

# String Periods in the Order-Preserving Model

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## Abstract

The order-preserving model (op-model, in short) was introduced quite recently but has already attracted significant attention because of its applications in data analysis. We introduce several types of periods in this setting (op-periods). Then we give algorithms to compute these periods in time  $O(n)$ ,  $O(n \log \log n)$ ,  $O(n \log^2 \log n / \log \log \log n)$ ,  $O(n \log n)$  depending on the type of periodicity. In the most general variant the number of different periods can be as big as  $\Omega(n^2)$ , and a compact representation is needed. Our algorithms require novel combinatorial insight into the properties of such periods.

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## 1 Introduction

Study of strings in the *order-preserving* model (*op-model*, in short) is a part of the so-called non-standard stringology. It is focused on pattern matching and repetition discovery problems in the shapes of number sequences. Here the shape of a sequence is given by the relative order of its elements. The applications of the op-model include finding trends in time series which appear naturally when considering e.g. the stock market or melody matching of two musical scores; see [32]. In such problems periodicity plays a crucial role.

One of motivations is given by the following scenario. Consider a sequence  $D$  of numbers that models a time series which is known to repeat the same shape every fixed period of time. For example, this could be certain stock market data or statistics data from a social network that is strongly dependent on the day of the week, i.e., repeats the same shape every consecutive week. Our goal is, given a fragment  $S$  of the sequence  $D$ , to discover such repeating shapes, called here *op-periods*, in  $S$ . We also consider some special cases of this setting. If the beginning of the sequence  $S$  is synchronized with the beginning of the repeating shape in  $D$ , we refer to the repeating shape as to an *initial* op-period. If the synchronization takes place also at the end of the sequence, we call the shape a *full* op-period. Finally, we also consider *sliding* op-periods that describe the case when every factor of the sequence  $D$  repeats the same shape every fixed period of time.

**Order-preserving model.** Let  $\llbracket a..b \rrbracket$  denote the set  $\{a, \dots, b\}$ . We say that two strings  $X = X[1] \dots X[n]$  and  $Y = Y[1] \dots Y[n]$  over an integer alphabet are *order-equivalent* (*equivalent* in short), written  $X \approx Y$ , iff  $\forall_{i,j \in \llbracket 1..n \rrbracket} X[i] < X[j] \Leftrightarrow Y[i] < Y[j]$ .

► **Example 1.**  $5275131035 \approx 647635956$ .

Order-equivalence is a special case of a substring consistent equivalence relation (SCER) that was defined in [37].

For a string  $S$  of length  $n$ , we can create a new string  $X$  of length  $n$  such that  $X[i]$  is equal to the number of distinct symbols in  $S$  that are not greater than  $S[i]$ . The string  $X$  is called the *shape* of  $S$  and is denoted by  $\text{shape}(S)$ . It is easy to observe that two strings  $S, T$  are order-equivalent if and only if they have the same shape.

► **Example 2.**  $\text{shape}(5275131035) = \text{shape}(647635956) = 425413634$ .

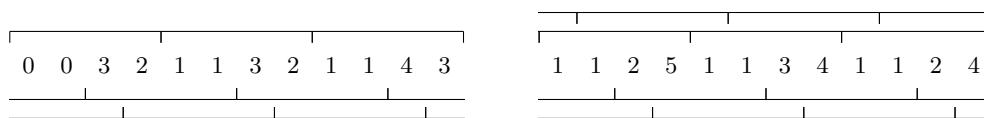
**Periods in the op-model.** We consider several notions of periodicity in the op-model, illustrated by Fig. 1. We say that a string  $S$  has a (general) *op-period*  $p$  with *shift*  $s \in \llbracket 0..p-1 \rrbracket$  if and only if  $p < |S|$  and  $S$  is a factor of a string  $V_1 V_2 \dots V_k$  such that:

$$|V_1| = \dots = |V_k| = p, \quad V_1 \approx \dots \approx V_k, \quad \text{and } S[s + 1..|S|] \text{ is a prefix of } V_2 \dots V_k.$$

The *shape* of the op-period is  $\text{shape}(V_1)$ . One op-period  $p$  can have several shifts; to avoid ambiguity, we sometimes denote the op-period as  $(p, s)$ . We define  $\text{Shifts}_p$  as the set of all shifts of the op-period  $p$ .

An op-period  $p$  is called *initial* if  $0 \in \text{Shifts}_p$ , *full* if it is initial and  $p$  divides  $|S|$ , and *sliding* if  $\text{Shifts}_p = \llbracket 0..p-1 \rrbracket$ . Initial and sliding op-periods are particular cases of block-based and sliding-window-based periods for SCER, both of which were introduced in [37].

**Models of periodicity.** In the standard model, a string  $S$  of length  $n$  has a period  $p$  iff  $S[i] = S[i+p]$  for all  $i = 1, \dots, n-p$ . The famous periodicity lemma of Fine and Wilf [26] states that a “long enough” string with periods  $p$  and  $q$  has also the period  $\text{gcd}(p, q)$ . The



■ **Figure 1** The string to the left has op-period 4 with three shifts:  $\text{Shifts}_4 = \llbracket 0..0 \rrbracket \cup \llbracket 2..3 \rrbracket$ . Due to the shift 0, the string has an initial—therefore, a full—op-period 4. The string to the right has op-period 4 with all four shifts:  $\text{Shifts}_4 = \llbracket 0..3 \rrbracket$ . In particular, 4 is a sliding op-period of the string. Notice that both strings (of length  $n = 12$ ) have (general, sliding) periods 4, but none of them has the order-border (in the sense of [36]) of length  $n - 4$ .

exact bound of being “long enough” is  $p + q - \gcd(p, q)$ . This result was generalized to arbitrary number of periods [9, 31, 40].

Periods were also considered in a number of non-standard models. Partial words, which are strings with don’t care symbols, possess quite interesting Fine–Wilf type properties, including probabilistic ones; see [4, 5, 6, 38, 39, 30]. In Section 2, we make use of periodicity graphs introduced in [38, 39]. In the abelian (jumbled) model, a version of the periodicity lemma was shown in [15] and extended in [7]. Also, algorithms for computing three types of periods analogous to full, initial, and general op-periods were designed [19, 24, 25, 33, 34, 35]. In the computation of full and initial op-periods we use some number-theoretic tools initially developed in [33, 34]. Remarkably, the fastest known algorithm for computing general periods in the abelian model has essentially quadratic time complexity [19, 35], whereas for the general op-periods we design a much more efficient solution. A version of the periodicity lemma for the parameterized model was proposed in [2].

Op-periods were first considered in [37] where initial and sliding op-periods were introduced and direct generalizations of the Fine–Wilf property to these kinds of op-periods were developed. A few distinctions between the op-periods and periods in other models should be mentioned. First, “to have a period 1” becomes a trivial property in the op-model. Second, all standard periods of a string have the “sliding” property; the first string in Fig. 1 demonstrates that this is not true for op-periods. The last distinction concerns borders. A standard period  $p$  in a string  $S$  of length  $n$  corresponds to a *border* of  $S$  of length  $n - p$ , which is both a prefix and a suffix of  $S$ . In the order-preserving setting, an analogue of a border is an *op-border*, that is, a prefix that is equivalent to the suffix of the same length. Op-borders have properties similar to standard borders and can be computed in  $O(n)$  time [36]. However, it is no longer the case that a (general, initial, full, or sliding) op-period must correspond to an op-border; see [37].

**Previous algorithmic study of the op-model.** The notion of order-equivalence was introduced in [32, 36]. (However, note the related combinatorial studies, originated in [22], on containment/avoidance of shapes in permutations.) Both [32, 36] studied pattern matching in the op-model (op-pattern matching) that consists in identifying all consecutive factors of a text that are order-equivalent to a given pattern. We assume that the alphabet is integer and, as usual, that it is polynomially bounded with respect to the length of the string, which means that a string can be sorted in linear time (cf. [16]). Under this assumption, for a text of length  $n$  and a pattern of length  $m$ , [32] solve the op-pattern matching problem in  $O(n + m \log m)$  time and [36] solve it in  $O(n + m)$  time. Other op-pattern matching algorithms were presented in [3, 14].

An index for op-pattern matching based on the suffix tree was developed in [18]. For a text of length  $n$  it uses  $O(n)$  space and answers op-pattern matching queries for a pattern of

length  $m$  in optimal,  $O(m)$  time (or  $O(m + Occ)$  time if we are to report all  $Occ$  occurrences). The index can be constructed in  $O(n \log \log n)$  expected time or  $O(n \log^2 \log n / \log \log \log n)$  worst-case time. We use the index itself and some of its applications from [18].

Other developments in this area include a multiple-pattern matching algorithm for the op-model [32], an approximate version of op-pattern matching [28], compressed index constructions [12, 21], a small-space index for op-pattern matching that supports only short queries [27], and a number of practical approaches [8, 10, 11, 13, 23].

**Our results.** We give algorithms to compute:

- all full op-periods in  $O(n)$  time;
- the smallest non-trivial initial op-period in  $O(n)$  time;
- all initial op-periods in  $O(n \log \log n)$  time;
- all sliding op-periods in  $O(n \log \log n)$  expected time or  $O(n \log^2 \log n / \log \log \log n)$  worst-case time (and linear space);
- all general op-periods with all their shifts (compactly represented) in  $O(n \log n)$  time and space. The output is the family of sets  $Shifts_p$  represented as unions of disjoint intervals. The total number of intervals, over all  $p$ , is  $O(n \log n)$ .

In the combinatorial part, we characterize the Fine–Wilf periodicity property (aka interaction property) in the op-model in the case of coprime periods. This result is at the core of the linear-time algorithm for the smallest initial op-period.

**Structure of the paper.** Combinatorial foundations of our study are given in Section 2. Then in Section 3 we recall known algorithms and data structures for the op-model and develop further algorithmic tools. The remaining sections are devoted to computation of the respective types of op-periods: full and initial op-periods in Section 4, the smallest non-trivial initial op-period in Section 5, all (general) op-periods in Section 6, and sliding op-periods in Section 7. Some proofs have been omitted due to space constraints; they can be found in the preprint [29].

## 2 Fine–Wilf Property for Op-Periods

The following result was shown as Theorem 2 in [37]. Note that if  $p$  and  $q$  are coprime, then the conclusion is void, as every string has the op-period 1.

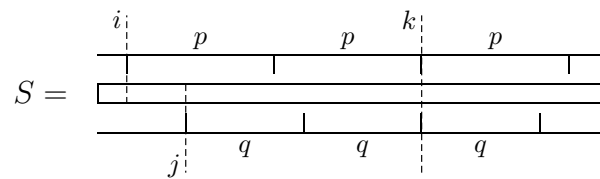
► **Theorem 3** ([37]). *Let  $p > q > 1$  and  $d = \gcd(p, q)$ . If a string  $S$  of length  $n \geq p + q - d$  has initial op-periods  $p$  and  $q$ , it has initial op-period  $d$ . Moreover, if  $S$  has length  $n \geq p + q - 1$  and sliding op-periods  $p$  and  $q$ , it has sliding op-period  $d$ .*

The aim of this section is to show a periodicity lemma in the case that  $\gcd(p, q) = 1$ .

### 2.1 Preliminary Notation

For a string  $S$  of length  $n$ , by  $S[i]$  (for  $1 \leq i \leq n$ ) we denote the  $i$ th letter of  $S$  and by  $S[i..j]$  we denote a *factor* of  $S$  equal to  $S[i] \dots S[j]$ . If  $i > j$ ,  $S[i..j]$  denotes the empty string  $\varepsilon$ .

A string which is strictly increasing, strictly decreasing, or constant, is called *strictly monotone*. A *strictly monotone op-period* of  $S$  is an op-period with a strictly monotone shape. Such an op-period is called increasing (decreasing, constant) if so is its shape. Clearly, any divisor of a strictly monotone op-period is a strictly monotone op-period as well. A string  $S$  is *2-monotone* if  $S = S_1 S_2$ , where  $S_1, S_2$  are strictly monotone in the same direction.



■ **Figure 2** Op-periods  $(p, i)$  and  $(q, j)$  synchronized at position  $k$ .

Below we assume that  $n > p > q > 1$ . Let a string  $S = S[1..n]$  have op-periods  $(p, i)$  and  $(q, j)$ . If there exists a number  $k \in \llbracket 1..n-1 \rrbracket$  such that  $k \bmod p = i$  and  $k \bmod q = j$ , we say that these op-periods are *synchronized* and  $k$  is a *synchronization point* (see Fig. 2).

► **Remark.** The proof of Theorem 3 can be easily adapted to prove the following.

► **Theorem 4.** *Let  $p > q > 1$  and  $d = \gcd(p, q)$ . If op-periods  $p$  and  $q$  of a string  $S$  of length  $n \geq p + q - 1$  are synchronized, then  $S$  has op-period  $d$ , synchronized with them.*

## 2.2 Periodicity Theorem For Coprime Periods

For a string  $S$ , by  $\text{trace}(S)$  we denote a string  $X$  of length  $|S| - 1$  over the alphabet  $\{+, 0, -\}$  such that:

$$X[i] = \begin{cases} + & \text{if } S[i] < S[i+1] \\ 0 & \text{if } S[i] = S[i+1] \\ - & \text{if } S[i] > S[i+1]. \end{cases}$$

► **Observation 5.**

- (1) *A string is strictly monotone iff its trace is a unary string.*
- (2) *If  $S$  has an op-period  $p$  with shift  $i$ , then  $\text{trace}(S)$  “almost” has a period  $p$ , namely,  $\text{trace}(S)[j] = \text{trace}(S)[k]$  for any  $j, k \in \llbracket 1..n-1 \rrbracket$  such that  $j = k \pmod p$  and  $j \neq i \pmod p$ . (This is because both  $\text{trace}(S)[j]$  and  $\text{trace}(S)[k]$  equal the sign of the difference between the same positions of the shape of the op-period of  $S$ .)*

► **Example 6.** Consider the string 758146245. It has an op-period  $(3, 1)$  with shape 231. The trace of this string is:

- + - + + - + +

The positions giving the remainder 1 modulo 3 are shown in gray; the sequence of the remaining positions is periodic.

It turns out that the existence of two coprime op-periods makes a string “almost” strictly monotone. One can use periodicity graphs [38, 39] to show the following result.

► **Theorem 7.** *Let  $S$  be a string of length  $n$  that has coprime op-periods  $p$  and  $q$  with shifts  $i$  and  $j$ , respectively, such that  $n > p > q > 1$ . Then:*

- (a) *if  $n > pq$ , then  $S$  has a strictly monotone op-period  $pq$ ;*
- (b) *if  $2p < n \leq pq$  and the op-periods are synchronized, then  $S$  is 2-monotone;*
- (c) *if  $p+q < n \leq 2p$  and the op-periods are synchronized, then  $(q, j)$  is a strictly monotone op-period of  $S$ ;*
- (d) *if  $n > \max\{2p, p+2q\}$  and the op-periods are not synchronized, then  $S$  is strictly monotone;*
- (e) *if  $n > 2p$ , the op-periods are not synchronized, and  $p$  is initial, then  $S$  is strictly monotone;*
- (f) *if  $p+q < n \leq 2p$  and  $p$  is initial, then  $(q, j)$  is a strictly monotone op-period of  $S$ .*

### 3 Algorithmic Toolbox for Op-Model

For a string  $S$  of length  $n$ , we introduce a table  $\text{op-PREF}[1..n]$  such that  $\text{op-PREF}[i]$  is the length of the longest prefix of  $S[i..n]$  that is equivalent to a prefix of  $S$ . It is a direct analogue of the  $\text{PREF}$  array used in standard string matching (see [20]) and can be computed similarly in  $O(n)$  time using one of the standard encodings for the op-model that were used in [14, 18, 36].

► **Lemma 8.** *For a string of length  $n$ , the op-PREF table can be computed in  $O(n)$  time.*

Let us mention an application of the op-PREF table that is used further in the algorithms. We denote by  $\text{op-LPP}_p(S)$  (“longest op-periodic prefix”) the length of the longest prefix of a string  $S$  having  $p$  as an initial op-period.

► **Lemma 9.** *For a string  $S$  of length  $n$ ,  $\text{op-LPP}_p(S)$  for a given  $p$  can be computed in  $O(\text{op-LPP}_p(S)/p + 1)$  time after  $O(n)$ -time preprocessing.*

**Proof.** We start by computing the op-PREF table for  $S$  in  $O(n)$  time. We assume that  $\text{op-PREF}[n+1] = 0$ . To compute  $\text{op-LPP}_p(S)$ , we iterate over positions  $i = p+1, 2p+1, \dots$  and for each of them check if  $\text{op-PREF}[i] \geq p$ . If  $i_0$  is the first position for which this condition is not satisfied (possibly because  $i_0 > n-p+1$ ), we have  $\text{op-LPP}_p(S) = i_0 + \text{op-PREF}[i_0] - 1$ . Clearly, this procedure works in the desired time complexity. ◀

For a string  $S$ , we define a *longest common extension* query  $\text{op-LCP}(i, j)$  in the order-preserving model as the maximum  $k \geq 0$  such that  $S[i..i+k-1] \approx S[j..j+k-1]$ . Symmetrically,  $\text{op-LCS}(i, j)$  is the maximum  $k \geq 0$  such that  $S[i-k+1..i] \approx S[j-k+1..j]$ .

Similarly as in the standard model [17], LCP-queries in the op-model can be answered using lowest common ancestor (LCA) queries in the op-suffix tree; see the following lemma.

► **Lemma 10.** *For a string of length  $n$ , after preprocessing in  $O(n \log \log n)$  expected time or in  $O(n \log^2 \log n / \log \log \log n)$  worst-case time one can answer op-LCP-queries in  $O(1)$  time.*

The factor  $S[i..i+2p-1]$  is called an order-preserving square (*op-square*) iff  $S[i..i+p-1] \approx S[i+p..i+2p-1]$ . For a string  $S$  of length  $n$ , we define the set

$$\text{op-Squares}_p = \{i \in [1..n-2p+1] : S[i..i+2p-1] \text{ is an op-square}\}.$$

Op-squares were first defined in [18] where an algorithm computing all the sets  $\text{op-Squares}_p$  for a string of length  $n$  in  $O(n \log n + \sum_p |\text{op-Squares}_p|)$  time was shown.

We say that an op-square  $S[i..i+2p-1]$  is *right shiftable* if  $S[i+1..i+2p]$  is an op-square and *right non-shiftable* otherwise. Similarly, we say that the op-square is *left shiftable* if  $S[i-1..i+2p-2]$  is an op-square and *left non-shiftable* otherwise. Using the approach of [18], one can show the following lemma.

► **Lemma 11.** *All the (left and right) non-shiftable op-squares in a string of length  $n$  can be computed in  $O(n \log n)$  time.*

### 4 Computing All Full and Initial Op-Periods

For a string  $S$  of length  $n$ , we define  $\text{op-PREF}'[i]$  for  $i = 0, \dots, n$  as:

$$\text{op-PREF}'[i] = \begin{cases} n & \text{if } \text{op-PREF}[i+1] = n-i \\ \text{op-PREF}[i+1] & \text{otherwise.} \end{cases}$$

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**Algorithm 1:** Computing All Initial Op-Periods of  $S$ .
 

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```

1  $T := \text{op-PREF}'$ ;
2 for  $j := n$  down to 2 do
3   foreach prime divisor  $q$  of  $j$  do
4      $P[j/q] := \min(P[j/q], P[j])$ ;
5 for  $p := 1$  to  $n$  do
6   if  $P[p] \geq p$  then  $p$  is an initial op-period;

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Here we assume that  $\text{op-PREF}[n+1] = 0$ . In the computation of full and initial op-periods we heavily rely on this table according to the following obvious observation.

► **Observation 12.**  $p$  is an initial op-period of a string  $S$  of length  $n$  if and only if  $\text{op-PREF}'[ip] \geq p$  for all  $i = 1, \dots, \lfloor n/p \rfloor$ .

## 4.1 Computing Initial Op-Periods

Let us introduce an auxiliary array  $P[0..n]$  such that:

$$P[p] = \min\{\text{op-PREF}'[ip] : i = 1, \dots, \lfloor n/p \rfloor\}.$$

Straight from Observation 12 we have:

► **Observation 13.**  $p$  is an initial period of  $S$  if and only if  $P[p] \geq p$ .

The table  $T$  could be computed straight from definition in  $O(n \log n)$  time. We improve this complexity to  $O(n \log \log n)$  by employing Eratosthenes's sieve. The sieve computes, in particular, for each  $j = 1, \dots, n$  a list of all distinct prime divisors of  $j$ . We use these divisors to compute the table via dynamic programming in a right-to-left scan, as shown in Algorithm 1.

► **Theorem 14.** All initial op-periods of a string of length  $n$  can be computed in  $O(n \log \log n)$  time.

**Proof.** By Lemma 8, the op-PREF table for the string—hence, the op-PREF' table—can be computed in  $O(n)$  time. Then we use Algorithm 1. Each prime number  $q \leq n$  has at most  $\frac{n}{q}$  multiples below  $n$ . Therefore, the complexity of Eratosthenes's sieve and the number of updates on the table  $T$  in the algorithm is  $\sum_{q \in \text{Primes}, q \leq n} \frac{n}{q} = O(n \log \log n)$ ; see [1]. ◀

## 4.2 Computing Full Op-Periods

Let us recall the following auxiliary data structure for efficient gcd-computations that was developed in [34]. We will only need a special case of this data structure to answer queries for  $\text{gcd}(x, n)$ .

► **Fact 15** (Theorem 4 in [34]). After  $O(n)$ -time preprocessing, given any  $x, y \in \{1, \dots, n\}$ , the value  $\text{gcd}(x, y)$  can be computed in constant time.

Let  $\text{Div}(i)$  denote the set of all positive divisors of  $i$ . In the case of full op-periods we only need to compute  $P[p]$  for  $p \in \text{Div}(n)$ . As in Algorithm 1, we start with  $T = \text{op-PREF}'$ . Then we perform a preprocessing phase that shifts the information stored in the array from

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**Algorithm 2:** Computing All Full Op-Periods of  $S$ .
 

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```

1  $T := \text{op-PREF}'$ ;
2 for  $i := 1$  to  $n$  do
3    $k := \text{gcd}(i, n)$ ;
4    $P[k] := \min(P[k], P[i])$ ;
5 foreach  $i \in \text{Div}(n)$  in decreasing order do
6   foreach  $d \in \text{Div}(i)$  do
7      $P[d] := \min(P[d], P[i])$ ;
8 foreach  $p \in \text{Div}(n)$  do
9   if  $P[p] \geq p$  then  $p$  is a full op-period;

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indices  $i \notin \text{Div}(n)$  to indices  $\text{gcd}(i, n) \in \text{Div}(n)$ . It is based on the fact that for  $d \in \text{Div}(n)$ ,  $d \mid i$  if and only if  $d \mid \text{gcd}(i, n)$ . Finally, we perform right-to-left processing as in Algorithm 1. However, this time we can afford to iterate over all divisors of elements from  $\text{Div}(n)$ . Thus we arrive at the pseudocode of Algorithm 2.

► **Theorem 16.** *All full op-periods of a string of length  $n$  can be computed in  $O(n)$  time.*

**Proof.** We apply Algorithm 2. The complexity of the first for-loop is  $O(n)$  by Fact 15. The second for-loop works in  $O(n)$  time as the sizes of the sets  $\text{Div}(n)$ ,  $\text{Div}(i)$  are  $O(\sqrt{n})$  and the elements of these sets can be enumerated in  $O(\sqrt{n})$  time as well. ◀

## 5 Computing Smallest Non-Trivial Initial Op-Period

If a string is not strictly monotone itself, it has  $O(n)$  such op-periods and they can all be computed in  $O(n)$  time. We use this as an auxiliary routine in the computation of the smallest initial op-period that is greater than 1.

► **Theorem 17.** *If a string of length  $n$  is not strictly monotone, all of its strictly monotone op-periods can be computed in  $O(n)$  time.*

Let us start with the following simple property.

► **Lemma 18.** *The shape of the smallest non-trivial initial op-period of a string has no shorter non-trivial full op-period.*

**Proof.** A full op-period of the initial op-period of a string  $S$  is an initial op-period of  $S$ . ◀

Now we can state a property of initial op-periods, implied by Theorem 7, that is the basis of the algorithm.

► **Lemma 19.** *If a string of length  $n$  has initial op-periods  $p > q > 1$  such that  $p + q < n$  and  $\text{gcd}(p, q) = 1$ , then  $q$  is strictly monotone.*

**Proof.** Let us consider three cases. If  $n > pq$ , then by Theorem 7(a), both  $p$  and  $q$  are strictly monotone. If  $2p < n \leq pq$ , then Theorem 7(e) implies that  $S[1..pq - 1]$  is strictly monotone, hence  $p$  and  $q$  are strictly monotone as well. Finally, if  $p + q < n \leq 2p$ , we have that  $q$  is strictly monotone by Theorem 7(f). ◀



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**Algorithm 3:** Computing the Smallest Non-Trivial Initial Op-Period of  $S$ .
 

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1 if  $S$  has a non-trivial strictly monotone op-period then
2   return smallest such op-period; ▷ Theorem 17
3  $p :=$  the length of the longest monotone prefix of  $S$  plus 1;
4 while  $p \leq n$  do
5    $k :=$  op-LPP $_p(S)$ ;
6   if  $k = n$  then return  $p$ ;
7    $p := \max(p + 1, k - p - 1)$ ;
8 return  $\min(p_{\text{mon}}, n)$ ;
```

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► **Theorem 20.** *The smallest initial op-period  $p > 1$  of a string  $S$  of length  $n$  can be computed in  $O(n)$  time.*

**Proof.** We follow the lines of Algorithm 3. If  $S$  is not strictly monotone itself, we can compute the smallest non-trivial strictly monotone initial op-period of  $S$  using Theorem 17. Otherwise, the smallest such op-period is 2. If  $S$  has a non-trivial strictly monotone initial op-period and the smallest such op-period is  $q > 1$ , then none of  $2, \dots, q - 1$  is an initial op-period of  $S$ . Hence, we can safely return  $q$ .

Let us now focus on the correctness of the while-loop. The invariant is that there is no initial op-period of  $S$  that is smaller than  $p$ . If the value of  $k = \text{op-LPP}_p(S)$  equals  $n$ , then  $p$  is an initial op-period of  $S$  and we can safely return it. Otherwise, we can advance  $p$  by 1. There is also no smallest initial op-period  $p'$  such that  $p < p' < k - p - 1$ . Indeed, Lemma 19 would imply that  $p$  is strictly monotone if  $\gcd(p, p') = 1$  (which is impossible due to the initial selection of  $p$ ) and Theorem 3 would imply an initial op-period of  $S[1..p']$  that is smaller than  $p'$  and divides  $p'$  if  $\gcd(p, p') > 1$  (which is impossible due to Lemma 18). This justifies the way  $p$  is increased.

Now let us consider the time complexity of the algorithm. The algorithm for strictly monotone op-periods of Theorem 17 works in  $O(n)$  time. By Lemma 9,  $k$  can be computed in  $O(k/p + 1)$  time. If  $k \leq 3p$ , this is  $O(1)$ . Otherwise,  $p$  at least doubles; let  $p'$  be the new value of  $p$ . Then  $O(k/p + 1) = O((p + p' - 1)/p + 1) = O(p' + 1)$ . The case that  $p$  doubles can take place at most  $O(\log n)$  times and the total sum of  $p'$  over such cases is  $O(n)$ . ◀

## 6 Computing All Op-Periods

An *interval representation* of a set  $X$  of integers is  $X = \llbracket i_1..j_1 \rrbracket \cup \llbracket i_2..j_2 \rrbracket \cup \dots \cup \llbracket i_k..j_k \rrbracket$  where  $j_1 + 1 < i_2, \dots, j_{k-1} + 1 < i_k$ ;  $k$  is called the *size* of the representation.

Our goal is to compute a *compact representation* of all the op-periods of a string that contains, for each op-period  $p$ , an interval representation of the set  $\text{Shifts}_p$ .

For an integer set  $X$ , by  $X \bmod p$  we denote the set  $\{x \bmod p : x \in X\}$ . The following technical lemma provides efficient operations on interval representations of sets.

► **Lemma 21.**

- (a) *Assume that  $X$  and  $Y$  are two sets with interval representations of sizes  $x$  and  $y$ , respectively. Then the interval representation of the set  $X \cap Y$  can be computed in  $O(x + y)$  time.*
- (b) *Assume that  $X_1, \dots, X_k \subseteq \llbracket 0..n \rrbracket$  are sets with interval representations of sizes  $x_1, \dots, x_k$  and  $p_1, \dots, p_k$  be positive integers. Then the interval representations of all the sets  $X_1 \bmod p_1, \dots, X_k \bmod p_k$  can be computed in  $O(x_1 + \dots + x_k + k + n)$  time.*

**Algorithm 4:** Computing a Compact Representation of All Op-Periods.

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1 Compute  $op\text{-Squares}_p$  for all  $p = 1, \dots, n$ ; ▷ Lemma 22
2 for  $p := 1$  to  $n$  do
3    $\mathcal{N}_p := \llbracket 1..n - 2p + 1 \rrbracket \setminus op\text{-Squares}_p$ ;
4    $k := \text{op-LCP}(1, p + 1)$ ;  $\ell := \text{op-LCS}(n, n - p)$ ;
5   if  $k = n - p$  then  $\mathcal{B}_p := \mathcal{C}_p := \llbracket 1..n \rrbracket$ ;
6   else  $\mathcal{B}_p := \llbracket 1..k \rrbracket$ ;  $\mathcal{C}_p := \llbracket n - \ell + 1..n \rrbracket$ ;
7 for  $p := 1$  to  $n$  simultaneously do
8    $\mathcal{N}_p := \{(x - 1) \bmod p : x \in \mathcal{N}_p\}$ ;  $\mathcal{B}_p := \mathcal{B}_p \bmod p$ ;  $\mathcal{C}_p := \mathcal{C}_p \bmod p$ ; ▷ Lemma 21(b)
9    $Shifts_1 := \llbracket 0 \rrbracket$ ;
10 for  $p := 2$  to  $n$  do
11    $\mathcal{A}_p := \llbracket 0..p - 1 \rrbracket \setminus \mathcal{N}_p$ ;
12    $Shifts_p := \mathcal{A}_p \cap \mathcal{B}_p \cap \mathcal{C}_p$ ; ▷ Lemma 21(a)
13 return  $Shifts_p$  for  $p = 1, \dots, n$ ;

```

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► **Lemma 22.** For a string of length  $n$ , interval representations of the sets  $op\text{-Squares}_p$  for all  $1 \leq p \leq n/2$  can be computed in  $O(n \log n)$  time.

**Proof.** Let us define the following two auxiliary sets.

$$\begin{aligned} \mathcal{L}_p &= \{i \in \llbracket 1..n - 2p + 1 \rrbracket : S[i..i + 2p - 1] \text{ is a left non-shiftable op-square}\} \\ \mathcal{R}_p &= \{i \in \llbracket 1..n - 2p + 1 \rrbracket : S[i..i + 2p - 1] \text{ is a right non-shiftable op-square}\}. \end{aligned}$$

By Lemma 11, all the sets  $\mathcal{L}_p$  and  $\mathcal{R}_p$  can be computed in  $O(n \log n)$  time. In particular,  $\sum_p |\mathcal{L}_p| = O(n \log n)$ .

Let us note that, for each  $p$ ,  $|\mathcal{L}_p| = |\mathcal{R}_p|$ . Thus let  $\mathcal{L}_p = \{\ell_1, \dots, \ell_k\}$  and  $\mathcal{R}_p = \{r_1, \dots, r_k\}$ . The interval representation of the set  $op\text{-Squares}_p$  is  $\llbracket \ell_1..r_1 \rrbracket \cup \dots \cup \llbracket \ell_k..r_k \rrbracket$ . Clearly, it can be computed in  $O(|\mathcal{L}_p|)$  time. ◀

We will use the following characterization of op-periods.

► **Observation 23.**  $p$  is an op-period of  $S$  with shift  $i$  if and only if all the following conditions hold:

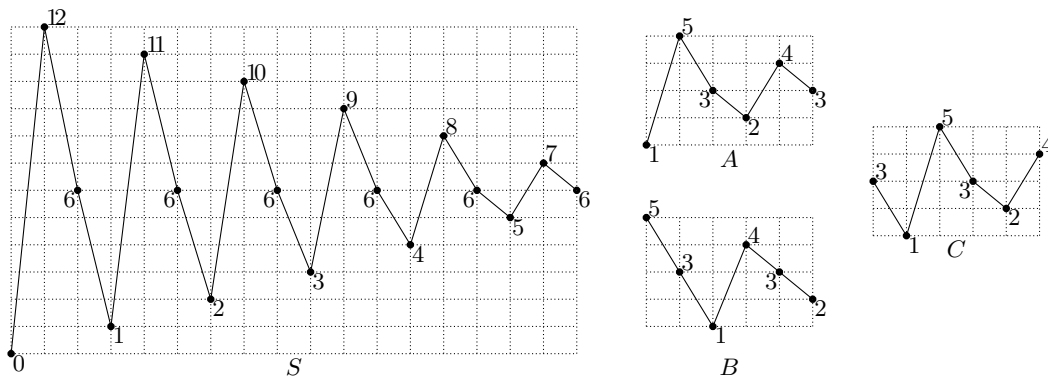
- (A)  $S[i + 1 + kp..i + (k + 2)p]$  is an op-square for every  $0 \leq k \leq (n - 2p - i)/p$ ,
- (B)  $\text{op-LCP}(1, p + 1) \geq \min(i, n - p)$ ,
- (C)  $\text{op-LCS}(n, n - p) \geq \min((n - i) \bmod p, n - p)$ .

► **Theorem 24.** A representation of size  $O(n \log n)$  of all the op-periods of a string of length  $n$  can be computed in  $O(n \log n)$  time.

**Proof.** We use Algorithm 4. The sets  $\mathcal{A}_p$ ,  $\mathcal{B}_p$ , and  $\mathcal{C}_p$  describe the sets of shifts  $i$  that satisfy conditions (A), (B), and (C) from Observation 23, respectively.

A crucial role is played by the set  $\mathcal{N}_p$  of all positions which are *not* the beginnings of op-squares of length  $2p$ . It is computed as a complement of the set  $op\text{-Squares}_p$ .

Operations “mod” on sets are performed simultaneously using Lemma 21(b). All sets  $\mathcal{A}_p$ ,  $\mathcal{B}_p$ ,  $\mathcal{C}_p$  have  $O(n \log n)$ -sized representations. This guarantees  $O(n \log n)$  time. ◀



■ **Figure 3** A string  $S = 0\ 12\ 6\ 1\ 11\ 6\ 2\ 10\ 6\ 3\ 9\ 6\ 4\ 8\ 6\ 5\ 7\ 6$  is graphically illustrated above (the  $i$ th point has coordinates  $(i, S[i])$ ). We have  $SH_6 = ABCABCABCA$ , where  $A = 1\ 5\ 3\ 2\ 4\ 3$ ,  $B = 5\ 3\ 1\ 4\ 3\ 2$ , and  $C = 3\ 1\ 5\ 3\ 2\ 4$ . The shortest period of  $SH_6$  is 3. Hence, 6 is a sliding op-period of  $S$ . Moreover, Lemma 27b implies that 3 is a period of  $SH_3$ , hence a sliding op-period of  $S$ .

## 7 Computing Sliding Op-Periods

For a string  $S$  of length  $n$ , we define a family of strings  $SH_1, \dots, SH_n$  such that  $SH_k[i] = \text{shape}(S[i..i+k-1])$  for  $1 \leq i \leq n-k+1$ . Note that the characters of the strings are shapes. Moreover, the total length of strings  $SH_k$  is quadratic in  $n$ , so we will not compute those strings explicitly. Instead, we use the following observation to test if two symbols are equal.

► **Observation 25.**  $SH_k[i] = SH_k[i']$  if and only if  $\text{op-LCP}(i, i') \geq k$ .

Sliding op-periods admit an elegant characterization based on  $SH_k$ ; see Figure 3.

► **Lemma 26.** An integer  $p$ ,  $1 \leq p \leq n$ , is a sliding op-period of  $S$  if and only if  $p \leq \frac{1}{2}n$  and  $p$  is a period of  $SH_p$ , or  $p > \frac{1}{2}n$  and  $S[1..n-p] \approx S[p+1..n]$ .

For a string  $X$ , we denote the shortest period of  $X$  by  $\text{per}(X)$ .

► **Lemma 27.** Suppose that  $p = \text{per}(SH_k[1..\ell]) < \ell$ . Then

- (a)  $p$  is also a period of  $SH_{k'}[1..\ell+k-k']$  for  $1 \leq k' \leq k$ ,
- (b)  $q = \text{per}(SH_k[1..\ell+1])$  satisfies  $p = q$  or  $p+q > \ell$ .

We introduce a two-dimensional table  $PER$ , where:

$$PER[k, \ell] = \text{per}(SH_k[1..\ell]) \text{ if } \text{per}(SH_k[1..\ell]) \leq \frac{1}{3}\ell, \text{ and } PER[k, \ell] = \perp (\text{undefined}) \text{ otherwise.}$$

The size of  $PER$  is quadratic in  $n$ . However, Algorithm 5 computes  $PER$  column after column, keeping only the current column  $P = PER[\cdot, \ell]$ . The total number of differences between consecutive columns is linear. Hence, any requested  $O(n)$  values  $PER[k, \ell]$  can be computed in  $O(n)$  time. We also use an analogous table  $PER^R$  for the reverse string  $S^R$ .

► **Lemma 28.** Algorithm 5 is correct, that is, it satisfies the invariant.

**Proof.** First, observe that the invariant is satisfied after the first iteration. This is because  $\text{per}(SH_k[1..1]) = 1$  for each  $k$  and the initial values are not changed during this iteration.

Thus, our task is to prove that the invariant is preserved after each subsequent  $\ell$ th iteration. Let  $t = \min\{k : PER[k, \ell-1] = \perp\}$  and  $t' = \min\{k : PER[k, \ell] = \perp\}$ .

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**Algorithm 5:** Computation of  $PER[\cdot, \ell]$  from  $PER[\cdot, \ell - 1]$ .

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```

1  $P[1..n] := [\perp, \dots, \perp]; t := 1; \ell' := 3;$ 
2 for  $\ell := 1$  to  $n$  do
3   if  $t > 1$  and  $SH_{t-1}[\ell] \neq SH_{t-1}[\ell - P[t - 1]]$  then
4      $t := t - 1; P[t] := \perp; \ell' := 2\ell;$ 
5   if  $\ell \geq \ell'$  then
6     while  $\text{per}(SH_t[1..\ell]) = \frac{1}{3}\ell$  do
7        $P[t] := \frac{1}{3}\ell; t := t + 1; \ell' := 2\ell;$ 
       $\triangleright$  Invariant:  $P[k] = PER[k, \ell]$ ,  $t = \min\{k : P[k] = \perp\}$ , and  $\text{per}(SH_t[1..\ell]) \geq \frac{1}{3}\ell'$ .

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**Algorithm 6:** Computing the sliding op-periods  $p \leq \frac{1}{2}n$ .

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1  $p := 1;$ 
2 while  $p \leq \frac{1}{2}n$  do
3   if  $(q := PER[p, n - 2p + 1]) = PER^R[p, n - 2p + 1] \neq \perp$  then
4     if  $p$  is a period of  $SH_p[1..p + q]$  then report  $p;$ 
5      $p := \min\{p' > p : p' \text{ is a period of } SH_p[1..p + 2q]\}$ 
6   else if  $PER[p, \lceil \frac{3}{4}(n - 2p + 1) \rceil] = PER^R[p, \lceil \frac{3}{4}(n - 2p + 1) \rceil] \neq \perp$  then  $p := p + 1;$ 
7   else
8     if  $p$  is a period of  $SH_p$  then report  $p;$ 
9      $p := \min\{p' > p : p' \text{ is a period of } SH_p\};$ 

```

---

First, we consider the values  $PER[k, \ell]$  for  $k < t$ . For this, we assume  $t > 1$  and denote  $p = PER[t - 1, \ell - 1]$ . Since  $p$  is a period of  $SH_{t-1}[1..\ell - 1]$ , Lemma 27a yields that  $p$  is also a period of  $SH_k[1..\ell]$  for  $k < t - 1$ . We apply Lemma 27b for  $p' = \text{per}(SH_k[1..\ell - 1])$ . Since  $p' + p \leq \ell - 1$ , we conclude that  $p' = \text{per}(SH_k[1..\ell])$ , i.e.,  $PER[k, \ell - 1] = p' = PER[k, \ell]$ . Now, we consider the value  $PER[t - 1, \ell]$ . Lemma 27b, applied for  $p = \text{per}(SH_{t-1}[1..\ell - 1])$  and  $q = \text{per}(SH_{t-1}[1..\ell])$ , yields  $p = q$  or  $p + q \geq \ell$ . To verify the first case, we check whether  $SH_{t-1}[\ell] = SH_{t-1}[\ell - p]$ . In the second case, we conclude that  $q \geq \frac{2}{3}\ell$ , so  $PER[t - 1, \ell] = \perp$  (and  $\ell' := 2\ell$  is also set correctly).

Next, we consider the values  $PER[k, \ell]$  for  $k \geq t$ . Since  $PER[k, \ell - 1] = \perp$ , we have  $PER[k, \ell] = \perp$  or  $PER[k, \ell] = \frac{1}{3}\ell$ . More precisely,  $PER[k, \ell] = \perp$  for  $k \geq t'$  and  $PER[k, \ell] = \frac{1}{3}\ell$  for  $t \leq k < t'$ . Thus, we check if  $\text{per}(SH_k[1..\ell]) = \frac{1}{3}\ell$  for subsequent values  $k \geq t$ . Since  $\text{per}(SH_t[1..\ell]) \geq \frac{1}{3}\ell'$ , no verification is needed if  $\ell < \ell'$ . To complete the proof, we need to show that the update  $\ell' := 2\ell$  is valid if  $t' > t$ . For a proof by contradiction suppose that  $r := \text{per}(SH_{t'}[1..\ell]) < \frac{2}{3}\ell$ . By Lemma 27a,  $r$  is a period of  $SH_t[1..\ell]$ . Since  $r + \frac{1}{3}\ell \leq \ell$ , Periodicity Lemma yields  $\frac{1}{3}\ell \mid r$ , and thus  $r = \frac{1}{3}\ell$ , which contradicts the definition of  $t'$ .  $\blacktriangleleft$

► **Lemma 29.** *Algorithm 5 can be implemented in time  $O(n)$  plus the time to answer  $O(n)$  op-LCP queries in  $S$ .*

► **Lemma 30.** *Algorithm 6 is correct, that is, it reports all sliding op-periods  $p \leq \frac{1}{2}n$  of  $S$ .*

**Proof.** Let  $p_i$  be the value of  $p$  at the beginning of the  $i$ th iteration of the while-loop and let  $\ell_i = n - 2p_i + 1$ . We shall prove that  $p_i$  is reported if and only if it is a sliding op-period and that there is no sliding op-period strictly between  $p_i$  and  $p_{i+1}$ .

First, suppose that  $q = \text{per}(SH_{p_i}[1..\ell_i]) = \text{per}(SH_{p_i}[p_i + 1..p_i + \ell_i]) \leq \frac{1}{3}\ell_i$ , i.e., we are in the first branch. If  $SH_{p_i}[1..q] = SH_{p_i}[p_i + 1..p_i + q]$ , then we must have  $SH_{p_i}[1..\ell_i] =$

$SH_{p_i}[p_i + 1..p_i + \ell_i]$ , i.e.,  $p_i$  is a period of  $SH_{p_i} = SH_{p_i}[1..p_i + \ell_i]$  and  $p_i$  is a sliding op-period due to Lemma 26. Moreover, any sliding op-period  $p' > p_i$  must be a period of  $SH_{p_i}$  (and, in particular, of  $SH_{p_i}[1..p_i + 2q]$ ) due to Lemma 27a. Consequently,  $p' \geq p_{i+1}$ , as claimed.

In the second branch we only need to prove that  $SH_{p_i}[1..\ell_i] \neq SH_{p_i}[p_i + 1..p_i + \ell_i]$ . For a proof by contradiction, suppose that we have an equality. The condition from Line 6 means that the length- $\lceil \frac{3}{4}\ell_i \rceil$  prefix and suffix of  $SH_{p_i}[1..\ell_i] = SH_{p_i}[p_i + 1..p_i + \ell_i]$  has the common shortest period  $q \leq \frac{1}{3}\lceil \frac{3}{4}\ell_i \rceil \leq \lceil \frac{1}{4}\ell_i \rceil$ . The prefix and the suffix overlap by at least  $\lceil \frac{1}{2}\ell_i \rceil$  characters, so we actually have  $q = \text{per}(SH_{p_i}[1..\ell_i]) = \text{per}(SH_{p_i}[p_i + 1..p_i + \ell_i])$ . Hence, in that case we would be in the first branch.

Finally, in the third branch we directly use Lemma 26 to check if  $p_i$  is a sliding op-period. Moreover, if  $p' > p_i$  is also a sliding op-period, then  $p'$  is a period of  $SH_{p_i}$ , i.e.,  $p' \geq p_{i+1}$ . ◀

Let us observe that  $PER[k, \ell]$  and  $PER^R[k, \ell]$  is used in Algorithm 6 only for  $\ell = n - 2k + 1$  or  $\ell = \lceil \frac{3}{4}(n - 2k + 1) \rceil$ . These  $O(n)$  values can be computed in  $O(n)$  time using Algorithm 5. In [29] we show the following lemma.

► **Lemma 31.** *Algorithm 6 can be implemented in time  $O(n)$  plus the time to answer  $O(n)$  op-LCP and op-LCS queries in  $S$ .*

► **Theorem 32.** *All sliding op-periods of a string of length  $n$  can be computed in  $O(n)$  space and  $O(n \log \log n)$  expected time or  $O(n \log^2 \log n / \log \log \log n)$  worst-case time.*

**Proof.** First, we apply Lemma 10 so that op-LCP and op-LCS queries can be answered in  $O(1)$  time. Next, we run Algorithm 6 to report sliding op-periods  $p \leq \frac{1}{2}n$ . Then, we iterate over  $p > \frac{1}{2}n$  and report  $p$  if  $\text{op-LCP}(1, p + 1) = n - p$ . Correctness follows from Lemmas 30 and 26. The overall time is  $O(n)$  (Lemma 31) plus the preprocessing time of Lemma 10. ◀

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