STRONGLY SEQUENTIALLY SEPARABLE FUNCTION SPACES, VIA SELECTION PRINCIPLES

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Abstract. A separable space is *strongly sequentially separable* if, for each countable dense set, every point in the space is a limit of a sequence from the dense set. We consider this and related properties, for the spaces of continous and Borel real-valued functions on Tychonoff spaces, with the topology of pointwise convergence. Our results solve a problem stated by Gartside, Lo, and Marsh.

1. Introduction

We apply methods of selection principles to a problem of Gartside, Lo, and Marsh [6, Problem 19].

By *space* we mean a Tychonoff topological space. A space is *Fréchet–Urysohn* if each point in the closure of a set is a limit of a sequence from the set. A separable space is *strongly sequentially separable (SSS)* [9] if, for each countable dense set, every point in the space is a limit of a sequence from the dense set. Every separable Fréchet–Urysohn space is strongly sequentially separable, but not conversely [1, Example 2.4].

For a space X, let $\mathrm{C}(X)$ and $\mathrm{B}(X)$ be the spaces of continuous and Borel, respectively, real-valued functions on X, with the topology of pointwise convergence. We are only concerned with uncountable spaces. In this case, the space $\mathrm{B}(X)$ is never Fréchet–Urysohn. Indeed, for an uncountable space, the constant function $\mathbf{1}$ is in the closure of the set of characteristic functions of finite subsets of the space and there is no sequence in the set converging to $\mathbf{1}$. Strong sequential separability is hereditary for separable dense subspaces. Thus, if the space $\mathrm{C}(X)$ is separable, we have the following implications.

$$\begin{tikzcd} \mathbb{R}^X \text{ is SSS} \arrow[r] & \mathrm{B}(X) \text{ is SSS} \arrow[r] & \mathrm{C}(X) \text{ is SSS} \\ & & \mathrm{C}(X) \text{ is Fréchet–Urysohn} \arrow[u] \end{tikzcd}$$

It is consistent that the properties in this diagram hold only for countable spaces X and are, thus, equivalent [6, Corollary 17]. This motivates the following problem [6, Problem 19].

Problem 1 (Gartside–Lo–Marsh)**.** *Is there, consistently, a space X such that the space $\mathrm{C}(X)$ is strongly sequentially separable but not Fréchet–Urysohn, and the space $\mathbb{R}^X$ is not strongly sequentially separable?*

We solve this problem, and all other problems suggested by the above diagram. To this end, we extend Arhangel'skiĭ's local-to-global duality, dualize these problems to ones concerning covering properties, and apply the theory of selection principles.

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2. Local-to-global Duality

A *cover* of a space is a family of *proper* subsets whose union is the entire space. For families $\mathbf{A}$ and $\mathbf{B}$ of covers of a space, the property that every cover in the family $\mathbf{A}$ has a subcover in the family $\mathbf{B}$ is denoted $\binom{\mathbf{A}}{\mathbf{B}}$. An *$\omega$-cover* is a cover such that each finite subset of the space is contained in some set from the cover. A *γ-cover* is an infinite cover such that each point of the space belongs to all but finitely many sets from the cover.

An *open cover* is a cover by open sets. Similarly, we define *Borel cover*, *clopen cover*, etc. Given a space, let Ω, Ω_{ctbl}, Ω_{coz}, Ω_{Bor} and Γ, be the families of *open* ω-covers, *countable open* ω-covers, *countable cozero* ω-covers, *countable Borel* ω-covers, and γ-covers, respectively.

The property $\binom{\Omega}{\Gamma}$ is the celebrated γ-property of Gerlits and Nagy, who proved that a space has this property if and only if the space $\mathrm{C}(X)$ is Fréchet–Urysohn [7, Theorem 2].

Lemma 2. *Let X be a space with a coarser second countable topology. The following assertions are equivalent:*

1. *The space $\mathrm{B}(X)$ is strongly sequentially separable.*
2. *The space X has the property $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.*

Proof. (1) $\Rightarrow$ (2): Since the space X has a coarser second countable topology, there is a countable dense set H in the space $\mathrm{C}(X)$ [12, Theorem 1] and hence H is dense in the space $\mathrm{B}(X)$. Let $\mathcal{U} \in \Omega_{\mathrm{Bor}}(X)$. For a Borel set $U \subseteq X$ and a function $h \in H$, let $f_{U,h} \in \mathrm{B}(X)$ be the function such that $f_{U,h} \restriction U := h \restriction U$ and $f_{U,h} \restriction (X \setminus U) := 1$. The set $D := \{ f_{U,h} : U \in \mathcal{U}, h \in H \}$ is a countable dense subset of $\mathrm{B}(X)$. By (1), there is a sequence $\{ f_{U_n,h_n} : n \in \mathbb{N} \}$ in the set D, converging to the zero function $\mathbf{0}$. Let F be a finite subset of X. The set $W := \{ f \in \mathrm{B}(X) : f[F] \subseteq (-1,1) \}$ is a neighborhood of $\mathbf{0}$ in $\mathrm{B}(X)$. For a natural number n, if $f_{U_n,h_n} \in W$, then $F \subseteq U_n$. Since all but finitely many elements of the sequence belong to the set W, we have $\{ U_n : n \in \mathbb{N} \} \in \Gamma(X)$. Thus, the space X satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.

(2) $\Rightarrow$ (1): The property $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ implies that every point in the closure of a countable set in $\mathrm{B}(X)$ is the limit of a sequence from that set [13, Lemma 2.8]. $\square$

Let $\mathbb{N}$ be the set of natural numbers. For infinite sets $a, b \subseteq \mathbb{N}$ we write $a \subseteq^* b$ if the set $a \setminus b$ is finite. A *pseudointersection* of a family of infinite sets is an infinite set a with $a \subseteq^* b$ for all sets b in the family. A subfamily of $[\mathbb{N}]^\infty$ is *centered* if the finite intersections of its elements, are infinite. Let $\mathfrak{p}$ be the minimal cardinality of a family of infinite subsets of $\mathbb{N}$ that is centered and has no pseudointersection. The hypothesis $\aleph_1 < \mathfrak{p}$ and its negation ($\aleph_1 = \mathfrak{p}$) are both consistent [4, Theorem 5.1]. Information about the cardinal number $\mathfrak{p}$ is available, for example, in van Douwen's survey [4].

Gartside, Lo and Marsh proved that a Tychonoff product $\mathbb{R}^X$ is strongly sequentially separable if and only if $|X| < \mathfrak{p}$ [6, Theorem 11]. They also proved that a function space $\mathrm{C}(X)$ is strongly sequentially separable if and only if the space X has a coarser second countable topology, and every coarser second countable topology for X satisfies $\binom{\Omega}{\Gamma}$ [6, Theorem 16]. The property $\binom{\Omega}{\Gamma}$ implies $\binom{\Omega_{\mathrm{coz}}}{\Gamma}$. Bonanzinga, Cammaroto and Matveev proved that a space X has the property $\binom{\Omega_{\mathrm{coz}}}{\Gamma}$ if and only if every coarser second countable topology for the space X has the property $\binom{\Omega}{\Gamma}$ [2, Theorem 54]. In summary, for spaces X with a coarser second countable topology, the diagram from the previous section dualizes to the following one.

$$\begin{tikzcd} |X|<\mathfrak{p} \arrow[r] & X \text{ satisfies } \binom{\Omega_{\mathrm{Bor}}}{\Gamma} \arrow[r] & X \text{ satisfies } \binom{\Omega_{\mathrm{coz}}}{\Gamma} \\ & & X \text{ satisfies } \binom{\Omega}{\Gamma} \arrow[u] \end{tikzcd}$$

Problem 1 is thus reduced to the following problem.

Problem 3. *Is there, consistently, a space X with a coarser second countable topology, that satisfies $\binom{\Omega_{\mathrm{coz}}}{\Gamma}$ but not $\binom{\Omega}{\Gamma}$, with $|X| \geq \mathfrak{p}$?*

We will solve this problem, as well as its variations.

3. The problems and their solutions

Consider the positions in the diagrams from the previous section. Write there "$\bullet$" if the property holds, and "$\circ$" if it does not. For example, sets $X \subseteq \mathbb{R}$ of cardinality smaller than $\mathfrak{p}$ realize the following setting.

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\ & & \bullet \arrow[u] \end{tikzcd}$$

that will be denoted $\bullet\,\bullet\,\substack{\bullet\\ \bullet}$ for brevity. We consider the consistency of all settings that are not ruled out by the implications in the diagram. These are the following settings:

$$\circ\,\circ\,\substack{\circ\\ \circ} \qquad \circ\,\circ\,\substack{\bullet\\ \circ} \qquad \circ\,\bullet\,\substack{\bullet\\ \circ} \qquad \circ\,\circ\,\substack{\bullet\\ \bullet} \qquad \bullet\,\bullet\,\substack{\bullet\\ \circ} \qquad \circ\,\bullet\,\substack{\bullet\\ \bullet} \qquad \bullet\,\bullet\,\substack{\bullet\\ \bullet}$$

Problem 1 asks whether either of the the settings $\circ\,\bullet\,\substack{\bullet\\ \circ}$ or $\circ\,\circ\,\substack{\bullet\\ \bullet}$ is consistent.

The following proposition is a variation of an earlier result [15, Example 4.7].

Proposition 4 ($\bullet\,\bullet\,\substack{\bullet\\ \circ}$)**.** *The following assertions are equivalent:*

1. *There is a space X such that the space $\mathbb{R}^X$ is strongly sequentially separable, but the space $\mathrm{C}(X)$ is not Fréchet–Urysohn.*
2. *$\aleph_1 < \mathfrak{p}$.*

Proof. Recall that the space $\mathbb{R}^X$ is strongly sequentially separable if and only if $|X| < \mathfrak{p}$.

(1) $\Rightarrow$ (2): The given space X has $|X| < \mathfrak{p}$. Had it been countable, the space $\mathrm{C}(X)$ would have been metrizable.

(2) $\Rightarrow$ (1): A discrete space of cardinality $\aleph_1$ is not Lindelöf, and thus does not satisfy $\binom{\Omega}{\Gamma}$. Apply duality. $\square$

The following folklore fact implies that discrete spaces of cardinality $\mathfrak{p}$ or greater have none of the studied properties.

Lemma 5 ($\circ\,\circ\,\substack{\circ\\ \circ}$)**.** *Let X be a set. Then $|X| < \mathfrak{p}$ if and only if every countable ω-cover consisting of subsets of X contains a γ-cover.*

Proof. ($\Rightarrow$) Let $\{ U_n : n \in \mathbb{N} \}$ be a countable ω-cover consisting of subsets of X. For each element $x \in X$, let $a_x := \{ n \in \mathbb{N} : x \in U_n \}$, an infinite subset of $\mathbb{N}$. Since $|X| < \mathfrak{p}$, the family $\{ a_x : x \in X \}$ has a pseudointersection a. Then $\{ U_n : n \in a \}$ is a γ-cover of X.

($\Leftarrow$) Assume that $|X| \geq \mathfrak{p}$. We may assume that $X \subseteq [\mathbb{N}]^\infty$ where $[\mathbb{N}]^\infty$ is the family of infinite subsets of $\mathbb{N}$. Then X is a family of infinite subsets of $\mathbb{N}$ of cardinality $\mathfrak{p}$, that is centered and has no pseudointersection. The family of sets $\{ U_n : n \in \mathbb{N} \}$, defined by $U_n := \{ x \in X : n \in x \}$ for natural numbers n, is a countable ω-cover of X and has no subfamily in Γ. $\square$

A theorem of Galvin and Miller [5, Theorem 2] asserts that, if $\mathfrak{p} = |\mathbb{R}|$, then there is a set $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{p}$, satisfying $\binom{\Omega}{\Gamma}$. The Galvin–Miller Theorem is refined by Theorem 6 of Orenshtein and Tsaban [14, Theorem 3.6]. Since this result is central to the remainder of this paper, we include here a simpler proof, due to the third named author [20].

We identify the Cantor space $\{0,1\}^\mathbb{N}$ with the family $\mathrm{P}(\mathbb{N})$ of all subsets of the set $\mathbb{N}$. Thus, we view the space $\mathrm{P}(\mathbb{N})$ as a subset of the real line. The space $\mathrm{P}(\mathbb{N})$ splits into two subspaces: the family of infinite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^\infty$, and the family of finite subsets of $\mathbb{N}$, denoted Fin. We identify every set $a \in [\mathbb{N}]^\infty$ with its increasing enumeration, an element of the Baire space $\mathbb{N}^\mathbb{N}$. Thus, for a natural number n, $a(n)$ is the n-th element in the increasing enumeration of the set a. This way, we have $[\mathbb{N}]^\infty \subseteq \mathbb{N}^\mathbb{N}$, and the topology of the space $[\mathbb{N}]^\infty$ (a subspace of the Cantor space $\mathrm{P}(\mathbb{N})$) coincides with the subspace topology induced by $\mathbb{N}^\mathbb{N}$. When an element of $[\mathbb{N}]^\infty$ is viewed as an element of $\mathbb{N}^\mathbb{N}$, we refer to it as a *function*.

For functions $a, b \in [\mathbb{N}]^\infty$, we write $a \leq^* b$ if the set $\{ n : b(n) < a(n) \}$ is finite. Let $A \subseteq [\mathbb{N}]^\infty$. For a function $b \in [\mathbb{N}]^\infty$, we write $A \leq^* b$ if $a \leq^* b$ for all functions $a \in A$. The set A is *unbounded* if there is no function $b \in [\mathbb{N}]^\infty$ with $A \leq^* b$. Let $\mathfrak{b}$ be the minimal cardinality of an unbounded set in $[\mathbb{N}]^\infty$. A set $\{ x_\alpha : \alpha < \mathfrak{b} \} \subseteq [\mathbb{N}]^\infty$ is an *unbounded tower* if it is unbounded and for all ordinal numbers $\alpha, \beta < \mathfrak{b}$ with $\alpha < \beta$, we have $x_\alpha \mathrel{^*{\supseteq}} x_\beta$. An unbounded tower of cardinality $\mathfrak{p}$ exists if (and only if) $\mathfrak{p} = \mathfrak{b}$ [14, Lemma 3.3].

Theorem 6 (Orenshtein–Tsaban)**.** *For each unbounded tower $T \subseteq [\mathbb{N}]^\infty$ of cardinality $\mathfrak{p}$, the set $T \cup \mathrm{Fin}$ of real numbers satisfies $\binom{\Omega}{\Gamma}$.*

In order to prove Theorem 6, we need the following notions and auxiliary results. Let n, m be natural numbers with $n < m$. Define $(n, m) := \{ i \in \mathbb{N} : n < i < m \}$. A set $a \in [\mathbb{N}]^\infty$ *omits* the interval (n, m) if $a \cap (n, m) = \emptyset$. For a space X, let $\Omega(X)$ be the family of all *open ω-covers* of X, and $\Gamma(X)$ be the family of all *open γ-covers* of X.

Lemma 7 (Galvin–Miller[5, Lemma 1.2])**.** *Let $\mathcal{U}$ be a family of open sets in $\mathrm{P}(\mathbb{N})$ such that $\mathcal{U} \in \Omega(\mathrm{Fin})$. There are a function $b \in [\mathbb{N}]^\infty$ and distinct sets $U_1, U_2, \ldots \in \mathcal{U}$ such that for each element $x \in [\mathbb{N}]^\infty$ and all natural numbers n:*

$$\textit{If } x \cap ((b(n), b(n+1)) = \emptyset, \textit{ then } x \in U_n.$$

Lemma 8 (Folklore [19, Lemma 2.13])**.** *Let Y be a subset of $[\mathbb{N}]^\infty$. The set Y is unbounded if and only if, for each function $b \in [\mathbb{N}]^\infty$, there is a set $a \in Y$ that omits infinitely many intervals $(b(n), b(n+1))$.*

Lemma 9. *Let $X \subseteq \mathrm{P}(\mathbb{N})$ be a set such that $\mathrm{Fin} \subseteq X$ and $|X| < \mathfrak{p}$. Let $\mathcal{U}$ be a family of open sets in $\mathrm{P}(\mathbb{N})$ such that $\mathcal{U} \in \Omega(X)$, and Y be an unbounded set in $[\mathbb{N}]^\infty$. There are a set $a \in Y$, and sets $U_1, U_2, \ldots \in \mathcal{U}$ such that $\{ U_n : n \in \mathbb{N} \} \in \Gamma(X)$, and for each element*

$x \in [\mathbb{N}]^\infty$ *and all natural numbers* n*:*

$$\textit{If } x \setminus \{1, \dots, n\} \subseteq a, \textit{ then } x \in \bigcap_{k \geq n} U_k.$$

Proof. Since $|X| < \mathfrak{p}$, the set X satisfies $\binom{\Omega}{\Gamma}$ [16, Proposition 2]. Let $\mathcal{V} \in \Gamma(X)$ be a subfamily of $\mathcal{U}$. By Lemma 7, there are a function $b \in [\mathbb{N}]^\infty$, and distinct sets $V_1, V_2, \dots \in \mathcal{V}$ such that for each element $x \in [\mathbb{N}]^\infty$, and all natural numbers i:

$$\text{If } x \cap (b(i), b(i+1)) = \emptyset, \text{ then } x \in V_i. \tag{1}$$

By Lemma 8, there is a set $a \in Y$ such that the set

$$c := \{ i \in \mathbb{N} : a \cap (b(i), b(i+1)) = \emptyset \}$$

is infinite. Fix a natural number n. Let k be a natural number with $n \leq k$, and $x \in [\mathbb{N}]^\infty$ be an element such that $x \setminus \{1, \dots, n\} \subseteq a$. Then $n \leq c(k)$, and we have

$$x \cap (b(c(k)), b(c(k)+1)) \subseteq a \cap (b(c(k)), b(c(k)+1)) = \emptyset.$$

By (1), we have $x \in V_{c(k)}$. Thus, $x \in \bigcap_{k \geq n} V_{c(k)}$.

Since $\mathcal{V} \in \Gamma(X)$, we have $\{ V_{c(i)} : i \in \mathbb{N} \} \in \Gamma(X)$. $\square$

Proof of Theorem 6. Let $\{ x_\alpha : \alpha < \mathfrak{b} \} \subseteq [\mathbb{N}]^\infty$ be an unbounded tower. Let $X := \mathrm{Fin} \cup \{ x_\alpha : \alpha < \mathfrak{b} \}$, and for ordinal numbers $\gamma < \mathfrak{b}$, let $X_\gamma := \mathrm{Fin} \cup \{ x_\alpha : \alpha < \gamma \}$. Let $\mathcal{U} \in \Omega(X)$. Fix an ordinal number $\gamma_0 < \mathfrak{b}$. By induction, for a natural number $m > 0$, we proceed as follows. By Lemma 9, there are an ordinal number $\gamma_m < \mathfrak{b}$, and a subfamily $\{ U_n^{(m)} : n \in \mathbb{N} \} \in \Gamma(X_{\gamma_{m-1}})$ of $\mathcal{U}$ such that, for each element $x \in [\mathbb{N}]^\infty$ and all natural numbers n:

$$\text{If } x \setminus \{1, \dots, n\} \subseteq x_{\gamma_m}, \text{ then } x \in \bigcap_{k \geq n} U_k^{(m)}. \tag{2}$$

Let $\gamma := \sup_n \gamma_n$. There is a function $g \in [\mathbb{N}]^\infty$ such that $x_\gamma \setminus \{1, \dots, g(n)\} \subseteq x_{\gamma_n}$ for all natural numbers n. Fix an ordinal number α with $\gamma \leq \alpha < \mathfrak{b}$. Since $x_\alpha \subseteq^* x_\gamma$, we have

$$x_\alpha \setminus \{1, \dots, g(n)\} \subseteq x_\gamma \setminus \{1, \dots, g(n)\} \subseteq x_{\gamma_n},$$

for all but finitely many natural numbers n. By (2), we have $x_\alpha \in \bigcap_{k \geq g(n)} U_k^{(n)}$ for all but finitely many natural numbers n. Thus, for any function $h \in [\mathbb{N}]^\infty$ with $g \leq^* h$, we have $\{ U_{h(n)}^{(n)} : n \in \mathbb{N} \} \in \Gamma(\{ x_\alpha : \gamma \leq \alpha < \mathfrak{b} \})$.

For each element $x \in X_\gamma$, and each natural number n, define

$$f_x(n) := \min \left\{ m \in \mathbb{N} : x \in \bigcap_{k \geq m} U_k^{(n)} \right\}$$

if the set is nonempty, and $f_x(n) := 0$ otherwise. Since $|X_\gamma| < \mathfrak{b}$, there is a function $h \in [\mathbb{N}]^\infty$ such that $\{ f_x : x \in X_\gamma \} \cup \{g\} \leq^* h$, and the sets $U_{h(n)}^{(n)}$ are distinct. Then $\{ U_{h(n)}^{(n)} : n \in \mathbb{N} \} \in \Gamma(X_\gamma)$. Since $\{ U_{h(n)}^{(n)} : n \in \mathbb{N} \} \in \Gamma(\{ x_\alpha : \gamma \leq \alpha < \mathfrak{b} \})$ as well, we have $\{ U_{h(n)}^{(n)} : n \in \mathbb{N} \} \in \Gamma(X)$. $\square$

4. Subsets of the Real, Michael, and Sorgenfrey line

The *Michael line* [10] is the set $\mathrm{P}(\mathbb{N})$, with the topology where the points of the set $[\mathbb{N}]^\infty$ are isolated, and the neighborhoods of the points of the set Fin are those induced by the Cantor space topology on $\mathrm{P}(\mathbb{N})$. The *Sorgenfrey line* [17] is the set $\mathbb{R}$ with the topology generated by the half-open intervals $[a, b)$, for $a, b \in \mathbb{R}$.

The forthcoming Theorem 10(2) solves the problem of Gartside–Lo–Marsh Problem (Problem 1). Recall that an unbounded tower in $[\mathbb{N}]^\infty$ of cardinality $\mathfrak{p}$ exists if and only if $\mathfrak{p} = \mathfrak{b}$. It is consistent that $\aleph_1 < \mathfrak{p} = \mathfrak{b}$, e.g., it holds assuming the Martin Axiom with the negation of the Continuum Hypothesis.

Theorem 10. *Let $T \subseteq [\mathbb{N}]^\infty$ be an unbounded tower of cardinality $\mathfrak{p}$.*

(1) $\left(\begin{smallmatrix}\circ & \circ & \bullet\\ & & \bullet\end{smallmatrix}\right)$ *As a subset of $\mathbb{R}$, the set $T \cup \mathrm{Fin}$ satisfies $\binom{\Omega}{\Gamma}$ but not $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.*

(2) $\left(\begin{smallmatrix}\circ & \circ & \bullet\\ & & \circ\end{smallmatrix}\right)$ *Assume that $\aleph_1 < \mathfrak{p}$. As a subset of the Michael line, the set $T \cup \mathrm{Fin}$ satisfies $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$ but neither $\binom{\Omega}{\Gamma}$ nor $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.*

Proof. (1) By Theorem 6, the set $T \cup \mathrm{Fin}$ satisfies $\binom{\Omega}{\Gamma}$. The set T is centered and has no pseudointersection. Thus, the set T does not satisfy $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ [18, Lemma 24]. Since the set T is a Borel subset of $T \cup \mathrm{Fin}$, and the property $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ is hereditary for Borel subsets [18, Theorem 48], the set $T \cup \mathrm{Fin}$ does not satisfy $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$, too.

(2) For a set $U \subseteq \mathrm{P}(\mathbb{N})$, let $\mathrm{Int}(U)$ be the interior of the set U in the Cantor space topology on $\mathrm{P}(\mathbb{N})$. If $\mathcal{U} \in \Omega(\mathrm{Fin})$ is a family of open sets in the Michael line, then $\{\, \mathrm{Int}(U) : U \in \mathcal{U} \,\} \in \Omega(\mathrm{Fin})$. Thus, Lemma 7 holds for a family of open sets in the Michael line. By Lemma 5, every space of cardinality smaller than $\mathfrak{p}$ satisfies $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$. Thus, Lemma 9 holds for a countable family of open sets in the Michael line. Consequently, the proof of Theorem 6 actually establishes that the set $T \cup \mathrm{Fin}$, as a subspace of the Michael line, satisfies $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$.

Write $T = \{\, x_\alpha : \alpha < \mathfrak{b} \,\}$ with $x_\alpha \subseteq^* x_\beta$ for $\beta < \alpha$. The set $A := \{\, x \in T : x_{\omega_1} \subseteq^* x \,\}$ has cardinality $\aleph_1$. The set A is F_σ in the Cantor space topology and, in particular, in the Michael line topology. Thus, the space $T \cup \mathrm{Fin}$ has an uncountable discrete F_σ subset. Since the Lindelöf property is hereditary for F_σ subsets, the space $T \cup \mathrm{Fin}$ is not Lindelöf. Every space with the property $\binom{\Omega}{\Gamma}$ is Lindelöf. Thus, the space $T \cup \mathrm{Fin}$ does not satisfy $\binom{\Omega}{\Gamma}$.

By (1), since every Borel set in the Cantor space is also Borel in the Michael line, the space $T \cup \mathrm{Fin}$ does not satisfy $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$. $\square$

Corollary 11. *Let $T \subseteq [\mathbb{N}]^\infty$ be an unbounded tower of cardinality $\mathfrak{p}$.*

(1) *For the real line topology, the space $\mathrm{C}(T \cup \mathrm{Fin})$ is Fréchet-Urysohn but the space $\mathrm{B}(T \cup \mathrm{Fin})$ is not strongly sequentially separable.*

(2) *Assume that $\aleph_1 < \mathfrak{p}$. For the Michael line topology, the space $\mathrm{C}(T \cup \mathrm{Fin})$ is strongly sequentially separable and not Fréchet–Urysohn, and the space $\mathrm{B}(T \cup \mathrm{Fin})$ is not strongly sequentially separable.* $\square$

Assuming the Continuum Hypothesis, there is an uncountable set of real numbers satisfying $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ ([3, Theorem 4.1], [11, Theorem 5]). If $\aleph_1 < \mathfrak{p}$, then any subset of real numbers of cardinality $\aleph_1$ satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ [18, Lemma 22, Theorem 27(1)].

Theorem 12. *Let $X \subseteq \mathbb{R}$ be an uncountable set satisfying $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.*

(1) *As a subset of $\mathbb{R}$, the set X satisfies $\binom{\Omega}{\Gamma}$.*

(2) *As a subset of the Sorgenfrey line, the set $-X \cup X$ satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ but not $\binom{\Omega}{\Gamma}$.*

In particular, if the Continuum Hypothesis holds, we obtain the setting $\begin{smallmatrix}\circ&\bullet&\bullet\\ &&\bullet\end{smallmatrix}$ *from (1), and the setting* $\begin{smallmatrix}\circ&\bullet&\bullet\\ &&\circ\end{smallmatrix}$ *from (2). If* $\aleph_1 < \mathfrak{p}$*, we obtain the settings* $\begin{smallmatrix}\bullet&\bullet&\bullet\\ &&\bullet\end{smallmatrix}$ *and* $\begin{smallmatrix}\bullet&\bullet&\bullet\\ &&\circ\end{smallmatrix}$*.*

Proof. (1) For subsets of $\mathbb{R}$, the property $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ implies $\binom{\Omega}{\Gamma}$.

(2) Let $Y \subseteq \mathbb{R}$ be an uncountable set satisfying $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$. The disjoint union $Y \sqcup Y$ satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$ as well: Let $\mathcal{U}$ be a countable Borel ω-cover of $Y \sqcup Y$. The family

$$\mathcal{V} := \{\, U \cap V : U \sqcup V \in \mathcal{U},\ U \subseteq Y \sqcup \emptyset, V \subseteq \emptyset \sqcup Y \,\}$$

is a countable Borel ω-cover of Y. Let $\mathcal{W} \subseteq \mathcal{V}$ be a γ-cover of Y. Then the family

$$\{\, U \sqcup V \in \mathcal{U} : U \cap V \in \mathcal{W},\ U \subseteq Y \sqcup \emptyset, V \subseteq \emptyset \sqcup Y \,\}$$

is a γ-cover of $Y \sqcup Y$.

The set $X := Y \cup \{\, -y : y \in Y \,\}$, a continuous image of the space $Y \sqcup Y$, satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$, too. Consider this set as a subspace of the Sorgenfrey line. Since the Borel sets in the real line and the Sorgenfrey line are the same, the space X satisfies $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$.

The product space $X \times X$ contains the uncountable closed discrete set $\{\, (x, -x) : x \in X \,\}$, and thus does not satisfy $\binom{\Omega}{\Gamma}$. The property $\binom{\Omega}{\Gamma}$ is preserved by finite powers [8, Theorem 3.6]. Thus, the space X does not satisfy $\binom{\Omega}{\Gamma}$. □

Corollary 13. *Let* $X \subseteq \mathbb{R}$ *be an uncountable set satisfying* $\binom{\Omega_{\mathrm{Bor}}}{\Gamma}$*. As a subset of the Sorgenfrey line, the space* $\mathrm{C}(-X \cup X)$ *is not Fréchet–Urysohn, but the space* $\mathrm{B}(-X \cup X)$ *is strongly sequentially separable.* □

5. Additional results

The space from Theorem 10(2) has the property $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$, that is formally stronger than $\binom{\Omega_{\mathrm{coz}}}{\Gamma}$. In the forthcoming Proposition 14, we show that the properties $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$ and $\binom{\Omega_{\mathrm{coz}}}{\Gamma}$ are different.

A family of sets is *almost disjoint* if the intersection of any pair of sets of this family is finite. For an almost disjoint family A in $[\mathbb{N}]^\infty$, the *Mrówka–Isbell* space $\Psi(A)$ is the set $A \cup \mathbb{N}$, with the points of $\mathbb{N}$ isolated, and with the sets $\{a\} \cup a \setminus b$ (for $b \in \mathrm{Fin}$) as neighborhoods of the points $a \in A$.

Proposition 14. *There is a maximal almost disjoint family* A *in* $[\mathbb{N}]^\infty$ *such that the Mrówka–Isbell space* $\Psi(A)$ *satisfies* $\binom{\Omega_{\mathrm{coz}}}{\Gamma}$ *but not* $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$*.*

Proof. There is a maximal almost disjoint family A in $[\mathbb{N}]^\infty$, of cardinality $|\mathbb{R}|$, such that the space $\Psi(A)$ satisfies $\binom{\Omega_{\mathrm{coz}}}{\Gamma}$ [2, Example 61 and Theorem 54]. Let $A = \{\, a_r : r \in \mathbb{R} \,\}$. Since $\mathbb{R}$ does not satisfy $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$, there is a family $\mathcal{U} \in \Omega_{\mathrm{ctbl}}(\mathbb{R})$ with no subfamily in $\Gamma(\mathbb{R})$. For each set $U \in \mathcal{U}$, let $U' := \{\, a_r : r \in U \,\} \cup \mathbb{N}$. The family $\{\, U' : U \in \mathcal{U} \,\}$ is in $\Omega_{\mathrm{ctbl}}(\Psi(A))$ and has no subfamily in $\Gamma(\Psi(A))$. Thus, the space $\Psi(A)$ does not satisfy $\binom{\Omega_{\mathrm{ctbl}}}{\Gamma}$. □

A space is *projectively* $\binom{\Omega}{\Gamma}$ if each continuous second countable image of this space satisfies $\binom{\Omega}{\Gamma}$ [2].

Proposition 15. *For a space* X*, the following assertions are equivalent:*

1. *The space* $\mathrm{C}(X)$ *is strongly sequentially separable.*
2. *The space* X *has a coarser second countable topology, and it is projectively* $\binom{\Omega}{\Gamma}$*.*

Proof. (1) $\Rightarrow$ (2): By a result of Noble [12, Theorem 1], the space X has a coarser second countable topology. In order to prove that the space X is projectively $\binom{\Omega}{\Gamma}$, we show that it satisfies the equivalent property $\binom{\Omega_{\mathrm{coz}}}{\Gamma}$ [2, Theorem 54]. Let $F \subseteq \mathrm{C}(X)$ be a countable set such that the family $\mathcal{U} = \{ f^{-1}[\mathbb{R} \setminus \{0\}] : f \in F \}$ is an ω-cover of X. Let $\mathcal{B}$ be a countable basis of $\mathbb{R}$, and $\mathcal{B}'$ be a countable basis of a coarser topology on X. Let Y be the set X with the topology generated by the family $\{ f^{-1}[B] : B \in \mathcal{B} \} \cup \mathcal{B}'$. The space Y is second countable. By a result of Gartside, Lo, and Marsh [6, Theorem 16], the space Y satisfies $\binom{\Omega}{\Gamma}$. Since $\mathcal{U} \in \Omega(Y)$, the family $\mathcal{U}$ contains a cover $\mathcal{V} \in \Gamma(Y)$. Thus, $\mathcal{V} \in \Gamma(X)$.

(2) $\Rightarrow$ (1): By (2), every coarser second countable topology for the space X satisfies $\binom{\Omega}{\Gamma}$. By a result of Gartside, Lo and Marsh [6, Theorem 16], the space $\mathrm{C}(X)$ is strongly sequentially separable. $\square$

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