathbf{p}_i, \mathbf{q}_i \in F^1$, an endomorphism ξ_i of F^1 and an identity $\mathbf{a}_i \approx \mathbf{b}_i$ from the system $\{\Phi, \alpha_k\}$ such that either $\mathbf{w}_i = \mathbf{p}_i\xi_i(\mathbf{a}_i)\mathbf{q}_i$ and $\mathbf{w}_{i+1} = \mathbf{p}_i\xi_i(\mathbf{b}_i)\mathbf{q}_i$ or $\mathbf{w}_i = \mathbf{p}_i\xi_i(\mathbf{b}_i)\mathbf{q}_i$ and $\mathbf{w}_{i+1} = \mathbf{p}_i\xi_i(\mathbf{a}_i)\mathbf{q}_i$. By induction we can assume without loss of generality that $\mathbf{u} = \mathbf{p}\xi(\mathbf{a})\mathbf{q}$ and $\mathbf{v} = \mathbf{p}\xi(\mathbf{b})\mathbf{q}$ for some possibly empty words $\mathbf{p}$ and $\mathbf{q}$, an endomorphism ξ of F^1 and an identity $\mathbf{a} \approx \mathbf{b} \in \{\Phi, \alpha_k\}$.

If $\mathbf{a} \approx \mathbf{b} \in \{xyx \approx xyx^2, x^2y \approx x^2yx\}$ then the required assertion is obvious because the first and second occurrences of the letters of $\mathbf{u}$ do not take part in replacing $\xi(\mathbf{a})$ to $\xi(\mathbf{b})$. Suppose now that $\mathbf{a} \approx \mathbf{b}$ coincides with the identity (4.4). Then, since $D(\mathbf{a}, x) = D(\mathbf{a}, y) = \infty$, Lemma 3.15 implies that the subword $\xi(\mathbf{a})$ of $\mathbf{u}$ located between $\mathbf{p}$ and $\mathbf{q}$ is contained in some s-block for all s. In particular, this subword is contained is some $(k-1)$-block. This implies that the claim (3.6) with $\ell = k$ is true.

Finally, suppose that $\mathbf{a} \approx \mathbf{b}$ coincides with α_k. Then

$$\xi(\mathbf{a}) = \mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1,$$

$$\xi(\mathbf{b}) = \mathbf{b}_k\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1$$

for some words $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k$ and $\mathbf{b}_k$, whence

$$\mathbf{u} = \mathbf{p}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q},$$

$$\mathbf{v} = \mathbf{p}\mathbf{b}_k\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{b}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q}.$$

By Lemma 6.5, $D(\mathbf{a}, x_k) = D(\mathbf{a}, y_k) = k$. Then Lemma 3.15 implies that the subword $\mathbf{a}_k\mathbf{b}_k$ of $\mathbf{u}$ located between $\mathbf{p}$ and $\mathbf{a}_{k-1}$ is contained in some $(k-1)$-block. This implies that the claim (3.6) with $\ell = k$ is true.

Sufficiency. Let us describe the outline of our further arguments. We note that sufficiency in Propositions 6.12, 6.14 and 6.17 will be proved below in the same scheme. Let $\mathbf{u} \approx \mathbf{v}$ be an identity which satisfies the hypothesis of the proposition. We start with a consideration of the $(k-1)$-decomposition of the word $\mathbf{u}$. Basing on Lemma 6.6 and using identities which hold in the variety $\mathbf{F}_k$, we show that any $(k-1)$-block of $\mathbf{u}$ can be replaced by a word of some "canonical form". We replace all $(k-1)$-blocks of $\mathbf{u}$ with getting some word $\mathbf{u}^\sharp$. After that we consider the word $\mathbf{v}$. It turns out that, up to identities in $\mathbf{F}_k$, this word has exactly the same $(k-1)$-blocks and $(k-1)$-dividers as the word $\mathbf{u}$. This allows us to change $(k-1)$-blocks of $\mathbf{v}$ in the same way as $(k-1)$-blocks of $\mathbf{u}$ with getting the word $\mathbf{u}^\sharp$ again. This evidently implies that the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{F}_k$.

Now we proceed to implement the above plan. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ satisfies the claims (2.1) and (3.6) with $\ell = k$. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$. Let us fix an index $i \in \{0, 1, \dots, m\}$. Lemma 6.2(ii) allows us to suppose that every letter from $\mathrm{con}(\mathbf{u}_i)$ occurs in $\mathbf{u}_i$ at most twice. Put $\mathrm{mul}(\mathbf{u}_i) = \{x_1, x_2 \dots, x_p\}$, $\mathrm{sim}(\mathbf{u}_i) = \{y_1, y_2, \dots, y_q\}$ and

$$\overline{\mathbf{u}_i} = x_1^2 x_2^2 \cdots x_p^2 y_1 y_2 \cdots y_q.$$

Note that $\overline{\mathbf{u}_i}$ is nothing but the "canonical form" of the $(k-1)$-block $\mathbf{u}_i$ mentioned above. Indeed, $\mathbf{u} = \mathbf{w}_1 \mathbf{u}_i \mathbf{w}_2$ for some possibly empty words $\mathbf{w}_1$ and $\mathbf{w}_2$. Lemmas 6.2(ii) and 6.6(iv) imply now that the variety $\mathbf{F}_k$ satisfies the identity

$$\mathbf{u} = \mathbf{w}_1 \mathbf{u}_i \mathbf{w}_2 \approx \mathbf{w}_1 \, \overline{\mathbf{u}_i} \, \mathbf{w}_2.$$

In particular, $\mathbf{F}_k$ satisfies the identities

$$\mathbf{u} = t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \mathbf{u}_m \approx t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \, \overline{\mathbf{u}_m} \, .$$

Put $\mathbf{u}' = t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \, \overline{\mathbf{u}_m}$. Note that the claims (2.1) and (3.6) with $\mathbf{v} = \mathbf{u}'$ and $\ell = k$ are true. Then Lemma 3.8 implies that the words $\mathbf{u}$ and $\mathbf{u}'$ are $(k-1)$-equivalent, i.e., the letters $t_0, t_1, \dots, t_m$ are $(k-1)$-dividers of the word $\mathbf{u}'$, while $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{m-1}, \overline{\mathbf{u}_m}$ are $(k-1)$-blocks of this word. After that, we can repeat literally arguments given above by changing the word $\mathbf{u}$ to the word $\mathbf{u}'$ and obtain the identities

$$\mathbf{u}' = t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \, \overline{\mathbf{u}_m} \approx t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \, \overline{\mathbf{u}_{m-1}} \, t_m \, \overline{\mathbf{u}_m}$$

hold in $\mathbf{F}_k$. Continuing this process, we get that $\mathbf{F}_k$ satisfies the identities

$$\begin{aligned} \mathbf{u} = {} & t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \mathbf{u}_m \approx t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \mathbf{u}_{m-1} t_m \, \overline{\mathbf{u}_m} \\ \approx {} & t_0 \mathbf{u}_0 t_1 \mathbf{u}_1 \cdots t_{m-1} \, \overline{\mathbf{u}_{m-1}} \, t_m \, \overline{\mathbf{u}_m} \approx \cdots \approx t_0 \, \overline{\mathbf{u}_0} \, t_1 \, \overline{\mathbf{u}_1} \, \cdots t_m \, \overline{\mathbf{u}_m} \, . \end{aligned} \tag{6.29}$$

Put $\mathbf{u}^\sharp = t_0 \, \overline{\mathbf{u}_0} \, t_1 \, \overline{\mathbf{u}_1} \, \cdots t_m \, \overline{\mathbf{u}_m}$.

One can return to the word $\mathbf{v}$. By Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). The claim (3.6) with $\ell = k$ implies that jth occurrence of a letter x in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$ if and only if jth occurrence of a letter x in $\mathbf{v}$ lies in the $(k-1)$-block $\mathbf{v}_i$ for any x and any $j = 1, 2$. We are going to check that $\mathrm{sim}(\mathbf{u}_i) = \mathrm{sim}(\mathbf{v}_i)$ and $\mathrm{mul}(\mathbf{u}_i) = \mathrm{mul}(\mathbf{v}_i)$. Let $x \in \mathrm{con}(\mathbf{u}_i)$. Lemma 6.2(ii) allows us to assume that $\mathrm{occ}_x(\mathbf{u}) \le 2$. There are three possibilities. First, if the first and the second occurrences of x in $\mathbf{u}$ lie in $\mathbf{u}_i$ then the first and the second occurrences of x in $\mathbf{v}$ lie in $\mathbf{v}_i$, whence $x \in \mathrm{mul}(\mathbf{u}_i)$ and $x \in \mathrm{mul}(\mathbf{v}_i)$. Second, if the first occurrences of x in $\mathbf{u}$ lies in $\mathbf{u}_i$ but the second one does not lie in $\mathbf{u}_i$ then the first occurrences of x in $\mathbf{v}$ lies in $\mathbf{v}_i$ but the second one does not lie in $\mathbf{v}_i$, whence $x \in \mathrm{sim}(\mathbf{u}_i)$ and $x \in \mathrm{sim}(\mathbf{v}_i)$. Finally, third, if first occurrence of x in $\mathbf{u}$ is located to the left of $\mathbf{u}_i$, while the second one lies in $\mathbf{u}_i$ then first occurrence of x in $\mathbf{v}$ is located to the left of $\mathbf{v}_i$, while the second one lies in $\mathbf{v}_i$. In this case we can apply the identity (4.9). This allows us to suppose that $x \in \mathrm{mul}(\mathbf{u}_i)$ and $x \in \mathrm{mul}(\mathbf{v}_i)$. Thus, $\mathrm{sim}(\mathbf{u}_i) = \mathrm{sim}(\mathbf{v}_i)$ and $\mathrm{mul}(\mathbf{u}_i) = \mathrm{mul}(\mathbf{v}_i)$. This implies that the $(k-1)$-blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ have the same "canonical form". Repeating literally arguments given above, we obtain the variety $\mathbf{F}_k$ satisfies the identities $\mathbf{v} \approx \mathbf{u}^\sharp \approx \mathbf{u}$.

(ii) *Necessity* follows from the already proved assertion (i) of this proposition and the evident inclusion $\mathbf{F}_k \subseteq \mathbf{K}$, while *sufficiency* is proved in the same way as in the assertion (i). □

Now we are well prepared to quickly complete the proof of the claim 3) of Proposition 6.1. Let $\mathbf{E} \subset \mathbf{X} \subset \mathbf{K}$. We have to verify that $\mathbf{X} \in [\mathbf{F}_k, \mathbf{F}_{k+1}]$ for some k. Suppose that $\mathbf{F}_1 \nsubseteq \mathbf{X}$. Then there is an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but does not hold in $\mathbf{F}_1$. Propositions 4.2 and 6.9(i) and the inclusion $\mathbf{E} \subseteq \mathbf{X}$ imply that the claims (2.1) and (4.2) hold, while the claim (3.6) with $\ell = 1$ is false. Let (3.4) be the 0-decomposition of $\mathbf{u}$. Then Lemma 3.8 applies with the conclusion that the 0-decomposition of $\mathbf{v}$ has the form (3.5). Since $\mathbf{u} \approx \mathbf{v}$ violates the claim (3.6) with $\ell = 1$ but satisfies (4.2), there is a letter x such that $h_2^0(\mathbf{u}, x) \ne h_2^0(\mathbf{v}, x)$. Put $t_i = h_2^0(\mathbf{u}, x)$ and $t_j = h_2^0(\mathbf{v}, x)$. We may assume without loss of generality that $j < i$. Since the claim (4.2) is true, we have $h_1^0(\mathbf{u}, x) = h_1^0(\mathbf{v}, x) = t_q$ for some q. Clearly, $q \le j$. Thus, the identity $\mathbf{u} \approx \mathbf{v}$ has the form

$$\mathbf{u}_1 t_q \mathbf{u}_2 x \mathbf{u}_3 t_i \mathbf{u}_4 x \mathbf{u}_5 \approx \mathbf{v}_1 t_q \mathbf{v}_2 x \mathbf{v}_3 x \mathbf{v}_4 t_i \mathbf{v}_5$$

for some possibly empty words $\mathbf{u}_s$ and $\mathbf{v}_s$ with $s = 1, 2, \dots, 5$. Substitute 1 for all letters occurring in the identity $\mathbf{u} \approx \mathbf{v}$ except x and t_i. Then we obtain an identity of the form $xt_ix^p \approx x^qt_ix^r$ where $p \ge 1$, $q \ge 2$ and $r \ge 0$. Now the identity (6.3) applies and we conclude that $\mathbf{X}$ satisfies $xt_ix \approx x^2t_i$. This fact together with the inclusion $\mathbf{X} \subseteq \mathbf{K}$ implies that $\mathbf{X} \subseteq \mathbf{E}$, contradicting the choice of $\mathbf{X}$. Thus, $\mathbf{F}_1 \subseteq \mathbf{X}$. If $\mathbf{X}$ contains an infinite number of varieties of the form $\mathbf{F}_k$ then Proposition 6.9 implies that $\mathbf{X} = \mathbf{K}$. Hence there is a natural number k such that $\mathbf{F}_k \subseteq \mathbf{X}$ but $\mathbf{F}_{k+1} \nsubseteq \mathbf{X}$. Then Proposition 6.9(i) implies that the claim (2.1) holds, while the claim (3.6) with $\ell = k + 1$ fails. Now we apply Lemma 6.8 and conclude that $\mathbf{X} \subseteq \mathbf{J}_k^k \subset \mathbf{F}_{k+1}$. Thus, $\mathbf{X} \in [\mathbf{F}_k, \mathbf{F}_{k+1}]$. The claim 3) of Proposition 6.1 is proved.

6.4. **Structure of the interval $[\mathbf{F}_k, \mathbf{F}_{k+1}]$.** Here we prove the claim 4) of Proposition 6.1. We divide this subsection into six subsubsections. In Subsubsections 6.4.1–6.4.5 we verify that an arbitrary variety from the interval $[\mathbf{F}_k, \mathbf{F}_{k+1}]$ coincides with one of the varieties $\mathbf{F}_k$, $\mathbf{H}_k$, $\mathbf{I}_k$, $\mathbf{J}_k^1$, $\mathbf{J}_k^2$, …, $\mathbf{J}_k^k$, $\mathbf{F}_{k+1}$. In Subsubsection 6.4.6 we check that all these varieties are pairwise different. These facts together with Lemma 6.4 imply the claim 4) of Proposition 6.1.

6.4.1. *If* $\mathbf{F}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ *then* $\mathbf{H}_k \subseteq \mathbf{X}$. The first step in the verification of the claim 4) of Proposition 6.1 is the following

Lemma 6.10. *If* $\mathbf{X}$ *is a monoid variety such that* $\mathbf{X} \in [\mathbf{F}_k, \mathbf{F}_{k+1}]$ *then either* $\mathbf{X} = \mathbf{F}_k$ *or* $\mathbf{X} \supseteq \mathbf{H}_k$.

To check this fact, we need

Lemma 6.11. *Let* $\mathbf{V}$ *be a monoid variety with* $\mathbf{F}_s \subseteq \mathbf{V} \subseteq \mathbf{K}$ *for some* s. *If* $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$ *such that* $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, b)$, $\ell_1(\mathbf{v}, b) < \ell_1(\mathbf{v}, a)$ *and* $D(\mathbf{u}, a) = D(\mathbf{u}, b) = s$ *for some* $a, b \in \mathrm{con}(\mathbf{u})$ *then* $\mathbf{V} = \mathbf{F}_s$.

Proof. Put $x_s = a$ and $y_s = b$. Since $\mathbf{F}_s \subseteq \mathbf{V}$, Proposition 6.9(i) implies that the claims (2.1) and (3.6) with $\ell = s$ are true. Suppose that

$$\ell_2(\mathbf{u}, x_s) < \ell_2(\mathbf{u}, y_s) \text{ and } \ell_2(\mathbf{v}, x_s) < \ell_2(\mathbf{v}, y_s). \tag{6.30}$$

Now Lemma 3.14 applies with the conclusion that there are letters $x_0, x_1, \dots, x_{s-1}$ such that $D(\mathbf{u}, x_r) = D(\mathbf{v}, x_r) = r$ for any $0 \le r < s$ and the identity $\mathbf{u} \approx \mathbf{v}$ has the form (3.7) for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2s+1}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2s+1}$.

One can verify that the first occurrences of x_s and y_s in $\mathbf{u}$ lie in the same $(s-1)$-block. Put $t_1 = h_1^{s-1}(\mathbf{u}, x_s)$ and $t_2 = h_1^{s-1}(\mathbf{u}, y_s)$. Arguing by contradiction, suppose that $t_1 \ne t_2$. Since $\ell_1(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, y_s)$, we have $\ell_1(\mathbf{u}, t_1) < \ell_1(\mathbf{u}, t_2)$. Lemma 3.8 with $k = s-1$ implies that $\ell_1(\mathbf{v}, t_1) < \ell_1(\mathbf{v}, t_2)$. In view of the claim (3.6) with $\ell = s$, $t_1 = h_1^{s-1}(\mathbf{v}, x_s)$ and $t_2 = h_1^{s-1}(\mathbf{v}, y_s)$. But this contradicts the fact that $\ell_1(\mathbf{v}, y_s) < \ell_1(\mathbf{v}, x_s)$. So, the first occurrences x_s and y_s in $\mathbf{u}$ lie in the same $(s-1)$-block. In particular, first occurrence of y_s in $\mathbf{u}$ precedes first occurrence of x_{s-1} in $\mathbf{u}$ because $\ell_1(\mathbf{u}, x_s) < \ell_1(\mathbf{u}, x_{s-1})$ and x_{s-1} is an $(s-1)$-divider. This implies that $\mathbf{u}_{2s} = \mathbf{u}'_{2s} y_s \mathbf{u}''_{2s}$ for some possibly empty words $\mathbf{u}'_{2s}$ and $\mathbf{u}''_{2s}$. Since first occurrence y_s in $\mathbf{v}$ precedes first occurrence of x_s in $\mathbf{v}$, we have $\mathbf{v}_{2s+1} = \mathbf{v}'_{2s+1} y_s \mathbf{v}''_{2s+1}$ for some possibly empty words $\mathbf{v}'_{2s+1}$ and $\mathbf{v}''_{2s+1}$.

Further, since $\ell_1(\mathbf{u}, y_s) < \ell_1(\mathbf{u}, x_{s-2})$, we apply Lemma 3.13 with $\mathbf{w} = \mathbf{u}$, $z = y_s$, $t = x_{s-2}$ and $r = s$ and obtain $\ell_2(\mathbf{u}, y_s) < \ell_1(\mathbf{u}, x_{s-2})$. This implies that $\mathbf{u}_{2s-2} = \mathbf{u}'_{2s-2} y_s \mathbf{u}''_{2s-2}$ for some possibly empty words $\mathbf{u}'_{2s-2}$ and $\mathbf{u}''_{2s-2}$. Analogously, we can verify that $\mathbf{v}_{2s-2} = \mathbf{v}'_{2s-2} y_s \mathbf{v}''_{2s-2}$ for some possibly empty words $\mathbf{v}'_{2s-2}$ and $\mathbf{v}''_{2s-2}$.

In view of the above, we have the identity $\mathbf{u} \approx \mathbf{v}$ has the form

$$\begin{aligned} &\mathbf{u}_{2s+1} x_s \mathbf{u}'_{2s} \overset{(1)}{y_s} \mathbf{u}'_{2s} x_{s-1} \mathbf{u}_{2s-1} x_s \mathbf{u}'_{2s-2} \overset{(2)}{y_s} \mathbf{u}'_{2s-2} x_{s-2} \mathbf{u}_{2s-3} x_{s-1} \cdots \\ &\quad \cdot \mathbf{u}_4 x_1 \mathbf{u}_3 x_2 \mathbf{u}_2 x_0 \mathbf{u}_1 x_1 \mathbf{u}_0 \\ \approx\ &\mathbf{v}'_{2s+1} \overset{(1)}{y_s} \mathbf{v}''_{2s+1} x_s \mathbf{v}_{2s} x_{s-1} \mathbf{v}_{2s-1} x_s \mathbf{v}'_{2s-2} \overset{(2)}{y_s} \mathbf{v}''_{2s-2} x_{s-2} \mathbf{v}_{2s-3} x_{s-1} \cdots \\ &\quad \cdot \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0. \end{aligned}$$

Lemma 6.2(ii) allows us to assume that the letters x_r with $1 \le r \le s$ and y_s occur twice in each of the words $\mathbf{u}$ and $\mathbf{v}$. Now substitute 1 for all letters occurring in this identity except $x_0, x_1, \dots, x_s$ and y_s. We get the identity

$$x_s y_s x_{s-1} x_s y_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \approx y_s x_s x_{s-1} x_s y_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1,$$

i.e., the identity α_s.

Suppose now that (6.30) is false. If $\ell_2(\mathbf{u}, x_s) < \ell_2(\mathbf{u}, y_s)$ but $\ell_2(\mathbf{v}, y_s) < \ell_2(\mathbf{v}, x_s)$ then the same considerations as above show that $\mathbf{V}$ satisfies the identity

$$x_s y_s x_{s-1} x_s y_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \approx \overset{(1)}{y_s}\overset{(1)}{x_s}\, x_{s-1}\, \overset{(2)}{y_s}\overset{(2)}{x_s}\, x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1.$$

According to Lemma 6.2(i), the variety $\mathbf{V}$ satisfies the identity σ_2. This allows us to transpose the second occurrences of the letters x_s and y_s in the right-hand side of the last identity. As a result, we get α_s as well.

Finally, if $\ell_2(\mathbf{u}, y_s) < \ell_2(\mathbf{u}, x_s)$ then we can repeat the above arguments but apply Lemmas 3.13 and 3.14 for the letter y_s rather than x_s. As a result, we obtain an identity of the form

$$x_s y_s x_{s-1} y_s x_s x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1 \approx y_s x_s x_{s-1} \mathbf{a} x_{s-2} x_{s-1} \cdots x_1 x_2 x_0 x_1$$

where

$$\mathbf{a} = \begin{cases} x_s y_s & \text{whenever } \ell_2(\mathbf{v}, x_s) < \ell_2(\mathbf{v}, y_s), \\ y_s x_s & \text{otherwise.} \end{cases}$$

In the case when $\mathbf{a} = x_s y_s$ this identity coincides with α_s, while otherwise we apply σ_2 once again and obtain α_s too. Thus, $\mathbf{V}$ satisfies α_s in any case, whence $\mathbf{V} \subseteq \mathbf{F}_s$. □

Proposition 6.12. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{H}_k$ *if and only if the claims* (2.1), (3.6) *and*

(6.31) *if either* $D(\mathbf{u}, x) \le \ell$ *or* $D(\mathbf{v}, x) \le \ell$ *then* $h_1^\ell(\mathbf{u}, x) = h_1^\ell(\mathbf{v}, x)$

with $\ell = k$ *are true.*

Proof. Necessity. Suppose that a non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{H}_k$. Proposition 6.9(i) and the inclusion $\mathbf{F}_k \subseteq \mathbf{H}_k$ imply that the claims (2.1) and (3.6) with $\ell = k$ are true. As in the proof of necessity in Proposition 6.9(i), we can assume that $\mathbf{u} = \mathbf{p}\xi(\mathbf{a})\mathbf{q}$ and $\mathbf{v} = \mathbf{p}\xi(\mathbf{b})\mathbf{q}$ for some possibly empty words $\mathbf{p}$ and $\mathbf{q}$, an endomorphism ξ of F^1 and an identity $\mathbf{a} \approx \mathbf{b} \in \{\Phi, \beta_k\}$.

If $\mathbf{a} \approx \mathbf{b} \in \Phi$ then the claim (3.6) is true for any ℓ by Proposition 6.9(ii). Evidently, this implies the required conclusion. Suppose now that $\mathbf{a} \approx \mathbf{b}$ coincides with β_k. Then

$$\xi(\mathbf{a}) = \mathbf{a}_{k+1}\mathbf{a}_k\mathbf{a}_{k+1}\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1} \cdots \mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1,$$
$$\xi(\mathbf{b}) = \mathbf{a}_k\mathbf{a}_{k+1}^2\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1} \cdots \mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1$$

for some words $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k$ and $\mathbf{a}_{k+1}$, whence

$$\mathbf{u} = \mathbf{p}\mathbf{a}_{k+1}\mathbf{a}_k\mathbf{a}_{k+1}\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1} \cdots \mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q},$$
$$\mathbf{v} = \mathbf{p}\mathbf{a}_k\mathbf{a}_{k+1}^2\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1} \cdots \mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q}.$$

By Lemma 6.5, $D(\mathbf{a}, x), D(\mathbf{a}, x_k) > k - 1$. Then Lemma 3.15 implies that the subword $\mathbf{a}_{k+1}\mathbf{a}_k\mathbf{a}_{k+1}$ of $\mathbf{u}$ located between $\mathbf{p}$ and $\mathbf{a}_{k-1}$ is contained in some $(k-1)$-block. Besides that, in view of Lemma 3.15, both occurrences of the word $\mathbf{a}_{k+1}$ in $\mathbf{u}$ do not contain any k-dividers of $\mathbf{u}$ because $D(\mathbf{u}, x) > k$

by Lemma 6.5. This means that the words $\mathbf{u}$ and $\mathbf{v}$ are k-equivalent. Now Lemma 3.8 applies with the conclusion that the claim (6.31) with $\ell = k$ is true.

Sufficiency. The outline of our arguments here is the same as in the proof of sufficiency in Proposition 6.9(i). But the canonical form of a $(k-1)$-block of $\mathbf{u}$ looks more complicated here than in that proposition.

Suppose that the claims (2.1), (3.6) and (6.31) with $\ell = k$ are true. Let (3.4) be the $(k-1)$-decomposition of $\mathbf{u}$. Let us fix an index $i \in \{0, 1, \dots, m\}$. Let

$$t_i\mathbf{u}_i = s_0\mathbf{a}_0 s_1\mathbf{a}_1 \cdots s_n\mathbf{a}_n \tag{6.32}$$

be the presentation of the word $t_i\mathbf{u}_i$ as the product of alternating k-dividers $s_0, s_1, \dots, s_n$ and k-blocks $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n$. Put $\mathbf{u}_i^* = \mathbf{a}_0\mathbf{a}_1 \cdots \mathbf{a}_n$. Let $\mathrm{con}(\mathbf{u}_i^*) = \{x_1, x_2, \dots, x_p\}$ and

$$\overline{\mathbf{u}_i} = x_1^2 x_2^2 \cdots x_p^2 s_1 s_2 \cdots s_n.$$

As we will see below, $\overline{\mathbf{u}_i}$ is nothing but the mentioned above "canonical form" of the $(k-1)$-block $\mathbf{u}_i$.

Clearly, $\mathbf{u} = \mathbf{w}_1\mathbf{u}_i\mathbf{w}_2$ for some possibly empty words $\mathbf{w}_1$ and $\mathbf{w}_2$. Suppose that $x \in \mathrm{con}(\mathbf{u}_i^*)$ but $x \notin \mathrm{con}(\mathbf{w}_1)$. If x is simple in $\mathbf{u}_i$ then x is a k-divider of $\mathbf{u}$ but this is not the case. Therefore, x is multiple in $\mathbf{u}_i$. Since $x \notin \mathrm{con}(\mathbf{w}_1)$, this means that the first and the second occurrences of x in $\mathbf{u}$ lie in the same $(k-1)$-block of $\mathbf{u}$, whence $D(\mathbf{u}, x) > k$. Further, Lemma 3.7 implies that $D(\mathbf{u}, s_j) = k$ for all $j = 1, \dots, n$. We see that if $a \in \mathrm{con}(\mathbf{u}_i^*)$ and $b \in \{s_1, s_2, \dots, s_n\}$ then either $a \in \mathrm{con}(\mathbf{w}_1)$ or $D(\mathbf{u}, a) \ne D(\mathbf{u}, b)$. Now the assertions (ii) and (iii) of Lemma 6.6 apply with the conclusion that the identities

$$\mathbf{u} = \mathbf{w}_1\mathbf{u}_i\mathbf{w}_2 \approx \mathbf{w}_1\mathbf{u}_i^* s_1 s_2 \cdots s_n\mathbf{w}_2$$

hold in $\mathbf{H}_k$. As we have seen above, if $x \in \mathrm{con}(\mathbf{u}_i^*) \setminus \mathrm{con}(\mathbf{w}_1)$ then $\mathrm{occ}_x(\mathbf{u}_i^*) \ge 2$. Further, if $x \in \mathrm{con}(\mathbf{w}_1) \cap \mathrm{con}(\mathbf{u}_i^*)$ then we can apply the identity (4.9) and obtain $\mathrm{occ}_x(\mathbf{u}_i^*) \ge 2$ too. Now Lemma 6.2(ii) applies with the conclusion that $\mathrm{occ}_x(\mathbf{u}_i^*) = 2$ for any $x \in \mathrm{con}(\mathbf{u}_i^*)$. Then Lemma 6.2(iii) applies and we conclude that the identities

$$\mathbf{u} \approx \mathbf{w}_1\mathbf{u}_i^* s_1 s_2 \cdots s_n\mathbf{w}_2 \approx \mathbf{w}_1\,\overline{\mathbf{u}_i}\,\mathbf{w}_2$$

hold in $\mathbf{H}_k$.

So, as in the proof of Proposition 6.9(i), using identities which hold in the variety $\mathbf{H}_k$, we can replace the $(k-1)$-blocks $\mathbf{u}_i$ of $\mathbf{u}$ successively, one after another, by the "canonical form" $\overline{\mathbf{u}_i}$ for $i = m, m-1, \dots, 0$. Then the variety $\mathbf{H}_k$ satisfies the identities (6.29). Put $\mathbf{u}^\sharp = t_0\,\overline{\mathbf{u}_0}\,t_1\,\overline{\mathbf{u}_1}\cdots t_m\,\overline{\mathbf{u}_m}$.

One can return to the word $\mathbf{v}$. By Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). By (6.31) and Lemma 3.8, the words $\mathbf{u}$ and $\mathbf{v}$ are k-equivalent. This means that the word $t_i\mathbf{v}_i$ is the product of alternating k-dividers $s_0, s_1, \dots, s_n$ and k-blocks $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n$, i.e.,

$$t_i\mathbf{v}_i = s_0\mathbf{b}_0 s_1\mathbf{b}_1 \cdots s_n\mathbf{b}_n. \tag{6.33}$$

The claim (3.6) with $\ell = k$ implies that jth occurrence of a letter x in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$ if and only if jth occurrence of a letter x in $\mathbf{v}$ lies in the $(k-1)$-block $\mathbf{v}_i$ for any x and any $j = 1, 2$. Also, Lemma 6.2(ii) allows us to assume that if the first and the second occurrences of the letter x in $\mathbf{u}$

do not lie in the $(k-1)$-block $\mathbf{u}_i$ then this letter does not occur in $\mathbf{u}_i$. Then $\mathrm{con}(\mathbf{u}_i^*) = \mathrm{con}(\mathbf{b}_0\mathbf{b}_1\cdots\mathbf{b}_n)$. This implies that the $(k-1)$-blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ have the same "canonical form". Repeating literally arguments given above, we obtain the variety $\mathbf{H}_k$ satisfies the identities $\mathbf{v} \approx \mathbf{u}^\sharp \approx \mathbf{u}$. □

Now we are well prepared to quickly complete the proof of Lemma 6.10. Let $\mathbf{F}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$. We have to verify that $\mathbf{X} \supseteq \mathbf{H}_k$. Arguing by contradiction, suppose that $\mathbf{H}_k \nsubseteq \mathbf{X}$. Then there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but does not hold in $\mathbf{H}_k$. Propositions 6.9(i) and 6.12 and the inclusion $\mathbf{F}_k \subset \mathbf{X}$ together imply that the claims (2.1) and (3.6) are true, while the claim (6.31) with $\ell = k$ is false. According to Lemma 3.10, the words $\mathbf{u}$ and $\mathbf{v}$ have the same set of k-dividers but $\mathbf{u}$ and $\mathbf{v}$ are not k-equivalent by Lemma 3.8. Then there are k-dividers a, b of the words $\mathbf{u}$, $\mathbf{v}$ such that $\ell_1(\mathbf{u}, a) < \ell_1(\mathbf{u}, b)$, while $\ell_1(\mathbf{v}, b) < \ell_1(\mathbf{v}, a)$. In view of Lemma 3.7, $D(\mathbf{u}, a), D(\mathbf{u}, b) \le k$. Suppose that $D(\mathbf{u}, a) = r < k$. According to Lemma 3.11, the claim (3.6) with $\ell = r$ is true. Then Lemma 3.12 implies that $D(\mathbf{v}, a) = r$. Also the words $\mathbf{u}$ and $\mathbf{v}$ are r-equivalent by Lemma 3.8. Put $c = h_1^r(\mathbf{u}, b)$. Since a is an r-divider of $\mathbf{u}$ by Lemma 3.7, first occurrence of a in $\mathbf{u}$ precedes first occurrence of c in $\mathbf{u}$. On the other hand, the claim (3.6) with $\ell = r$ implies that $c = h_1^r(\mathbf{v}, b)$, whence $\ell_1(\mathbf{v}, c) < \ell_1(\mathbf{v}, a)$. This contradicts the fact that the words $\mathbf{u}$ and $\mathbf{v}$ are r-equivalent. So, $D(\mathbf{u}, a) = k$. Analogously, $D(\mathbf{u}, b) = k$. Now Lemma 6.11 with $s = k$ applies and we conclude that $\mathbf{X} \subseteq \mathbf{F}_k$, a contradiction. Lemma 6.10 is proved. □

6.4.2. *If* $\mathbf{H}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ *then* $\mathbf{I}_k \subseteq \mathbf{X}$. The second step in the verification of the claim 4) of Proposition 6.1 is the following

Lemma 6.13. *If* $\mathbf{X}$ *is a monoid variety such that* $\mathbf{X} \in [\mathbf{H}_k, \mathbf{F}_{k+1}]$ *then either* $\mathbf{X} = \mathbf{H}_k$ *or* $\mathbf{X} \supseteq \mathbf{I}_k$.

To check this fact, we need the following auxiliary result.

Proposition 6.14. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{I}_k$ *if and only if the claims* (2.1)*,* (3.6) *and*

$$h_1^\ell(\mathbf{u}, x) = h_1^\ell(\mathbf{v}, x) \text{ for all } x \in \mathrm{con}(\mathbf{u}) \tag{6.34}$$

with $\ell = k$ *are true.*

Proof. Necessity. Suppose that a non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{I}_k$. Proposition 6.12 and the inclusion $\mathbf{H}_k \subseteq \mathbf{I}_k$ imply that the claims (2.1) and (3.6) with $\ell = k$ are true. As in the proof of Proposition 6.9(i), we can assume that $\mathbf{u} = \mathbf{p}\xi(\mathbf{a})\mathbf{q}$ and $\mathbf{v} = \mathbf{p}\xi(\mathbf{b})\mathbf{q}$ for some possibly empty words $\mathbf{p}$ and $\mathbf{q}$, an endomorphism ξ of F^1 and an identity $\mathbf{a} \approx \mathbf{b} \in \{\Phi, \gamma_k\}$.

If $\mathbf{a} \approx \mathbf{b} \in \Phi$ then the claim (3.6) is true for any ℓ by Proposition 6.9(ii). Evidently, this implies the required conclusion. Suppose now that $\mathbf{a} \approx \mathbf{b}$ coincides with γ_k. Then

$$\xi(\mathbf{a}) = \mathbf{b}_1\mathbf{b}_0\mathbf{b}_1\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1,$$
$$\xi(\mathbf{b}) = \mathbf{b}_1\mathbf{b}_0\mathbf{a}_k\mathbf{b}_1\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1$$

for some words $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k$ and $\mathbf{b}_0, \mathbf{b}_1$, whence

$$\mathbf{u} = \mathbf{p}\mathbf{b}_1\mathbf{b}_0\mathbf{b}_1\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q},$$
$$\mathbf{v} = \mathbf{p}\mathbf{b}_1\mathbf{b}_0\mathbf{a}_k\mathbf{b}_1\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q}.$$

By Lemma 6.5, $D(\mathbf{a}, x_k) = k$. Then Lemma 3.15 implies that the subword $\mathbf{a}_k$ of $\mathbf{u}$ located between $\mathbf{b}_1$ and $\mathbf{a}_{k-1}$ does not contain any $(k-1)$-divider. Also, obviously, the subword $\mathbf{b}_1$ of $\mathbf{u}$ located between $\mathbf{b}_0$ and $\mathbf{a}_k$ does not contain any s-divider for all s. Therefore, the subword $\mathbf{b}_1\mathbf{a}_k$ of $\mathbf{u}$ located between $\mathbf{b}_0$ and $\mathbf{a}_{k-1}$ lies in some $(k-1)$-block. It is evident that the subword $\mathbf{b}_1$ of $\mathbf{u}$ located between $\mathbf{b}_0$ and $\mathbf{a}_k$ does not contain first occurrence of any letter in $\mathbf{u}$. This implies that the claim (6.34) with $\ell = k$ is true.

Sufficiency. As in the proof of Proposition 6.12, the outline of our arguments here is similar to one from the proof of sufficiency in Proposition 6.9(i). But the canonical form of the block here is even more complicated than in Proposition 6.12.

Suppose that the claims (2.1), (3.6) and (6.34) with $\ell = k$ are true. As in the proof of sufficiency in Proposition 6.12, we suppose that (3.4) is the $(k-1)$-decomposition of $\mathbf{u}$, while (6.32) is the representation of $t_i\mathbf{u}_i$ as the product of alternating k-dividers $s_0, s_1, \dots, s_n$ and k-blocks $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n$.

For any $j = 0, 1, \dots, n$, we put

$$X_j = \{x \in \mathrm{con}(\mathbf{a}_j) \mid \text{ first occurrence of } x \text{ in } \mathbf{u} \text{ lies in } \mathbf{a}_j\}.$$

Note that the set X_j may be defined by another (of course, equivalent) way. Namely, it is clear that a letter x lies in X_j if and only if it occurs in the k-block $\mathbf{a}_j$ and the $(1,k)$-restrictor of x in $\mathbf{u}$ coincides with the k-divider of $\mathbf{u}$ that immediately precedes $\mathbf{a}_j$. In other words,

$$X_j = \bigl\{x \in \mathrm{con}(\mathbf{a}_j) \mid s_j = h_1^k(\mathbf{u}, x)\bigr\}.$$

Put $X = X_0 \cup X_1 \cup \cdots \cup X_n$, $\mathbf{a}_j' = (\mathbf{a}_j)_X$ and $\mathbf{u}_i^* = \mathbf{a}_0'\mathbf{a}_1'\cdots\mathbf{a}_n'$. Let $X_j = \{x_{j1}, x_{j2}, \dots, x_{jq_j}\}$, $\mathrm{con}(\mathbf{u}_i^*) = \{c_1, c_2, \dots, c_p\}$ and

$$\overline{\mathbf{u}_i} = (c_1c_2\cdots c_p)\cdot(x_{01}^2\cdots x_{0q_0}^2)\cdot(s_1x_{11}^2\cdots x_{1q_1}^2)\cdot(s_2x_{21}^2\cdots x_{2q_2}^2)\cdots(s_nx_{n1}^2\cdots x_{nq_n}^2).$$

As we will see below, $\overline{\mathbf{u}_i}$ is nothing but the "canonical form" of the $(k-1)$-block $\mathbf{u}_i$ mentioned above.

Clearly, $\mathbf{u} = \mathbf{w}_1\mathbf{u}_i\mathbf{w}_2$ for some possibly empty words $\mathbf{w}_1$ and $\mathbf{w}_2$. Let $x \in X_j$. If x is simple in $\mathbf{u}_i$ then either x coincides with one of the k-dividers $s_1, s_2, \dots, s_n$ or $x \in \mathrm{con}(\mathbf{w}_1)$. But both these variants contradict the choice of x. Therefore, x is multiple in $\mathbf{u}_i$. In view of the identy (6.3), we can assume that $\mathrm{occ}_x(\mathbf{u}_i) = 2$. Thus, $\mathbf{u} = \mathbf{a}x\mathbf{b}x\mathbf{c}$ for possibly empty words $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $x\mathbf{b}x$ is a subword of $\mathbf{u}_i$. One can verify that the variety $\mathbf{I}_k$ satisfies the identity $\mathbf{u} \approx \mathbf{a}x^2\mathbf{b}\mathbf{c}$. If $\mathbf{b} = \lambda$ then this claim is evident. Let now $\mathbf{b} \ne \lambda$. Then we apply Lemma 6.6(ii) and successively transpose second occurrence of x in $\mathbf{u}$ with all the letters of the word $\mathbf{b}$ from right to left. Thus, we conclude that $\mathbf{I}_k$ satisfies the identity $\mathbf{u} \approx \mathbf{a}x^2\mathbf{b}\mathbf{c}$. We can assume without loss of generality that $\ell_1(\mathbf{u}, x_{j0}) < \ell_1(\mathbf{u}, x_{j1}) < \cdots < \ell_1(\mathbf{u}, x_{jq_j})$. Therefore, $\mathbf{I_k}$ satisfies the identity

$$\mathbf{u} \approx \mathbf{w}_1\cdot(x_{01}^2\cdots x_{0q_0}^2\mathbf{a}_0')\cdot(s_1x_{11}^2\cdots x_{1q_1}^2\mathbf{a}_1')\cdots(s_nx_{n1}^2\cdots x_{nq_n}^2\mathbf{a}_n')\cdot\mathbf{w}_2. \tag{6.35}$$

The definition of the set X and words of the form $\mathbf{a}_j'$ imply that $x \in \mathrm{con}(\mathbf{w}_1)$ for any $x \in \mathrm{con}(\mathbf{u}_i^*)$. Now we can apply Lemma 6.6(ii) and obtain that the identity

$$\mathbf{u} \approx \mathbf{w}_1 \cdot \mathbf{u}_i^* \cdot (x_{01}^2 \cdots x_{0q_0}^2) \cdot (s_1 x_{11}^2 \cdots x_{1q_1}^2) \cdots (s_n x_{n1}^2 \cdots x_{nq_n}^2) \cdot \mathbf{w}_2$$

holds in $\mathbf{I}_k$. As we have seen above, $\mathrm{con}(\mathbf{u}_i^*) \subseteq \mathrm{con}(\mathbf{w}_1)$. Then we can apply the identity (6.3) and obtain the word $\mathbf{u}_i^*$ is linear. Then Lemma 6.2(i) applies and we conclude that $\mathbf{I}_k$ satisfies the identities

$$\begin{aligned}\mathbf{u} &\approx \mathbf{w}_1 \cdot (c_1 c_2 \cdots c_p) \cdot (x_{01}^2 \cdots x_{0q_0}^2) \cdot (s_1 x_{11}^2 \cdots x_{1q_1}^2) \cdots (s_n x_{n1}^2 \cdots x_{nq_n}^2) \cdot \mathbf{w}_2\\ &= \mathbf{w}_1 \overline{\mathbf{u}_i}\, \mathbf{w}_2.\end{aligned}$$

So, as in the proof of Proposition 6.9(i), using identities which hold in the variety $\mathbf{I}_k$, we can replace the $(k-1)$-blocks $\mathbf{u}_i$ of $\mathbf{u}$ successively, one after another, by the "canonical form" $\overline{\mathbf{u}_i}$ for $i = m, m-1, \ldots, 0$. Then the variety $\mathbf{I}_k$ satisfies the identities (6.29). Put $\mathbf{u}^\sharp = t_0\, \overline{\mathbf{u}_0}\, t_1\, \overline{\mathbf{u}_1} \cdots t_m\, \overline{\mathbf{u}_m}$.

One can return to the word $\mathbf{v}$. By Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). Furthermore, the claim (6.34) with $\ell = k$ and Lemma 3.8 imply that (6.33) is a representation of $t_i \mathbf{v}_i$ as the product of alternating k-dividers $s_0, s_1, \ldots, s_n$ and k-blocks $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$. The claim (6.34) implies that

$$X_j = \{x \in \mathrm{con}(\mathbf{b}_j) \mid s_j = h_1^k(\mathbf{v}, x)\}$$

for all $j = 0, 1, \ldots, r_i$. Put $\mathbf{b}_j' = (\mathbf{b}_j)_X$. The claim (3.6) with $\ell = k$ implies that jth occurrence of a letter x in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$ if and only if jth occurrence of a letter x in $\mathbf{v}$ lies in the $(k-1)$-block $\mathbf{v}_i$ for any x and any $j = 1, 2$. Also, Lemma 6.2(ii) allows us to assume that if the first and the second occurrences of the letter x in $\mathbf{u}$ do not lie in the $(k-1)$-block $\mathbf{u}_i$ then this letter does not occur in $\mathbf{u}_i$. Then $\mathrm{con}(\mathbf{u}_i^*) = \mathrm{con}(\mathbf{b}_0'\mathbf{b}_1' \cdots \mathbf{b}_n')$. This implies that the $(k-1)$-blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ have the same "canonical form". Repeating literally arguments given above, we obtain the variety $\mathbf{I}_k$ satisfies the identities $\mathbf{v} \approx \mathbf{u}^\sharp \approx \mathbf{u}$. □

Now we are well prepared to quickly complete the proof of Lemma 6.13. Let $\mathbf{H}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$. We have to verify that $\mathbf{X} \supseteq \mathbf{I}_k$. Arguing by contradiction, suppose that $\mathbf{I}_k \nsubseteq \mathbf{X}$. Then there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but does not hold in $\mathbf{I}_k$. Then Propositions 6.12 and 6.14 and the inclusion $\mathbf{H}_k \subset \mathbf{X}$ together imply that the claims (2.1), (3.6) and (6.31) with $\ell = k$ are true, while the claim (6.34) with $\ell = k$ is false. Let (3.4) be the k-decomposition of the word $\mathbf{u}$. The claim (6.31) and Lemma 3.8 imply that the k-decomposition of $\mathbf{v}$ has the form (3.5). Since the claim (6.34) is false, there is a letter x such that $h_1^k(\mathbf{u}, x) \ne h_1^k(\mathbf{v}, x)$. Put $t_i = h_1^k(\mathbf{u}, x)$ and $t_j = h_1^k(\mathbf{v}, x)$. Then $i \ne j$. We can assume without loss of generality that $i < j$. Then $\ell_1(\mathbf{u}, x) < \ell_1(\mathbf{u}, t_j)$, while $\ell_1(\mathbf{v}, t_j) < \ell_1(\mathbf{u}, x)$. Lemma 3.7 implies that $D(\mathbf{u}, t_j) \le k$. Put $D(\mathbf{u}, t_j) = r$. If $r = 0$ then t_j is a 0-divider of $\mathbf{u}$. The claim (2.1) implies that t_j is a 0-divider of $\mathbf{v}$ too. Then $t_j = h_1^0(\mathbf{v}, x)$ but $t_j \ne h_1^0(\mathbf{u}, x)$. In view of Lemma 3.11, the claim (3.6) with $\ell = p$ is true for any $1 \le p \le k$, a contradiction. Thus, $r \ge 1$. Now Lemma 6.7(i) with $s = r$ and $x_s = t_j$ applies and we conclude that $\mathbf{X} \subseteq \mathbf{H}_r \subseteq \mathbf{H}_k$, a contradiction. Lemma 6.13 is proved. □

6.4.3. *If* $\mathbf{I}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ *then* $\mathbf{J}_k^1 \subseteq \mathbf{X}$. The third step in the verification of the claim 4) of Proposition 6.1 is the following

Lemma 6.15. *If* $\mathbf{X}$ *is a monoid variety such that* $\mathbf{X} \in [\mathbf{I}_k, \mathbf{F}_{k+1}]$ *then either* $\mathbf{X} = \mathbf{I}_k$ *or* $\mathbf{X} \supseteq \mathbf{J}_k^1$.

To check this fact, we need the following

Lemma 6.16. *Let* $\mathbf{V}$ *be a monoid variety with* $\mathbf{V} \subseteq \mathbf{K}$ *and* ℓ *a natural number. Suppose that* $\mathbf{V}$ *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$.

(i) *If the claims* (2.1), (3.6) *and* (6.34) *are true but the claim*

$$\text{(6.36)} \qquad \text{if } x \in \operatorname{con}(\mathbf{u}) \text{ and } D(\mathbf{u}, x) \le m \text{ then } h_2^\ell(\mathbf{u}, x) = h_2^\ell(\mathbf{v}, x)$$

with $m = 1$ *is false then* $\mathbf{V} \subseteq \mathbf{I}_\ell$.

(ii) *If the claims* (2.1), (3.6), (6.34) *and* (6.36) *with* $m = r$ *are true but the claim* (6.36) *with* $m = r + 1$ *is false then* $\mathbf{V} \subseteq \mathbf{J}_\ell^r$.

Proof. Proofs of the assertions (i) and (ii) have the same initial part. Suppose that the variety $\mathbf{V}$ satisfies the hypothesis of one of these two claims. Then the claims (2.1), (3.6) and (6.34) are true. Let m be the least natural number such that the claim (6.36) is false. Then there is a letter y_m such that $D(\mathbf{u}, y_m) = m$ and $h_2^\ell(\mathbf{u}, y_m) \ne h_2^\ell(\mathbf{v}, y_m)$. Put $x_\ell = h_2^\ell(\mathbf{u}, y_m)$ and $z_\ell = h_2^\ell(\mathbf{v}, y_m)$. In view of Lemma 3.7, $D(\mathbf{u}, x_\ell)$, $D(\mathbf{u}, z_\ell) \le \ell$. Note that either $D(\mathbf{u}, x_\ell) = \ell$ or $D(\mathbf{u}, z_\ell) = \ell$. Indeed, if $D(\mathbf{u}, x_\ell)$, $D(\mathbf{u}, z_\ell) < \ell$ then $D(\mathbf{v}, x_\ell), D(\mathbf{v}, z_\ell) < \ell$ by Lemma 3.12. Then x_ℓ and z_ℓ are $(\ell-1)$-dividers of $\mathbf{u}$ and $\mathbf{v}$, whence $x_\ell = h_2^{\ell-1}(\mathbf{u}, y_m)$ and $z_\ell = h_2^{\ell-1}(\mathbf{v}, y_m)$. But this contradicts the claim (3.6). Suppose without loss of generality that $D(\mathbf{u}, x_\ell) = \ell$. By symmetry, we may assume that first occurrence of z_ℓ in the word $\mathbf{u}$ precedes first occurrence of x_ℓ in this word. Since the claim (6.34) is true, $\ell_1(\mathbf{v}, z_\ell) < \ell_1(\mathbf{v}, x_\ell)$ by Lemma 3.8. This implies that $\ell_2(\mathbf{v}, y_m) < \ell_1(\mathbf{v}, x_\ell)$.

Now we apply Lemma 3.14 with $x_s = x_\ell$ and $s = \ell$ and conclude that there are letters $x_0, x_1, \ldots, x_{\ell-1}$ such that, for any $p = 0, 1, \ldots, \ell-1$ and $q = 0, 1, \ldots, \ell-2$, $D(\mathbf{u}, x_p) = D(\mathbf{v}, x_p) = p$ and the inequalities

$$\ell_1(\mathbf{w}, x_{p+1}) < \ell_1(\mathbf{w}, x_p) < \ell_2(\mathbf{w}, x_{p+1}) \text{ and } \ell_2(\mathbf{w}, x_{q+2}) < \ell_1(\mathbf{w}, x_q)$$

hold where $\mathbf{w}$ is any of the words $\mathbf{u}$ or $\mathbf{v}$.

Put $y_{m-1} = h_2^{m-1}(\mathbf{u}, y_m)$. According to Lemma 3.9, $D(\mathbf{u}, y_{m-1}) = m - 1$ and $\ell_1(\mathbf{u}, y_m) < \ell_1(\mathbf{u}, y_{m-1})$. Besides that, the claim (3.6) and Lemma 3.11 imply that $h_2^{m-1}(\mathbf{v}, y_m) = h_2^{m-1}(\mathbf{u}, y_m) = y_{m-1}$. Now we apply Lemma 3.9 again and obtain $D(\mathbf{v}, y_{m-1}) = m - 1$ and $\ell_1(\mathbf{v}, y_m) < \ell_1(\mathbf{v}, y_{m-1})$. In view of Lemma 3.7, the letter $x_{\ell-1}$ is an ℓ-divider of $\mathbf{u}$, whence $\ell_2(\mathbf{u}, y_m) < \ell_1(\mathbf{u}, x_{\ell-1})$ because $x_\ell = h_2^\ell(\mathbf{u}, y_m)$ and $\ell_1(\mathbf{u}, x_\ell) < \ell_1(\mathbf{u}, x_{\ell-1})$.

Lemma 6.2(ii) allows us to assume that the letters y_m and x_p with $1 \le p \le \ell$ occur twice in each of the words $\mathbf{u}$ and $\mathbf{v}$. Further considerations are divided into two cases corresponding to the statements (i) and (ii).

Case 1: $m = 1$. In view of the above, we have the identity $\mathbf{u} \approx \mathbf{v}$ has the form

$$\begin{aligned}&\mathbf{u}_{2\ell+4}y_1\mathbf{u}_{2\ell+3}y_0\mathbf{u}_{2\ell+2}x_\ell\mathbf{u}_{2\ell+1}y_1\mathbf{u}_{2\ell}x_{\ell-1}\mathbf{u}_{2\ell-1}x_\ell\mathbf{u}_{2\ell-2}x_{\ell-2}\mathbf{u}_{2\ell-3}x_{\ell-1}\cdots\\&\cdot\mathbf{u}_4x_1\mathbf{u}_3x_2\mathbf{u}_2x_0\mathbf{u}_1x_1\mathbf{u}_0\\ \approx\ &\mathbf{v}_{2\ell+4}y_1\mathbf{v}_{2\ell+3}y_0\mathbf{v}_{2\ell+2}y_1\mathbf{v}_{2\ell+1}x_\ell\mathbf{v}_{2\ell}x_{\ell-1}\mathbf{v}_{2\ell-1}x_\ell\mathbf{v}_{2\ell-2}x_{\ell-2}\mathbf{v}_{2\ell-3}x_{\ell-1}\cdots\\&\cdot\mathbf{v}_4x_1\mathbf{v}_3x_2\mathbf{v}_2x_0\mathbf{v}_1x_1\mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2\ell+4}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2\ell+4}$ such that $x_s, y_0,\ y_1 \notin \mathrm{con}(\mathbf{u}_i\mathbf{v}_i)$ for $0 \le s \le \ell$ and $0 \le i \le 2\ell + 4$. Now substitute 1 for all letters occurring in this identity except $x_0, x_1, \dots, x_\ell, y_0$ and y_1. We get the identity

$$y_1y_0x_\ell y_1x_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_1x_2x_0x_1 \approx y_1y_0y_1x_\ell x_{\ell-1}x_\ell x_{\ell-2}x_{\ell-1}\cdots x_1x_2x_0x_1,$$

i.e., the identity γ_ℓ. The claim (i) is proved.

Case 2: $m > 1$. Now we will prove that $\ell_2(\mathbf{v}, x_m) < \ell_2(\mathbf{v}, y_{m-1})$ and $\ell_2(\mathbf{u}, x_m) < \ell_2(\mathbf{u}, y_{m-1})$. Put $y_{m-2} = h_2^{m-2}(\mathbf{v}, y_{m-1})$. Since $D(\mathbf{v}, y_{m-1}) = m - 1$, Lemma 3.9 implies that $D(\mathbf{v}, y_{m-2}) = m - 2$ and $\ell_1(\mathbf{v}, y_{m-1}) < \ell_1(\mathbf{v}, y_{m-2})$. Recall that $\ell_1(\mathbf{v}, y_m) < \ell_1(\mathbf{v}, y_{m-1})$, whence $\ell_1(\mathbf{v}, y_m) < \ell_1(\mathbf{v}, y_{m-2})$. Since $D(\mathbf{v}, y_m) = m$, we can apply Lemma 3.13 and conclude that $\ell_2(\mathbf{v}, y_m) < \ell_1(\mathbf{v}, y_{m-2})$. First occurrence of x_ℓ in $\mathbf{v}$ precedes second occurrence of y_m, whence $\ell_1(\mathbf{v}, x_\ell) < \ell_1(\mathbf{v}, y_{m-2})$. Then Lemma 3.13 implies that $\ell_2(\mathbf{v}, x_\ell) < \ell_1(\mathbf{v}, y_{m-2})$. This implies that $\ell_1(\mathbf{v}, x_{\ell-1}) < \ell_2(\mathbf{v}, x_\ell) < \ell_1(\mathbf{v}, y_{m-2})$. If $\ell - 1 \ge m$ then Lemma 3.13 applies with the conclusion that $\ell_2(\mathbf{v}, x_{\ell-1}) < \ell_1(\mathbf{v}, y_{m-2})$. Continuing this process, we eventually obtain $\ell_2(\mathbf{v}, x_m) < \ell_1(\mathbf{v}, y_{m-2})$. In particular, $\ell_1(\mathbf{v}, x_m) < \ell_1(\mathbf{v}, y_{m-2})$. In view of Lemma 3.7, the letters x_m and y_{m-2} are ℓ-dividers of $\mathbf{v}$. Now Lemma 3.8 applies with the conclusion that $\ell_1(\mathbf{u}, x_m) < \ell_1(\mathbf{u}, y_{m-2})$. Then Lemma 3.13 shows that $\ell_2(\mathbf{u}, x_m) < \ell_1(\mathbf{u}, y_{m-2})$. The choice of y_{m-2} implies that first occurrence of y_{m-2} in $\mathbf{v}$ precedes second occurrence of y_{m-1}. Therefore, $\ell_2(\mathbf{v}, x_m) < \ell_2(\mathbf{v}, y_{m-1})$. In view of the claim (3.6) and Lemma 3.11, $h_2^{m-2}(\mathbf{u}, y_{m-1}) = h_2^{m-2}(\mathbf{v}, y_{m-1}) = y_{m-2}$, whence $\ell_2(\mathbf{u}, x_m) < \ell_2(\mathbf{u}, y_{m-1})$.

Let now $m > 2$. Note that

$$\ell_1(\mathbf{u}, y_{m-1}) < \ell_2(\mathbf{u}, y_m) < \ell_1(\mathbf{u}, x_\ell) < \ell_1(\mathbf{u}, x_{\ell-1}) < \cdots < \ell_1(\mathbf{u}, x_{m-3}).$$

If $\ell_1(\mathbf{u}, x_{m-3}) < \ell_2(\mathbf{u}, y_{m-1})$ then the letter x_{m-3} lies between the first and the second occurrences of y_{m-1} in $\mathbf{u}$. Since x_{m-3} is an $(m-3)$-divider of $\mathbf{u}$, we obtain a contradiction with the equality $D(\mathbf{u}, y_{m-1}) = m - 1$. Thus, $\ell_2(\mathbf{u}, y_{m-1}) < \ell_1(\mathbf{u}, x_{m-3})$ whenever $m > 2$.

Further, there are three possibilities for second occurrence of the letter y_{m-1} in $\mathbf{u}$:

(6.37) $$\ell_2(\mathbf{u}, y_{m-1}) < \ell_1(\mathbf{u}, x_{m-2});$$

(6.38) $$\ell_1(\mathbf{u}, x_{m-2}) < \ell_2(\mathbf{u}, y_{m-1}) < \ell_2(\mathbf{u}, x_{m-1});$$

(6.39) $$\ell_2(\mathbf{u}, x_{m-1}) < \ell_2(\mathbf{u}, y_{m-1}).$$

Suppose that the claim (6.37) is true. Then $\ell_1(\mathbf{u}, y_{m-2}) < \ell_1(\mathbf{u}, x_{m-2})$. In view of Lemma 3.8, $\ell_1(\mathbf{v}, y_{m-2}) < \ell_1(\mathbf{v}, x_{m-2})$. Since $y_{m-2} = h_2^{m-2}(\mathbf{v}, y_{m-1})$

by Lemma 3.11, we have $\ell_2(\mathbf{v}, y_{m-1}) < \ell_1(\mathbf{v}, x_{m-2})$. Now if $m > 2$ then we apply Lemma 3.14 with $x_s = y_{m-2}$ and $s = m-2$ and conclude that there are letters $y_0, y_1, \dots, y_{m-3}$ such that $D(\mathbf{u}, y_p) = D(\mathbf{v}, y_p) = p$ and, for any $p = 0, 1, \dots, m-2$, the inequalities

$$\ell_1(\mathbf{w}, y_{p+1}) < \ell_1(\mathbf{w}, y_p) < \ell_2(\mathbf{w}, y_{p+1}) \text{ and } \ell_2(\mathbf{w}, y_{p+2}) < \ell_1(\mathbf{w}, y_p)$$

hold where $\mathbf{w}$ is any of the words $\mathbf{u}$ or $\mathbf{v}$. Lemma 6.2(ii) allows us to assume that the letters y_p with $1 \le p \le m$ occur twice in each of the words $\mathbf{u}$ and $\mathbf{v}$. In view of the above, we have the identity $\mathbf{u} \approx \mathbf{v}$ has the form

$$\begin{aligned}
&\mathbf{u}_{2\ell+5} y_m \mathbf{u}_{2\ell+4} y_{m-1} \mathbf{u}_{2\ell+3} x_\ell \mathbf{u}_{2\ell+2} y_m \mathbf{u}_{2\ell+1} x_{\ell-1} \mathbf{u}_{2\ell} x_\ell \mathbf{u}_{2\ell-1} x_{\ell-2} \mathbf{u}_{2\ell-2} x_{\ell-1} \cdots \\
&\cdot \mathbf{u}_{2m+1} y_{m-2} \mathbf{u}_{2m} y_{m-1} \mathbf{u}_{2m-1} x_{m-1} \mathbf{u}_{2m-2} y_{m-3} \mathbf{u}_{2m-2} y_{m-2} \cdots \\
&\cdot \mathbf{u}_4 y_1 \mathbf{u}_3 y_2 \mathbf{u}_2 y_0 \mathbf{u}_1 y_1 \mathbf{u}_0 \\
\approx\ &\mathbf{v}_{2\ell+5} y_m \mathbf{v}_{2\ell+4} y_{m-1} \mathbf{v}_{2\ell+3} y_m \mathbf{v}_{2\ell+2} x_\ell \mathbf{v}_{2\ell+1} x_{\ell-1} \mathbf{v}_{2\ell} x_\ell \mathbf{v}_{2\ell-1} x_{\ell-2} \mathbf{v}_{2\ell-2} x_{\ell-1} \cdots \\
&\cdot \mathbf{v}_{2m+1} y_{m-2} \mathbf{v}_{2m} y_{m-1} \mathbf{v}_{2m-1} x_{m-1} \mathbf{v}_{2m-2} y_{m-3} \mathbf{v}_{2m-2} y_{m-2} \cdots \\
&\cdot \mathbf{v}_4 y_1 \mathbf{v}_3 y_2 \mathbf{v}_2 y_0 \mathbf{v}_1 y_1 \mathbf{v}_0
\end{aligned}$$

for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2\ell+5}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2\ell+5}$ such that $x_s, y_t \notin \mathrm{con}(\mathbf{u}_i \mathbf{v}_i)$ for $m-1 \le s \le \ell$, $0 \le t \le m$ and $0 \le i \le 2\ell+5$. Now substitute 1 for all letters occurring in this identity except $y_0, y_1, \dots, y_m$ and $x_{m-1}, x_m, \dots, x_\ell$. We get the identity

$$\begin{aligned}
&y_m y_{m-1} x_\ell y_m x_{\ell-1} x_\ell x_{\ell-2} x_{\ell-1} \cdots y_{m-2} y_{m-1} x_{m-1} y_{m-3} y_{m-2} \cdots y_1 y_2 y_0 y_1 \\
\approx\ &y_m y_{m-1} y_m x_\ell x_{\ell-1} x_\ell x_{\ell-2} x_{\ell-1} \cdots y_{m-2} y_{m-1} x_{m-1} y_{m-3} y_{m-2} \cdots y_1 y_2 y_0 y_1.
\end{aligned}$$

Now we rename in this identity y_i in x_i for $i = 0, 1, \dots, m-2$ and obtain the identity

$$\begin{aligned}
&y_m \overset{(1)}{y_{m-1}} x_\ell y_m x_{\ell-1} x_\ell x_{\ell-2} x_{\ell-1} \cdots x_{m-2} \overset{(2)}{y_{m-1}} \overset{(2)}{x_{m-1}} x_{m-3} x_{m-2} \cdots \\
&\cdot x_1 x_2 x_0 x_1 \\
\approx\ &y_m \overset{(1)}{y_{m-1}} y_m x_\ell x_{\ell-1} x_\ell x_{\ell-2} x_{\ell-1} \cdots x_{m-2} \overset{(2)}{y_{m-1}} \overset{(2)}{x_{m-1}} x_{m-3} x_{m-2} \cdots \\
&\cdot x_1 x_2 x_0 x_1.
\end{aligned} \tag{6.40}$$

In view of Lemma 6.2(i), we may use the identity σ_2. This identity allows us to swap the second occurrences of the letters x_{m-1} and y_{m-1} in both the sides of the identity (6.40). As a result, we get δ_ℓ^{m-1}.

Suppose now that the claim (6.38) is true. If $\ell_2(\mathbf{v}, y_{m-1}) < \ell_1(\mathbf{v}, x_{m-2})$ then $\ell_1(\mathbf{v}, y_{m-2}) < \ell_1(\mathbf{v}, x_{m-2})$. In view of Lemma 3.8, $\ell_1(\mathbf{u}, y_{m-2}) < \ell_1(\mathbf{u}, x_{m-2})$. Since $y_{m-2} = h_2^{m-2}(\mathbf{v}, y_{m-1})$ by Lemma 3.11, we have $\ell_2(\mathbf{u}, y_{m-1}) < \ell_1(\mathbf{u}, x_{m-2})$. This contradicts the claim (6.38). Thus, $\ell_1(\mathbf{v}, x_{m-2}) < \ell_2(\mathbf{v}, y_{m-1})$.

Suppose that $\ell_2(\mathbf{v}, y_{m-1}) < \ell_2(\mathbf{v}, x_{m-1})$. Then the identity $\mathbf{u} \approx \mathbf{v}$ has the form

$$\begin{aligned}&\mathbf{u}_{2\ell+5} y_m \mathbf{u}_{2\ell+4} y_{m-1} \mathbf{u}_{2\ell+3} x_\ell \mathbf{u}_{2\ell+2} y_m \mathbf{u}_{2\ell+1} x_{\ell-1} \mathbf{u}_{2\ell} x_\ell \mathbf{u}_{2\ell-1} x_{\ell-2} \mathbf{u}_{2\ell-2} x_{\ell-1} \cdots \\ &\quad\cdot \mathbf{u}_{2m+1} x_{m-2} \mathbf{u}_{2m} y_{m-1} \mathbf{u}_{2m-1} x_{m-1} \mathbf{u}_{2m-2} x_{m-3} \mathbf{u}_{2m-2} x_{m-2} \cdots \\ &\quad\cdot \mathbf{u}_4 x_1 \mathbf{u}_3 x_2 \mathbf{u}_2 x_0 \mathbf{u}_1 x_1 \mathbf{u}_0 \\ \approx\ &\mathbf{v}_{2\ell+5} y_m \mathbf{v}_{2\ell+4} y_{m-1} \mathbf{v}_{2\ell+3} y_m \mathbf{v}_{2\ell+2} x_\ell \mathbf{v}_{2\ell+1} x_{\ell-1} \mathbf{v}_{2\ell} x_\ell \mathbf{v}_{2\ell-1} x_{\ell-2} \mathbf{v}_{2\ell-2} x_{\ell-1} \cdots \\ &\quad\cdot \mathbf{v}_{2m+1} x_{m-2} \mathbf{v}_{2m} y_{m-1} \mathbf{v}_{2m-1} x_{m-1} \mathbf{v}_{2m-2} x_{m-3} \mathbf{v}_{2m-2} x_{m-2} \cdots \\ &\quad\cdot \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2\ell+5}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2\ell+5}$ such that $x_s, y_{m-1},\ y_m \notin \mathrm{con}(\mathbf{u}_i \mathbf{v}_i)$ for $0 \le s \le \ell$ and $0 \le i \le 2\ell + 5$. Now substitute 1 for all letters occurring in this identity except $y_{m-1}, y_m, x_0, x_1, \dots, x_\ell$. We get the identity (6.40). As above, combining this identity with σ_2, we get δ_ℓ^{m-1}.

If $\ell_2(\mathbf{v}, x_{m-1}) < \ell_2(\mathbf{v}, y_{m-1})$ then the same arguments as above show that the identity

$$\begin{aligned}&y_m \overset{(1)}{y_{m-1}} x_\ell y_m x_{\ell-1} x_\ell x_{\ell-2} x_{\ell-1} \cdots x_{m-2} \overset{(2)}{y_{m-1}} \overset{(2)}{x_{m-1}} x_{m-3} x_{m-2} \cdots x_1 x_2 x_0 x_1 \\ \approx\ &y_m y_{m-1} y_m x_\ell x_{\ell-1} x_\ell x_{\ell-2} x_{\ell-1} \cdots x_{m-2} x_{m-1} y_{m-1} x_{m-3} x_{m-2} \cdots x_1 x_2 x_0 x_1\end{aligned}$$

holds in $\mathbf{V}$. Now we apply the identity σ_2 to the left-hand side of the last identity and get δ_ℓ^{m-1}.

Finally, suppose that the claim (6.39) is true. Suppose that $\ell_2(\mathbf{v}, y_{m-1}) < \ell_2(\mathbf{v}, x_{m-1})$. Then the identity $\mathbf{u} \approx \mathbf{v}$ has the form

$$\begin{aligned}&\mathbf{u}_{2\ell+5} y_m \mathbf{u}_{2\ell+4} y_{m-1} \mathbf{u}_{2\ell+3} x_\ell \mathbf{u}_{2\ell+2} y_m \mathbf{u}_{2\ell+1} x_{\ell-1} \mathbf{u}_{2\ell} x_\ell \mathbf{u}_{2\ell-1} x_{\ell-2} \mathbf{u}_{2\ell-2} x_{\ell-1} \cdots \\ &\quad\cdot \mathbf{u}_{2m+1} x_{m-2} \mathbf{u}_{2m} x_{m-1} \mathbf{u}_{2m-1} y_{m-1} \mathbf{u}_{2m-2} x_{m-3} \mathbf{u}_{2m-2} x_{m-2} \cdots \\ &\quad\cdot \mathbf{u}_4 x_1 \mathbf{u}_3 x_2 \mathbf{u}_2 x_0 \mathbf{u}_1 x_1 \mathbf{u}_0 \\ \approx\ &\mathbf{v}_{2\ell+5} y_m \mathbf{v}_{2\ell+4} y_{m-1} \mathbf{v}_{2\ell+3} y_m \mathbf{v}_{2\ell+2} x_\ell \mathbf{v}_{2\ell+1} x_{\ell-1} \mathbf{v}_{2\ell} x_\ell \mathbf{v}_{2\ell-1} x_{\ell-2} \mathbf{v}_{2\ell-2} x_{\ell-1} \cdots \\ &\quad\cdot \mathbf{v}_{2m+1} x_{m-2} \mathbf{v}_{2m} y_{m-1} \mathbf{v}_{2m-1} x_{m-1} \mathbf{v}_{2m-2} x_{m-3} \mathbf{v}_{2m-2} x_{m-2} \cdots \\ &\quad\cdot \mathbf{v}_4 x_1 \mathbf{v}_3 x_2 \mathbf{v}_2 x_0 \mathbf{v}_1 x_1 \mathbf{v}_0\end{aligned}$$

for some possibly empty words $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{2\ell+5}$ and $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{2\ell+5}$ such that $x_s, y_{m-1},\ y_m \notin \mathrm{con}(\mathbf{u}_i \mathbf{v}_i)$ for $0 \le s \le \ell$ and $0 \le i \le 2\ell + 5$. Now substitute 1 for all letters occurring in this identity except $y_{m-1}, y_m, x_0, x_1, \dots, x_\ell$. We get the identity

$$\begin{aligned}&y_m y_{m-1} x_\ell y_m x_{\ell-1} x_\ell x_{\ell-2} x_{\ell-1} \cdots x_{m-2} x_{m-1} y_{m-1} x_{m-3} x_{m-2} \cdots x_1 x_2 x_0 x_1 \\ \approx\ &y_m \overset{(1)}{y_{m-1}} y_m x_\ell x_{\ell-1} x_\ell x_{\ell-2} x_{\ell-1} \cdots x_{m-2} \overset{(2)}{y_{m-1}} \overset{(2)}{x_{m-1}} x_{m-3} x_{m-2} \cdots x_1 x_2 x_0 x_1.\end{aligned}$$

Applying once again σ_2 to the right-hand side of this identity, we get δ_ℓ^{m-1}.

If $\ell_2(\mathbf{v}, x_{m-1}) < \ell_2(\mathbf{v}, y_{m-1})$ then the same arguments as above show that the identity δ_ℓ^{m-1} holds in $\mathbf{V}$. □

Proposition 6.17. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{J}_k^r$ *if and only if the claims* (2.1)*,* (3.6)*,* (6.34) *and* (6.36) *with* $\ell = k$ *and* $m = r$ *are true.*

Proof. Necessity. Suppose that a non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{J}_k^r$. The claims (2.1), (3.6) and (6.34) with $\ell = k$ follow from Proposition 6.14 and the inclusion $\mathbf{I}_k \subseteq \mathbf{J}_k^r$. It remains to verify that the claim (6.36) with $\ell = k$ and $m = r$ is true. As in the proof of Proposition 6.9(i), we can assume that $\mathbf{u} = \mathbf{p}\xi(\mathbf{a})\mathbf{q}$ and $\mathbf{v} = \mathbf{p}\xi(\mathbf{b})\mathbf{q}$ for some possibly empty words $\mathbf{p}$ and $\mathbf{q}$, an endomorphism ξ of F^1 and an identity $\mathbf{a} \approx \mathbf{b} \in \{\Phi, \delta_k^r\}$.

If $\mathbf{a} \approx \mathbf{b} \in \Phi$ then the claim (3.6) is true for any ℓ by Proposition 6.9(ii). Evidently, this implies the required conclusion. Suppose now that $\mathbf{a} \approx \mathbf{b}$ coincides with δ_k^r. Then

$$\xi(\mathbf{a}) = \mathbf{b}_{r+1}\mathbf{b}_r\mathbf{b}_{r+1}\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_{r-1}\mathbf{a}_r\mathbf{b}_r\mathbf{a}_{r-2}\mathbf{a}_{r-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1,$$

$$\xi(\mathbf{b}) = \mathbf{b}_{r+1}\mathbf{b}_r\mathbf{a}_k\mathbf{b}_{r+1}\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_{r-1}\mathbf{a}_r\mathbf{b}_r\mathbf{a}_{r-2}\mathbf{a}_{r-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1$$

for some words $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_k$ and $\mathbf{b}_r, \mathbf{b}_{r+1}$, whence

$$\mathbf{u} = \mathbf{p}\mathbf{b}_{r+1}\mathbf{b}_r\mathbf{b}_{r+1}\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_{r-1}\mathbf{a}_r\mathbf{b}_r\mathbf{a}_{r-2}\mathbf{a}_{r-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q},$$

$$\mathbf{v} = \mathbf{p}\mathbf{b}_{r+1}\mathbf{b}_r\mathbf{a}_k\mathbf{b}_{r+1}\mathbf{a}_{k-1}\mathbf{a}_k\mathbf{a}_{k-2}\mathbf{a}_{k-1}\cdots\mathbf{a}_{r-1}\mathbf{a}_r\mathbf{b}_r\mathbf{a}_{r-2}\mathbf{a}_{r-1}\cdots\mathbf{a}_1\mathbf{a}_2\mathbf{a}_0\mathbf{a}_1\mathbf{q}.$$

By Lemma 6.5, $D(\mathbf{a}, x_k) = k$. Then Lemma 3.15 implies that the subword $\mathbf{a}_k$ of $\mathbf{u}$ located between $\mathbf{b}_{r+1}$ and $\mathbf{a}_{k-1}$ does not contain any $(k-1)$-divider. Also, obviously, the subword $\mathbf{b}_{r+1}$ of $\mathbf{u}$ located between $\mathbf{b}_r$ and $\mathbf{a}_k$ does not contain any s-divider for all s. Therefore, the subword $\mathbf{b}_{r+1}\mathbf{a}_k$ of $\mathbf{u}$ located between $\mathbf{b}_r$ and $\mathbf{a}_{k-1}$ lies in some $(k-1)$-block. Now we apply Lemma 3.15 again and obtain the subword $\mathbf{b}_{r+1}$ located between $\mathbf{p}$ and $\mathbf{b}_r$ does not contain any s-divider for all $s \le r$. Hence if second occurrence in $\mathbf{u}$ of some letter lies in the subword $\mathbf{b}_{r+1}$ located between $\mathbf{b}_r$ and $\mathbf{a}_k$ then the depth of this letter is more than r. This implies that the claim (6.36) with $\ell = k$ and $m = r$ is true.

Sufficiency. As in the proofs of Propositions 6.12 and 6.14, the outline of our arguments here is similar to one from the proof of sufficiency in Proposition 6.9(i). But the canonical form of the block here is even more complicated than in Proposition 6.14.

Suppose that the claims (2.1), (3.6), (6.34) and (6.36) with $\ell = k$ and $m = r$ are true. As in the proof of sufficiency in Proposition 6.12, we suppose that (3.4) is the $(k-1)$-decomposition of $\mathbf{u}$, while (6.32) is the representation of $t_i\mathbf{u}_i$ as the product of alternating k-dividers $s_0, s_1, \ldots, s_n$ and k-blocks $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$.

Clearly, $\mathbf{u} = \mathbf{w}_1\mathbf{u}_i\mathbf{w}_2$ for some possibly empty words $\mathbf{w}_1$ and $\mathbf{w}_2$. For $j = 0, 1, \ldots, n$, we put

$$X_j = \{x \in \mathrm{con}(\mathbf{a}_j) \mid \text{ first occurrence of } x \text{ in } \mathbf{u} \text{ lies in } \mathbf{a}_j\}.$$

Let $X_j = \{x_{j1}, x_{j2}, \ldots, x_{jq_j}\}$, $X = X_0 \cup X_1 \cup \cdots \cup X_n$, $\mathbf{a}_j' = (\mathbf{a}_j)_X$. As in the proof of sufficiency in Proposition 6.14, we can verify that

$$X_j = \bigl\{x \in \mathrm{con}(\mathbf{a}_j) \mid s_j = h_1^k(\mathbf{u}, x)\bigr\}.$$

For any $j = 0, 1, \ldots, n$, we put

$$Z_j = \{z \in \mathrm{con}(\mathbf{a}_j') \mid D(\mathbf{u}, z) \le r\},$$

$Z = Z_0 \cup Z_1 \cup \cdots \cup Z_n$, $\mathbf{a}_j'' = (\mathbf{a}_j')_Z$ and $\mathbf{u}_i^* = \mathbf{a}_0''\mathbf{a}_1''\cdots\mathbf{a}_n''$. Let $Z_j = \{z_{j1}, z_{j2}, \ldots, z_{jh_j}\}$, $\mathrm{con}(\mathbf{u}_i^*) = \{c_1, c_2, \ldots, c_p\}$ and

$$\begin{aligned}\overline{\mathbf{u}_i} &= (c_1c_2\cdots c_p)\cdot(x_{01}^2\cdots x_{0q_0}^2 z_{01}\cdots z_{0h_0})\cdot(s_1x_{11}^2\cdots x_{1q_1}^2 z_{11}\cdots z_{1h_1})\cdots\\ &\quad\cdot(s_nx_{n1}^2\cdots x_{nq_n}^2 z_{n1}\cdots z_{nh_n}).\end{aligned}$$

As we will see below, $\overline{\mathbf{u}_i}$ is nothing but the mentioned above "canonical form" of the $(k-1)$-block $\mathbf{u}_i$.

As in the proof of sufficiency in Proposition 6.14, we can verify that $\mathbf{J}_k^r$ satisfies the identity (6.35). The definition of the set X and words of the form $\mathbf{a}_j'$ imply that $z \in \mathrm{con}(\mathbf{w}_1)$ for any $z \in \mathrm{con}(\mathbf{a}_0'\mathbf{a}_1' \cdots \mathbf{a}_n')$. This implies that if $z \in Z_j$ then we can assume that $\mathrm{occ}_z(\mathbf{u}_i) = 1$ because $\mathbf{J}_k^r$ satisfies the identity (6.3) by Lemma 6.2(ii). Then we can assume without loss of generality that $\ell_1(\mathbf{u}, z_{j1}) < \ell_1(\mathbf{u}, z_{j2}) < \cdots < \ell_1(\mathbf{u}, z_{jh_j})$. Since $z \in \mathrm{con}(\mathbf{w}_1)$ and $D(\mathbf{u}, z) > r$ for any $z \in \mathrm{con}(\mathbf{a}_j'')$, we apply Lemma 6.6(i) with $m = r$ and obtain the identity

$$\begin{aligned}\mathbf{u} \approx \mathbf{w}_1 \cdot \mathbf{u}_i^* \cdot (x_{01}^2 \cdots x_{0q_0}^2 z_{01} \cdots z_{0h_0}) \cdot (s_1 x_{11}^2 \cdots x_{1q_1}^2 z_{11} \cdots z_{1h_1}) \cdots \\ \cdot (s_n x_{n1}^2 \cdots x_{nq_n}^2 z_{n1} \cdots z_{nh_n}) \cdot \mathbf{w}_2\end{aligned}$$

holds in $\mathbf{J}_k^r$. As we have seen above, $\mathrm{con}(\mathbf{u}_i^*) \subseteq \mathrm{con}(\mathbf{w}_1)$. Then we can apply the identity (6.3) and obtain the word $\mathbf{u}_i^*$ is linear. Then Lemma 6.2(i) applies and we conclude that $\mathbf{J}_k^r$ satisfies the identities

$$\begin{aligned}\mathbf{u} &\approx \mathbf{w}_1 \cdot (c_1 c_2 \cdots c_p) \cdot (x_{01}^2 \cdots x_{0q_0}^2 z_{01} \cdots z_{0h_0}) \cdot (s_1 x_{11}^2 \cdots x_{1q_1}^2 z_{11} \cdots z_{1h_1}) \cdots \\ &\quad \cdot (s_n x_{n1}^2 \cdots x_{nq_n}^2 z_{n1} \cdots z_{nh_n}) \cdot \mathbf{w}_2 \\ &= \mathbf{w}_1 \overline{\mathbf{u}_i}\, \mathbf{w}_2.\end{aligned}$$

So, as in the proof of Proposition 6.9(i), using identities which hold in the variety $\mathbf{J}_k^r$, we can replace the $(k-1)$-blocks $\mathbf{u}_i$ of $\mathbf{u}$ successively, one after another, by the "canonical form" $\overline{\mathbf{u}_i}$ for $i = m, m-1, \ldots, 0$. Then the variety $\mathbf{J}_k^r$ satisfies the identities (6.29). Put $\mathbf{u}^\sharp = t_0\, \overline{\mathbf{u}_0}\, t_1\, \overline{\mathbf{u}_1} \cdots t_m\, \overline{\mathbf{u}_m}$.

One can return to the word $\mathbf{v}$. By Lemma 3.8, the $(k-1)$-decomposition of $\mathbf{v}$ has the form (3.5). Furthermore, the claim (6.34) with $\ell = k$ and Lemma 3.8 imply that (6.33) is a representation of $t_i\mathbf{v}_i$ as the product of alternating k-dividers $s_0, s_1, \ldots, s_n$ and k-blocks $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$. The claim (6.34) with $\ell = k$ implies that

$$X_j = \bigl\{x \in \mathrm{con}(\mathbf{b}_j) \mid s_j = h_1^k(\mathbf{v}, x)\bigr\}$$

for all $j = 0, 1, \ldots, n$. Put $\mathbf{b}_j' = (\mathbf{b}_j)_X$. In view of the claim (6.36) with $\ell = k$ and $m = r$, we have

$$Z_j = \{z \in \mathrm{con}(\mathbf{b}_j') \mid D(\mathbf{v}, z) \le r\}$$

for all $j = 0, 1, \ldots, n$. Put $\mathbf{b}_j'' = (\mathbf{b}_j')_Z$. The claim (3.6) with $\ell = k$ implies that jth occurrence of a letter x in $\mathbf{u}$ lies in the $(k-1)$-block $\mathbf{u}_i$ if and only if jth occurrence of a letter x in $\mathbf{v}$ lies in the $(k-1)$-block $\mathbf{v}_i$ for any x and any $j = 1, 2$. Also, Lemma 6.2(ii) allows us to assume that if the first and the second occurrences of the letter x in $\mathbf{u}$ do not lie in the $(k-1)$-block $\mathbf{u}_i$ then this letter does not occur in $\mathbf{u}_i$. Then $\mathrm{con}(\mathbf{u}_i^*) = \mathrm{con}(\mathbf{b}_0''\mathbf{b}_1'' \cdots \mathbf{b}_n'')$. This implies that the $(k-1)$-blocks $\mathbf{u}_i$ and $\mathbf{v}_i$ have the same "canonical form". Repeating literally arguments given above, we obtain the variety $\mathbf{J}_k^r$ satisfies the identities $\mathbf{v} \approx \mathbf{u}^\sharp \approx \mathbf{u}$. □

Now we are well prepared to quickly complete the proof of Lemma 6.15. Let $\mathbf{I}_k \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$. We have to verify that $\mathbf{X} \supseteq \mathbf{J}_k^1$. Arguing by contradiction, suppose that $\mathbf{J}_k^1 \nsubseteq \mathbf{X}$. Then there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$

but does not hold in $\mathbf{J}_k^1$. Then Propositions 6.14 and 6.17 and the inclusion $\mathbf{I}_k \subset \mathbf{X}$ together imply that the claims (2.1), (3.6) and (6.34) are true, while the claim (6.36) with $m = 1$ is false. Then Lemma 6.16(i) implies that $\mathbf{X} \subseteq \mathbf{I}_k$, a contradiction. Lemma 6.15 is proved. □

6.4.4. *If $\mathbf{J}_k^m \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ with $1 \le m < k$ then $\mathbf{J}_k^{m+1} \subseteq \mathbf{X}$.* The fourth step in the verification of the claim 4) of Proposition 6.1 is the following

Lemma 6.18. *If $\mathbf{X}$ is a monoid variety such that $\mathbf{X} \in [\mathbf{J}_k^m, \mathbf{F}_{k+1}]$ for some $1 \le m < k$ then either $\mathbf{X} = \mathbf{J}_k^m$ or $\mathbf{X} \supseteq \mathbf{J}_k^{m+1}$.*

Proof. Let $1 \le m < k$, $\mathbf{J}_k^m \subset \mathbf{X} \subseteq \mathbf{F}_{k+1}$ and $\mathbf{J}_k^{m+1} \nsubseteq \mathbf{X}$. Then there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but does not hold in $\mathbf{J}_k^{m+1}$. Then Proposition 6.17 and the inclusion $\mathbf{J}_k^m \subset \mathbf{X}$ imply that the claims (2.1), (3.6), (6.34) and (6.36) with $\ell = k$ are true, while the claim

$$\text{if } x \in \operatorname{con}(\mathbf{u}) \text{ and } D(\mathbf{u}, x) \le m+1 \text{ then } h_2^k(\mathbf{u}, x) = h_2^k(\mathbf{v}, x)$$

is false. Then Lemma 6.16(ii) implies that $\mathbf{X} \subseteq \mathbf{J}_k^m$, a contradiction. We see that either $\mathbf{X} = \mathbf{J}_k^m$ or $\mathbf{J}_k^{m+1} \subseteq \mathbf{X}$. □

6.4.5. *The interval $[\mathbf{J}_k^k, \mathbf{F}_{k+1}]$ consists of $\mathbf{J}_k^k$ and $\mathbf{F}_{k+1}$ only.* The fifth step in the verification of the claim 4) of Proposition 6.1 is the following

Lemma 6.19. *If $\mathbf{X}$ is a monoid variety such that $\mathbf{X} \in [\mathbf{J}_k^k, \mathbf{F}_{k+1}]$ then either $\mathbf{X} = \mathbf{J}_k^k$ or $\mathbf{X} = \mathbf{F}_{k+1}$.*

Proof. Suppose that $\mathbf{J}_k^k \subset \mathbf{X} \subset \mathbf{F}_{k+1}$. Since $\mathbf{F}_{k+1} \nsubseteq \mathbf{X}$, there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{X}$ but does not hold in $\mathbf{F}_{k+1}$. Propositions 6.9(i) and 6.17 and the inclusion $\mathbf{J}_k^k \subset \mathbf{X}$ together imply that the claims (2.1), (3.6), (6.34) and the claim (6.36) with $\ell = m = k$ are true, while $h_2^k(\mathbf{u}, x) \ne h_2^k(\mathbf{v}, x)$ for some letter $x \in \operatorname{con}(\mathbf{u})$ such that $D(\mathbf{u}, x) > k$. Then we apply Lemma 6.8 for the variety $\mathbf{F}_{k+1}$ and obtain $\mathbf{X} \subseteq \mathbf{J}_k^k$, a contradiction. □

6.4.6. *All inclusions are strict.* Here we are going to verify the inclusions (6.1). To achieve this goal, we use Lemma 6.5 and Table 6.1 without explicit references. We note that non-strict inclusions (6.5) are true by Lemma 6.4. If $\mathbf{u} \approx \mathbf{v}$ is the identity α_k then $D(\mathbf{u}, x_k) = k$ but $h_1^k(\mathbf{u}, x_k) = \lambda$ and $h_1^k(\mathbf{v}, x_k) = y_k$. Then Proposition 6.12 implies that $\mathbf{F}_k \subset \mathbf{H}_k$. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ coincides with the identity β_k. Then $h_1^k(\mathbf{u}, x) = \lambda$, while $h_1^k(\mathbf{v}, x) = x_k$. We apply Proposition 6.14 and obtain $\mathbf{H}_k \subset \mathbf{I}_k$. Let now $\mathbf{u} \approx \mathbf{v}$ be equal γ_k. In this case $D(\mathbf{u}, y_1) = 1$ but $h_2^k(\mathbf{u}, y_1) = y_0$ and $h_2^k(\mathbf{v}, y_1) = x_k$. In view of Proposition 6.17, $\mathbf{I}_k \subset \mathbf{J}_k^1$. Suppose now that $\mathbf{u} \approx \mathbf{v}$ coincides with the identity δ_k^m for some $1 \le m < k$. Then $D(\mathbf{u}, y_{m+1}) = m+1$ but $h_2^k(\mathbf{u}, y_{m+1}) = y_m$ and $h_2^k(\mathbf{v}, y_{m+1}) = x_k$. Now we apply Proposition 6.17 again and obtain $\mathbf{J}_k^m \subset \mathbf{J}_k^{m+1}$. Finally, suppose that $\mathbf{u} \approx \mathbf{v}$ is the identity δ_k^k. Since $h_2^k(\mathbf{u}, y_{k+1}) = y_k$ and $h_2^k(\mathbf{v}, y_{k+1}) = x_k$, Proposition 6.9(i) implies that $\mathbf{J}_k^k \subset \mathbf{F}_{k+1}$.

Thus, we have proved the inclusions (6.1). Therefore, the varieties $\mathbf{F}_k$, $\mathbf{H}_k$, $\mathbf{I}_k$, $\mathbf{J}_k^1$, $\mathbf{J}_k^2$, $\dots$, $\mathbf{J}_k^k$ and $\mathbf{F}_{k+1}$ are pairwise different. This fact and Lemmas 6.10, 6.13, 6.15, 6.18 (with $m = 1, 2, \dots, k-1$) and 6.19 together implies the claim 4) of Proposition 6.1. In view of Lemma 2.10(ii) and the results of Subsections 6.1 and 6.3, we complete the proof of Proposition 6.1. □

Lemmas 2.8 and 2.9(ii), Corollary 4.7, Propositions 5.1, 5.2 and 6.1, and the dual of Propositions 5.2 and 6.1 together imply the "if" part of Theorem 1.1.

Recall that the "only if" part of Theorem 1.1 was verified in Section 4. Thus, Theorem 1.1 is completely proved. □

7. Corollaries

First of all, we indicate the exhaustive list of non-group chain varieties of monoids. Theorem 1.1 together with Lemmas 2.8 and 2.9(ii), Corollary 4.7, Propositions 5.1, 5.2 and 6.1, and the dual of Propositions 5.2 and 6.1 implies the following

Corollary 7.1. *The varieties* $\mathbf{C}_n$, $\mathbf{D}_k$, $\mathbf{D}$, $\mathbf{E}$, $\overleftarrow{\mathbf{E}}$, $\mathbf{F}_k$, $\overleftarrow{\mathbf{F}_k}$, $\mathbf{H}_k$, $\overleftarrow{\mathbf{H}_k}$, $\mathbf{I}_k$, $\overleftarrow{\mathbf{I}_k}$, $\mathbf{J}_k^m$, $\overleftarrow{\mathbf{J}_k^m}$, $\mathbf{K}$, $\overleftarrow{\mathbf{K}}$, $\mathbf{L}$, $\mathbf{LRB}$, $\mathbf{M}$, $\overleftarrow{\mathbf{M}}$, $\mathbf{N}$, $\overleftarrow{\mathbf{N}}$, $\mathbf{RRB}$, $\mathbf{SL}$ *where* $n \ge 2$, $k \in \mathbb{N}$ *and* $1 \le m \le k$ *and only they are non-group chain varieties of monoids.* □

The set of all non-group chain varieties of monoids ordered by inclusion together with the variety $\mathbf{T}$ is shown in Fig. 7.1. It is interesting to compare this figure with the diagram of the partially ordered set of all non-group chain varieties of semigroups (as we have already mentioned in Section 1, such varieties were completely determined in [22]). This diagram is shown in Fig. 7.2 where $\mathbf{LZ} = \mathrm{var}\{xy \approx x\}$, $\mathbf{RZ} = \mathrm{var}\{xy \approx y\}$, $\mathbf{ZM} = \mathrm{var}\{xy \approx 0\}$, $\mathbf{N}_k = \mathrm{var}\{x^2 \approx x_1x_2\cdots x_k \approx 0,\ xy \approx yx\}$ for all $k \ge 3$, $\mathbf{N}_\omega = \mathrm{var}\{x^2 \approx 0,\ xy \approx yx\}$, $\mathbf{N}_3^2 = \mathrm{var}\{x^2 \approx xyz \approx 0\}$ and $\mathbf{N}_3^c = \mathrm{var}\{xyz \approx 0,\ xy \approx yx\}$ (here $\mathrm{var}\,\Sigma$ means the semigroup variety given by Σ; as is usually done when considering semigroup varieties, we write the symbolic identity $\mathbf{w} \approx 0$ as a short form of the identity system $\mathbf{w}x \approx x\mathbf{w} \approx \mathbf{w}$ where $x \notin \mathrm{con}(\mathbf{w})$).

We see that, out of the group case, there are 1 countably infinite series and 6 "sporadic" chain semigroup varieties, but 10 countably infinite series and 12 "sporadic" chain monoid varieties. Namely, we have the countably infinite series $\mathbf{N}_k$ (including $\mathbf{ZM}$ as $\mathbf{N}_2$) and sporadic varieties $\mathbf{LZ}$, $\mathbf{RZ}$, $\mathbf{SL}$, $\mathbf{N}_3^2$, $\mathbf{N}_3^c$, $\mathbf{N}_\omega$ in the semigroup case, and countably infinite series $\mathbf{C}_n$ (including $\mathbf{SL}$ as $\mathbf{C}_1$), $\mathbf{D}_k$, $\mathbf{F}_k$, $\overleftarrow{\mathbf{F}_k}$, $\mathbf{H}_k$, $\overleftarrow{\mathbf{H}_k}$, $\mathbf{I}_k$, $\overleftarrow{\mathbf{I}_k}$, $\mathbf{J}_k^m$, $\overleftarrow{\mathbf{J}_k^m}$ and sporadic varieties $\mathbf{D}$, $\mathbf{E}$, $\overleftarrow{\mathbf{E}}$, $\mathbf{K}$, $\overleftarrow{\mathbf{K}}$, $\mathbf{L}$, $\mathbf{LRB}$, $\mathbf{M}$, $\overleftarrow{\mathbf{M}}$, $\mathbf{N}$, $\overleftarrow{\mathbf{N}}$, $\mathbf{RRB}$ in the monoid case. One can say that the number of non-group chain varieties in the case of monoids is much larger (in some informal sense) than in the case of semigroups. Consequently, the partially ordered set of non-group chain varieties in the former case is much more complicated than in the latter one.

As we have already mentioned in Section 1, a non-group chain variety of semigroups is contained in a maximal chain variety, while this is not the case for monoid varieties. The following two corollaries indicate cases when the analog of the semigroup statement is true. Fig. 7.1 shows that the following is true.

Corollary 7.2. *A non-group chain variety of monoids* $\mathbf{V}$ *is contained in some maximal chain variety if and only if* $\mathbf{C}_3 \nsubseteq \mathbf{V}$. □

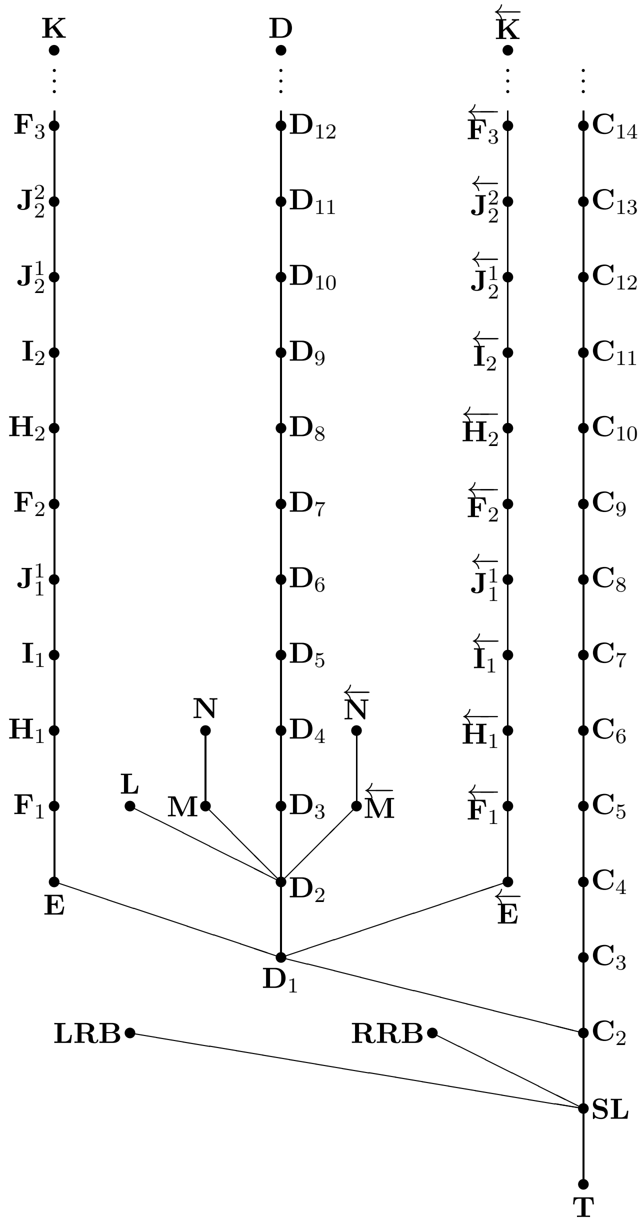

FIGURE 7.1. All non-group chain varieties of monoids

Theorem 1.1 shows that commutative non-group chain varieties of monoids are exhausted by the varieties **SL** and $\mathbf{C}_n$ with $n \geq 2$. This claim and Fig. 7.1 imply the following

Corollary 7.3. *A non-commutative non-group chain variety of monoids is contained in some maximal chain variety.* □

In the following corollary we mention the variety **O** introduced in Subsection 4.3.

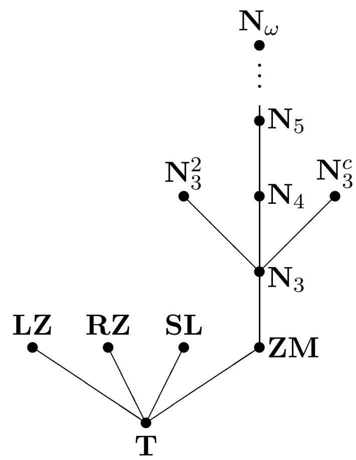

FIGURE 7.2. All non-group chain varieties of semigroups

Corollary 7.4. *Let* $\mathbf{X}$ *be a monoid variety with* $\mathbf{L} \subset \mathbf{X} \subseteq \mathbf{O}$. *Then* $\mathbf{X}$ *is not a chain variety and does not contain a just non-chain subvariety.*

Proof. Theorem 1.1 immediately implies that there are no chain monoid varieties that properly contain $\mathbf{L}$, whence $\mathbf{X}$ is not a chain variety. It remains to check that $\mathbf{X}$ does not contain a just non-chain subvariety. Arguing by contradiction, suppose that $\mathbf{X}$ contains such a subvariety $\mathbf{Y}$. In view of Theorem 1.1, any chain subvariety of the variety $\mathbf{O}$ is contained in $\mathbf{L}$. In particular, $\mathbf{O}$ (and therefore, $\mathbf{Y}$) does not contain incomparable chain subvarieties. On the other hand, being a non-chain variety, $\mathbf{Y}$ contains at least two incomparable subvarieties. These two varieties are proper subvarieties of $\mathbf{Y}$, whence they are chain varieties. We have a contradiction. □

The following question seems to be very interesting.

Question 7.5. Is it true that a non-chain non-group monoid variety $\mathbf{X}$ with $\mathbf{X} \nsubseteq \mathbf{O}$ contains a just non-chain subvariety?

Recall that a variety of universal algebras is called *locally finite* if all its finitely generated members are finite. A variety is called *finitely generated* if it is generated by a finite algebra. Clearly, if a variety is contained in some finitely generated variety then it is locally finite.

Corollary 7.6. *An arbitrary non-group chain monoid variety is contained in some finitely generated variety; in particular, it is locally finite.*

Proof. Clearly, it suffices to verify that each of the varieties listed in Theorem 1.1 is contained in a finitely generated variety. It is well known that a proper variety of band monoids is finitely generated [5]. In particular, the varieties $\mathbf{LRB}$ and $\mathbf{RRB}$ have this property. It is evident that the monoid $S(\mathbf{w})$ is finite for any word $\mathbf{w}$. Then Lemmas 2.4 and 4.6 provide the required conclusion for the varieties $\mathbf{C}_n$ and $\mathbf{L}$ respectively. The fact that the variety $\overleftarrow{\mathbf{N}}$ is finitely generated follows from Example 1 in Erratum to [8]. By symmetry, it remains to consider the varieties $\mathbf{D}$ and $\mathbf{K}$.

The variety **D** is not finitely generated by [14, Theorem 2], but it is shown in [15, Example 5.3] that **D** is a subvariety of the variety generated by the well-known 6-element Brandt monoid $B_2^1 = B_2 \cup \{1\}$ where

$$B_2 = \langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b\rangle = \{a, b, ab, ba, 0\}.$$

Finally, it is easy to see that if a monoid M belongs to **K** and consists of k elements then M satisfies the identity α_k. Therefore, any finitely generated subvariety of **K** is contained in $\mathbf{F}_k$ for some k. In particular, the variety **K** is not finitely generated. But Lemma 6.2 implies that $\mathbf{K} \subseteq \operatorname{var}\{xyxzx \approx xyxz,\ \sigma_2\}$. To complete our considerations, it remains to note that the variety $\operatorname{var}\{xyxzx \approx xyxz,\ \sigma_2\}$ is generated by the 5-element monoid

$$\langle a, b \mid a^2 = ab = a,\ b^2a = b^2\rangle \cup \{1\} = \{a, b, ba, b^2, 1\}.$$

This fact is proved in [17, Corollary 6.6]. □

Analog of Corollary 7.6 for arbitrary chain varieties of monoids (including group ones) does not hold. Indeed, as we have already mentioned in Section 1, it is verified in [11] that there are uncountably many non-locally finite chain varieties of groups. But explicit examples of such varieties have not yet been specified anywhere.

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References

[1] J. Almeida, *Finite Semigroups and Universal Algebra*, World Scientific, Singapore, 1994.
[2] V.A. Artamonov, *Chain varieties of groups*, Trudy Seminara Imeni I.G.Petrovskogo (Proc. of the Seminar Named After I.G.Petrovsky), **3** (1978), 3–8 [Russian].
[3] S. Burris and E. Nelson, *Embedding the dual of* Π_m *in the lattice of equational classes of commutative semigroups*, Proc. Amer. Math. Soc., **30** (1971), 37–39.
[4] S. Burris and E. Nelson, *Embedding the dual of* Π_∞ *in the lattice of equational classes of semigroups*, Algebra Universalis, **1** (1971), 248–254.
[5] J.A. Gerhard, *Some subdirectly irreducible idempotent semigroups*, Semigroup Forum, **5** (1972), 362–369.
[6] S.V. Gusev, *On the lattice of overcommutative varieties of monoids*, Izv. Vyssh. Uchebn. Zaved. Matem., No.5 (2018), 28–32 [Russian; Engl. translation is available at: https://arxiv.org/abs/1702.08749].
[7] T.J. Head, *The varieties of commutative monoids*, Nieuw Arch. Wiskunde. III Ser., **16** (1968), 203–206.
[8] M. Jackson, *Finiteness properties of varieties and the restriction to finite algebras*, Semigroup Forum, **70** (2005), 154–187; *Erratum to "Finiteness properties of varieties and the restriction to finite algebras"*, Semigroup Forum, **96** (2018), 197–198.
[9] M. Jackson and E. W.H. Lee, *Monoid varieties with extreme properties*, Trans. Amer. Math. Soc., doi: https://doi.org/10.1090/tran/7091.
[10] M. Jackson and O. Sapir, *Finitely based, finite sets of words*, Int. J. Algebra and Comput., **10** (2000), 683–708.

[11] P.A. Kozhevnikov, *On nonfinitely based varieties of groups of large prime exponent*, Commun. Algebra, **40** (2012), 2628–2644.
[12] E. W.H. Lee, *Varieties generated by* 2*-testable monoids*, Studia Sci. Math. Hungar., **49** (2012), 366–389.
[13] E. W.H. Lee, *Maximal Specht varieties of monoids*, Moscow Math. J., **12** (2012), 787–802.
[14] E. W.H. Lee, *Almost Cross varieties of aperiodic monoids with central idempotents*, Beiträge zur Algebra und Geometrie, **54** (2013), 121–129.
[15] E. W.H. Lee, *Inherently non-finitely generated varieties of aperiodic monoids with central idempotents*, Zapiski Nauchnykh Seminarov POMI (Notes of Scientific Seminars of the St. Petersburg Branch of the Math. Institute of the Russ. Acad. of Sci.), **423** (2014), 166–182.
[16] E. W.H. Lee, *On certain Cross varieties of aperiodic monoids with central idempotents*, Results Math., **66** (2014), 491–510.
[17] E. W.H. Lee and J.R. Li, *Minimal non-finitely based monoids*, Dissert. Math., **475** (2011), 65 p.
[18] P. Perkins, *Bases for equational theories of semigroups*, J. Algebra, **11** (1969), 298–314.
[19] Gy. Pollák, *Some lattices of varieties containing elements without cover*, Quad. Ric. Sci., **109** (1981), 91–96.
[20] O. Sapir, *Non-finitely based monoids*, Semigroup Forum, **90** (2015), 557–586.
[21] L.N. Shevrin, B.M. Vernikov and M.V. Volkov, *Lattices of semigroup varieties*, Izv. Vyssh. Uchebn. Zaved. Matem., No.3 (2009), 3–36 [Russian; Engl. translation: Russian Math. (Iz. VUZ), **53**, No.3 (2009), 1–28].
[22] E.V. Sukhanov, *Almost linear semigroup varieties*, Matem. Zametki, **32** (1982), 469–476 [Russian; Engl. translation: Math. Notes, **32** (1982), 714–717].
[23] B.M. Vernikov, *Special elements in lattices of semigroup varieties*, Acta Sci. Math. (Szeged), **81** (2015), 79–109.
[24] S.L. Wismath, *The lattice of varieties and pseudovarieties of band monoids*, Semigroup Forum, **33** (1986), 187–198.

Ural Federal University, Institute of Natural Sciences and Mathematics, Lenina 51, 620000 Ekaterinburg, Russia

E-mail addresses: sergey.gusb@gmail.com, bvernikov@gmail.com