A NEW EXAMPLE OF A LIMIT VARIETY OF MONOIDS

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Abstract. A variety of universal algebras is called limit if it is non-finitely based but all its proper subvarieties are finitely based. Until recently, only two explicit examples of limit varieties of monoids, constructed by Jackson, were known. Recently Zhang and Luo found the third example of such a variety. In our work, one more example of a limit variety of monoids is given.

1. Introduction and summary

A variety of algebras is called *finitely based* if it has a finite basis of its identities, otherwise, the variety is said to be *non-finitely based*. Much attention is paid to the study of finitely based and non-finitely based varieties of algebras of various types. In particular, the finitely based and non-finitely based varieties of semigroups and monoids have been the subject of an intensive research (see the surveys [11, 12]).

A variety is called a *limit variety* if it is non-finitely based but every proper subvariety is finitely based. Limit varieties play an important role because each non-finitely based variety contains some limit subvariety. There are continuum many limit varieties of groups [5]. When studying varieties of semigroups and monoids, the varieties play an important role that are away from group varieties in a sense. We mainly mean varieties of *aperiodic* monoids, i.e., monoids that have trivial subgroups only. A few explicit examples of limit varieties of monoids are known so far, and all these varieties consist of aperiodic monoids. In [3], Jackson found the first two examples of such varieties $\mathbf{J}_1$ and $\mathbf{J}_2$. Lee established that only $\mathbf{J}_1$ and $\mathbf{J}_2$ are limit varieties within several classes of monoid varieties [7, 8]. In 2013, Zhang found a non-finitely based variety $\mathbf{L}$ of aperiodic monoids that does not contain the varieties $\mathbf{J}_1$ and $\mathbf{J}_2$ [14] and, therefore, she proved that there exists a limit variety of monoids that differs from $\mathbf{J}_1$ and $\mathbf{J}_2$. Just recently, Zhang and Luo identified an explicit example of such variety [15]. In this article we exhibit another example of a limit variety of aperiodic monoids. Note that our limit variety is not contained in the variety $\mathbf{L}$.

In order to formulate the main result of the article, we need some notation. The free monoid over a countably infinite alphabet is denoted by F^1. As usual, elements of F^1 and the alphabet are called *words* and *letters* respectively. Words and letters are denoted by small Latin letters. However, words unlike letters are written in bold. Expressions like to $\mathbf{u} \approx \mathbf{v}$ are used for identities, whereas $\mathbf{u} = \mathbf{v}$ means that the words $\mathbf{u}$ and $\mathbf{v}$ coincide. As usual, the symbol $\mathbb{N}$ stands for the set of all natural

2010 *Mathematics Subject Classification.* 20M07.

Key words and phrases. Monoid, variety, limit variety, finite basis problem, finitely based variety, non-finitely based variety.

The work is supported by the Ministry of Education and Science of the Russian Federation (project 1.6018.2017/8.9) and by Russian Foundation for Basic Research (grant 17-01-00551).

numbers. For an arbitrary $n \in \mathbb{N}$, we denote by S_n the full symmetric group on the set $\{1, 2, \dots, n\}$. If $\pi \in S_n$ then we put

$$\mathbf{w}_n[\pi] = x\, z_{1\pi} z_{2\pi} \cdots z_{n\pi}\, x \left(\prod_{i=1}^{n} t_i z_i \right) \quad \text{and} \quad \mathbf{w}'_n[\pi] = x^2\, z_{1\pi} z_{2\pi} \cdots z_{n\pi} \left(\prod_{i=1}^{n} t_i z_i \right).$$

We fix notation for the following identity system:

$$\Phi = \{xyx \approx xyx^2,\ x^2y^2 \approx y^2x^2,\ xyzxy \approx yxzxy\}.$$

For an identity system Σ, we denote by var Σ the variety of monoids given by Σ. Put

$$\mathbf{J} = \operatorname{var}\{\Phi,\ xyxztx \approx xyxzxtx,\ \mathbf{w}_n[\pi] \approx \mathbf{w}'_n[\pi] \mid n \in \mathbb{N},\ \pi \in S_n\}.$$

A variety is called *finitely generated* if it is generated by a finite algebra.

The main result of the paper is the following

Theorem 1.1. *The variety* $\mathbf{J}$ *is a finitely generated limit variety of monoids.*

Recall that a variety is called *Cross* if it is finitely based, finitely generated and small. A non-Cross variety is said to be *almost Cross* if all its proper subvarieties are Cross. As we will see below, Theorem 1.1 implies

Corollary 1.2. *The variety* $\mathbf{J}$ *is almost Cross.*

We note that only a few papers with new explicit examples of almost Cross varieties of aperiodic monoids are known. They are [2–4, 9, 13, 15].

If $\mathbf{X}$ is a monoid variety then we denote by $\overleftarrow{\mathbf{X}}$ the variety *dual to* $\mathbf{X}$, i.e., the variety consisting of monoids antiisomorphic to monoids from $\mathbf{X}$. A monoid variety $\mathbf{X}$ is called *self-dual* if $\mathbf{X} = \overleftarrow{\mathbf{X}}$. We note that all the limit monoid varieties from [3, 15] are self-dual, while the variety $\mathbf{J}$ is non-self-dual. So, Theorem 1.1 implies that the variety $\overleftarrow{\mathbf{J}}$ is a finitely generated limit variety of monoids too.

The article consists of four sections. Section 2 contains definitions, notation and auxiliary results. In Section 3 we describe the subvariety lattice of the variety $\mathbf{J}$, while Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminaries

Recall that a variety of universal algebras is called *locally finite* if all of its finitely generated members are finite. Varieties with a finite subvariety lattice are called *small*.

Lemma 2.1 ([3, Lemma 6.1])**.** *Every small locally finite variety of algebras is finitely generated.* □

As usual, $\mathrm{End}(F^1)$ denotes the endomorphism monoid of the monoid F^1. The following statement is the specialization for monoids of a well-known universal-algebraic fact.

Lemma 2.2. *The identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety of monoids given by an identity system* Σ *if and only if there exists a sequence of words*

$$\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m \tag{2.1}$$

such that $\mathbf{u} = \mathbf{v}_0$, $\mathbf{v}_m = \mathbf{v}$ *and, for any* $0 \le i < m$, *there are words* $\mathbf{a}_i, \mathbf{b}_i \in F^1$, *an endomorphism* $\xi_i \in \mathrm{End}(F^1)$ *and an identity* $\mathbf{s}_i \approx \mathbf{t}_i \in \Sigma$ *such that either* $\mathbf{v}_i = \mathbf{a}_i \xi_i(\mathbf{s}_i) \mathbf{b}_i$ *and* $\mathbf{v}_{i+1} = \mathbf{a}_i \xi_i(\mathbf{t}_i) \mathbf{b}_i$ *or* $\mathbf{v}_i = \mathbf{a}_i \xi_i(\mathbf{t}_i) \mathbf{b}_i$ *and* $\mathbf{v}_{i+1} = \mathbf{a}_i \xi_i(\mathbf{s}_i) \mathbf{b}_i$. □

A letter is called *simple* [*multiple*] *in a word* $\mathbf{w}$ if it occurs in $\mathbf{w}$ once [at least twice]. The set of all simple [multiple] letters in a word $\mathbf{w}$ is denoted by $\mathrm{sim}(\mathbf{w})$ [respectively $\mathrm{mul}(\mathbf{w})$]. The *content* of a word $\mathbf{w}$, i.e., the set of all letters occurring in $\mathbf{w}$, is denoted by $\mathrm{con}(\mathbf{w})$. We denote the empty word by λ. The number of occurrences of the letter x in $\mathbf{w}$ is denoted by $\mathrm{occ}_x(\mathbf{w})$. For a word $\mathbf{w}$ and letters $x_1, x_2, \ldots, x_k \in \mathrm{con}(\mathbf{w})$, let $\mathbf{w}(x_1, x_2, \ldots, x_k)$ be the word obtained from $\mathbf{w}$ by deleting from $\mathbf{w}$ all letters except $x_1, x_2, \ldots, x_k$.

Let $\mathbf{w}$ be a word and $\mathrm{sim}(\mathbf{w}) = \{t_1, t_2, \ldots, t_m\}$. We can assume without loss of generality that $\mathbf{w}(t_1, t_2, \ldots, t_m) = t_1 t_2 \cdots t_m$. Then $\mathbf{w} = t_0 \mathbf{w}_0 t_1 \mathbf{w}_1 \cdots t_m \mathbf{w}_m$ where $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m$ are possibly empty words and $t_0 = \lambda$. The words $\mathbf{w}_0$, $\mathbf{w}_1$, $\ldots$, $\mathbf{w}_m$ are called *blocks* of a word $\mathbf{w}$, while $t_0, t_1, \ldots, t_m$ are said to be *dividers* of $\mathbf{w}$. The representation of the word $\mathbf{w}$ as a product of alternating dividers and blocks, starting with the divider t_0 and ending with the block $\mathbf{w}_m$ is called a *decomposition* of the word $\mathbf{w}$. For a given word $\mathbf{w}$, a letter $x \in \mathrm{con}(\mathbf{w})$ and a natural number $i \le \mathrm{occ}_x(\mathbf{w})$, we denote by $h_i(\mathbf{w}, x)$ the right-most divider of $\mathbf{w}$ that precedes the ith occurrence of x in $\mathbf{w}$, and by $t(\mathbf{w}, x)$ the right-most divider of $\mathbf{w}$ that precedes the latest occurrence of x in $\mathbf{w}$.

Example 2.3. Let $\mathbf{w} = yxsxy^2tzy$. Then $\mathrm{sim}(\mathbf{w}) = \{s, t, z\}$ and $\mathrm{mul}(\mathbf{w}) = \{x, y\}$. Therefore, the decomposition of $\mathbf{w}$ has the form

$$\lambda \cdot \underline{yx} \cdot s \cdot \underline{xy^2} \cdot t \cdot \underline{\lambda} \cdot z \cdot \underline{y} \tag{2.2}$$

(here we underline blocks to distinguish them from dividers). Then we have that $h_1(\mathbf{w}, x) = h_1(\mathbf{w}, y) = h_1(\mathbf{w}, s) = t(\mathbf{w}, s) = \lambda$, $h_2(\mathbf{w}, x) = t(\mathbf{w}, x) = h_2(\mathbf{w}, y) = h_3(\mathbf{w}, y) = h_1(\mathbf{w}, t) = t(\mathbf{w}, t) = s$, $h_1(\mathbf{w}, z) = t(\mathbf{w}, z) = t$ and $h_4(\mathbf{w}, y) = t(\mathbf{w}, y) = z$.

Put

$$\begin{aligned} \mathbf{E} &= \mathrm{var}\{x^2 \approx x^3,\, x^2 y \approx xyx,\, x^2 y^2 \approx y^2 x^2\}, \\ \mathbf{F} &= \mathrm{var}\{\Phi,\, xyxz \approx xyxzx\}. \end{aligned}$$

The following statement implies, in particular, that $\mathbf{E} \subset \mathbf{F}$.

Lemma 2.4 ([2, Propositions 4.2 and 6.9(i)])**.** *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds:*

(i) *in the variety* $\mathbf{E}$ *if and only if*

$$\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v}) \text{ and } \mathrm{mul}(\mathbf{u}) = \mathrm{mul}(\mathbf{v}) \tag{2.3}$$

and

$$h_1(\mathbf{u}, x) = h_1(\mathbf{v}, x) \text{ for all } x \in \mathrm{con}(\mathbf{u}); \tag{2.4}$$

(ii) *in the variety* $\mathbf{F}$ *if and only if*

$$h_2(\mathbf{u}, x) = h_2(\mathbf{v}, x) \text{ for all } x \in \mathrm{con}(\mathbf{u}) \tag{2.5}$$

and the claims (2.3) *and* (2.4) *are true.* □

Lemma 2.4 implies the following two statements.

Corollary 2.5. *A non-trivial identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{F} \vee \overleftarrow{\mathbf{E}}$ *if and only if*

$$t(\mathbf{u}, x) = t(\mathbf{v}, x) \text{ for all } x \in \mathrm{con}(\mathbf{u}) \tag{2.6}$$

and the claims (2.3)*–*(2.5) *are true.* □

Corollary 2.6. *Let* $\mathbf{u} \approx \mathbf{v}$ *be an identity that holds in the variety* $\mathbf{E}$. *Suppose that*

$$t_0\mathbf{u}_0t_1\mathbf{u}_1\cdots t_m\mathbf{u}_m \tag{2.7}$$

is the decomposition of $\mathbf{u}$. *Then the decomposition of* $\mathbf{v}$ *has the form*

$$t_0\mathbf{v}_0t_1\mathbf{v}_1\cdots t_m\mathbf{v}_m. \tag{2.8}$$

Proof. In view of Lemma 2.4(i), the claims (2.3) and (2.4) are true. Taking into account the claim (2.3), we obtain that $\mathrm{sim}(\mathbf{v}) = \{t_1, t_2, \dots, t_m\}$. Then the claim (2.4) implies that $\mathbf{v}(t_1, t_2, \dots, t_m) = t_1t_2\cdots t_m$, and we are done. □

If $\mathbf{u}$ and $\mathbf{v}$ are words and ε is an identity then we will write $\mathbf{u} \stackrel{\varepsilon}{\approx} \mathbf{v}$ in the case when the identity $\mathbf{u} \approx \mathbf{v}$ follows from ε.

Lemma 2.7. *The identities*

$$xyzx \approx xyxzx, \tag{2.9}$$

$$x^2y^2 \approx y^2x^2 \tag{2.10}$$

form an identity basis of the variety $\mathbf{E} \vee \overleftarrow{\mathbf{E}}$.

Proof. Consider the semigroup

$$B_0 = \langle a, b, c \mid a^2 = a, b^2 = b, ab = ba = 0, ac = cb = c\rangle = \{a, b, c, 0\}.$$

It follows from [6, Proposition 1.7(i),(ii) and Figure 4] that the variety $\mathbf{E} \vee \overleftarrow{\mathbf{E}}$ is generated by the monoid B_0^1, i.e., the semigroup B_0 with a new identity element adjoined. The identities (2.9) and

$$xzytxy \approx xzytyx, \tag{2.11}$$

$$xyzxty \approx yxzxty, \tag{2.12}$$

$$xzxyty \approx xzyxty \tag{2.13}$$

form an identity basis of the monoid B_0^1 by [1, Proposition 3.1(i)]. We note that the identity (2.11) follows from the identities (2.9) and (2.10) because

$$xzytxy \stackrel{(2.9)}{\approx} xzytx^2y^2 \stackrel{(2.10)}{\approx} xzyty^2x^2 \stackrel{(2.9)}{\approx} xzytyx.$$

Analogously, the identities (2.9) and (2.10) imply the identities (2.12) and (2.13). It remains to note that the identities (2.9) and (2.10) hold in $\mathbf{E} \vee \overleftarrow{\mathbf{E}}$. □

The set of all letters that occur precisely k times in a word $\mathbf{w}$ is denoted by $\mathrm{con}_k(\mathbf{w})$.

Lemma 2.8. *The identities* (2.10), (2.12), (2.13) *and*

$$xyx \approx xyx^2, \tag{2.14}$$

$$xyxztx \approx xyxzxtx \tag{2.15}$$

form an identity basis of the variety $\mathbf{F} \vee \overleftarrow{\mathbf{E}}$.

Proof. The identities (2.10), (2.12), (2.13), (2.14) and (2.15) hold in $\mathbf{F} \vee \overleftarrow{\mathbf{E}}$ by Corollary 2.5. Note that (2.11) follows from (2.10) and (2.14) because

$$xzytxy \stackrel{(2.14)}{\approx} xzytx^2y^2 \stackrel{(2.10)}{\approx} xzyty^2x^2 \stackrel{(2.14)}{\approx} xzytyx.$$

Let $\mathbf{u} \approx \mathbf{v}$ be an identity that holds in $\mathbf{F} \vee \overleftarrow{\mathbf{E}}$. The identities (2.14) and (2.15) allow us to assume that

$$\mathrm{mul}(\mathbf{u}) = \mathrm{con}_3(\mathbf{u}) \quad \text{and} \quad \mathrm{mul}(\mathbf{v}) = \mathrm{con}_3(\mathbf{v}). \tag{2.16}$$

In view of Corollary 2.6, if (2.7) is the decomposition of $\mathbf{u}$ then the decomposition of $\mathbf{v}$ has the form (2.8). This fact, Corollary 2.5 and the claim (2.16) imply that

$$\mathrm{con}_j(\mathbf{u}_i) = \mathrm{con}_j(\mathbf{v}_i) \quad \text{for any} \quad i = 0, 1, \dots, m \quad \text{and} \quad j = 1, 2, 3. \tag{2.17}$$

Note that the identity (2.11) [respectively (2.12)] allows us to swap the adjacent non-first [respectively the non-latest] occurrences of two multiple letters, while the identity (2.13) allows us to swap a non-first occurrence and a non-latest occurrence of two multiple letters whenever these occurrences are adjacent to each other. So, since the claim (2.17) is true, the identities (2.11)–(2.13) imply the identities

$$\mathbf{u} = t_0\mathbf{u}_0t_1\mathbf{u}_1 \cdots t_m\mathbf{u}_m \approx t_0\mathbf{v}_0t_1\mathbf{u}_1 \cdots t_m\mathbf{u}_m \approx \cdots \approx t_0\mathbf{v}_0t_1\mathbf{v}_1 \cdots t_m\mathbf{v}_m = \mathbf{v},$$

and we are done. □

3. The subvariety lattice of J

The trivial variety of monoids is denoted by $\mathbf{T}$, while $\mathbf{SL}$ denotes the variety of all semilattice monoids. Put also

$$\mathbf{C} = \mathrm{var}\{x^2 \approx x^3,\, xy \approx yx\},$$
$$\mathbf{D} = \mathrm{var}\{x^2 \approx x^3,\, x^2y \approx xyx \approx yx^2\},$$
$$\mathbf{H} = \mathrm{var}\{\Phi,\, xyxztx \approx xyxzxtx,\, x^2yty \approx xyxty \approx yx^2ty\},$$
$$\mathbf{I} = \mathrm{var}\{\Phi,\, xyxztx \approx xyxzxtx,\, xzxyty \approx xzyxty\}.$$

The subvariety lattice of a monoid variety $\mathbf{X}$ is denoted by $L(\mathbf{X})$.

The goal of this section is to prove the following

Proposition 3.1. *The lattice $L(\mathbf{J})$ has the form shown in Fig. 1.*

To verify Proposition 3.1, we need several assertions.

Lemma 3.2. *Let $\mathbf{X}$ be a monoid variety that satisfies the identities* (2.10), (2.14), (2.15) *and*

$$xyzxy \approx yxzxy. \tag{3.1}$$

Then the lattice $L(\mathbf{X})$ is the set-theoretical union of the lattice $L(\mathbf{F} \vee \overleftarrow{\mathbf{E}})$ and the interval $[\mathbf{F} \vee \overleftarrow{\mathbf{E}}, \mathbf{X}]$. The lattice $L(\mathbf{F} \vee \overleftarrow{\mathbf{E}})$ has the form shown in Fig. 1.

Proof. Let $\mathbf{V}$ be a subvariety of the variety $\mathbf{X}$. We need to verify that if $\mathbf{F} \vee \overleftarrow{\mathbf{E}} \nsubseteq \mathbf{V}$ then $\mathbf{V}$ coincides with one of the varieties $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\overleftarrow{\mathbf{E}}$, $\mathbf{E} \vee \overleftarrow{\mathbf{E}}$ and $\mathbf{F}$. Clearly, $\mathbf{V}$ does not contain either $\mathbf{F}$ or $\overleftarrow{\mathbf{E}}$. A variety of monoids is called *completely regular* if it consists of *completely regular monoids* (i.e., unions of groups). If $\mathbf{V}$ is completely regular then it is a variety of *bands*, i.e. idempotent monoids because every aperiodic completely regular variety is a variety of bands. Evidently, every variety of bands with the identity (2.10) is commutative. Therefore, $\mathbf{V}$ is one of the varieties $\mathbf{T}$ or $\mathbf{SL}$. So, we can assume that $\mathbf{V}$ is non-completely regular.

Suppose that $\overleftarrow{\mathbf{E}} \nsubseteq \mathbf{V}$. It is verified in [2, the dual to Lemma 4.3] that if $\mathbf{Y}$ is a non-completely regular variety of monoids that satisfies the identity

$$x^2 \approx x^3 \tag{3.2}$$

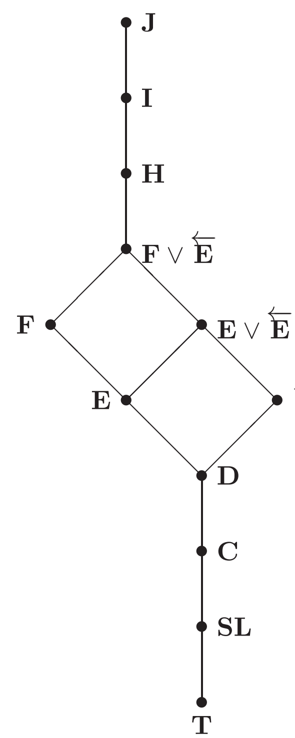

FIGURE 1. The lattice $L(\mathbf{J})$

and does not contain the variety $\overleftarrow{\mathbf{E}}$ then $\mathbf{Y}$ satisfies the identity

$$x^2y \approx x^2yx^2. \tag{3.3}$$

This fact implies that the identity (3.3) holds in $\mathbf{V}$. Then the identities

$$xyxz \stackrel{(2.14)}{\approx} xyx^2z \stackrel{(3.3)}{\approx} xyx^2zx^2 \stackrel{(2.14)}{\approx} xyxzx$$

hold in $\mathbf{V}$. Thus $\mathbf{V} \subseteq \mathbf{F}$. It is proved in [2, Proposition 6.1] that the lattice $L(\mathbf{F})$ has the form shown in Fig. 1. Hence $\mathbf{V}$ is one of the varieties $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ or $\mathbf{F}$. So, we can assume that $\overleftarrow{\mathbf{E}} \subseteq \mathbf{V}$.

It follows that $\mathbf{F} \nsubseteq \mathbf{V}$. Then there exists an identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{V}$ but does not hold in $\mathbf{F}$. If (2.7) is the decomposition of $\mathbf{u}$ then the decomposition of $\mathbf{v}$ has the form (2.8) by the dual to Corollary 2.6. Lemma 2.4(ii) and the dual to Lemma 2.4(i) imply that one of the claims (2.4) and (2.5) is false. Then there are a letter $x \in \mathrm{mul}(\mathbf{u})$ and $k \in \{1, 2\}$ such that $h_k(\mathbf{u}, x) \neq h_k(\mathbf{v}, x)$. If $k = 1$ then we multiply the identity $\mathbf{u} \approx \mathbf{v}$ by xt on the left where $t \notin \mathrm{con}(\mathbf{u})$. So, we can believe that $k = 2$. Suppose that $h_2(\mathbf{u}, x) = t_i$ and $h_2(\mathbf{v}, x) = t_j$ where $i \neq j$. We can assume without any loss that $i > j$. Then $\mathbf{V}$ satisfies the identity

$$\mathbf{u}(x, t_i) = xt_ix^p \approx x^qt_ix^r = \mathbf{v}(x, t_i)$$

where $p \geq 1$, $q \geq 2$ and $r \geq 0$. If $r < 2$ or $p < 2$ then we multiply this identity by x^2 on the right. Taking into account the identity (3.2), we get that $\mathbf{V}$ satisfies the identity

$$xyx^2 \approx x^2yx^2. \tag{3.4}$$

Then the identities

$$xyzx \overset{(2.14)}{\approx} xyzx^2 \overset{(3.4)}{\approx} x^2yzx^2 \overset{(2.15)}{\approx} x^2yxzx^2 \overset{(3.4)}{\approx} xyxzx^2 \overset{(2.14)}{\approx} xyxzx$$

hold in $\mathbf{V}$. In view of Lemma 2.7, we have that $\mathbf{V} \subseteq \mathbf{E} \vee \overleftarrow{\mathbf{E}}$. It is proved in [6, Section 5] that the lattice $L(\mathbf{E} \vee \overleftarrow{\mathbf{E}})$ has the form shown in Fig. 1. This fact implies that $\mathbf{V}$ is one of the varieties $\overleftarrow{\mathbf{E}}$ and $\mathbf{E} \vee \overleftarrow{\mathbf{E}}$. □

Lemma 3.3. *Let* $\mathbf{w} = \mathbf{v}_1 a \mathbf{v}_2 a \mathbf{v}_3$ *where* $\mathbf{v}_1$, $\mathbf{v}_2$ *and* $\mathbf{v}_3$ *are possibly empty words. Suppose that* $\mathrm{con}(\mathbf{v}_2) \subseteq \mathrm{mul}(\mathbf{w})$. *Then* $\mathbf{J}$ *satisfies the identity* $\mathbf{w} \approx \mathbf{v}_1 a^2 \mathbf{v}_2 \mathbf{v}_3$.

Proof. Put $X = \mathrm{con}(\mathbf{v}_2) \setminus \mathrm{con}(\mathbf{v}_3)$. We use induction on the cardinality of the set X and aim to verify that $\mathbf{J}$ satisfies the identity $\mathbf{w} \approx \mathbf{v}_1 a^2 \mathbf{v}_2 \mathbf{v}_3$.

Induction base. Let the set X be empty. Then $\mathrm{con}(\mathbf{v}_2) \subseteq \mathrm{con}(\mathbf{v}_3)$. We can believe that $\mathrm{occ}_x(\mathbf{v}_2) \le \mathrm{occ}_x(\mathbf{v}_3)$ for every $x \in \mathrm{con}(\mathbf{v}_2)$ because $\mathbf{J}$ satisfies the identity (2.14). Then we can rename the letters and assume that $\mathbf{v}_2 = z_{1\pi} z_{2\pi} \cdots z_{n\pi}$ for some n and $\pi \in S_n$ and the latest occurrence of z_i precedes the latest occurrence of z_j in $\mathbf{v}_3$ whenever $i < j$. It follows that there are letters $t_1, t_2, \ldots, t_n \notin \mathrm{con}(\mathbf{w})$, a mapping ξ from $\{t_1, t_2, \ldots, t_n\}$ to F^1 and a word $\mathbf{v} \in F^1$ such that

$$\mathbf{v}_1 a \mathbf{v}_2\, a\, \mathbf{v}_3 = \mathbf{v}_1 a\, z_{1\pi} z_{2\pi} \cdots z_{n\pi}\, a \left(\prod_{i=1}^{n} \xi(t_i) z_i \right) \mathbf{v},$$

$$\mathbf{v}_1 a^2 \mathbf{v}_2\, \mathbf{v}_3 = \mathbf{v}_1 a^2\, z_{1\pi} z_{2\pi} \cdots z_{n\pi} \left(\prod_{i=1}^{n} \xi(t_i) z_i \right) \mathbf{v}.$$

We can extend ξ to an endomorphis of F^1 so that $\xi(z_i) = z_i$ for all $i = 1, 2, \ldots, n$. So, we can assume without any loss that $\xi \in \mathrm{End}(F^1)$ and $\xi(z_i) = z_i$ for all $i = 1, 2, \ldots, n$. Then

$$\mathbf{w} = \mathbf{v}_1 a \mathbf{v}_2\, a\, \mathbf{v}_3 = \mathbf{v}_1\, \xi(\mathbf{w}_n[\pi])\, \mathbf{v} \ \text{ and } \mathbf{v}_1 a^2 \mathbf{v}_2\, \mathbf{v}_3 = \mathbf{v}_1\, \xi(\mathbf{w}_n'[\pi])\, \mathbf{v},$$

whence $\mathbf{J}$ satisfies the identity $\mathbf{w} = \mathbf{v}_1 a^2 \mathbf{v}_2 \mathbf{v}_3$.

Induction step. Let now X be non-empty. Then there is a letter $x \in \mathrm{con}(\mathbf{v}_2)$ and the possibly empty words $\mathbf{v}_2'$ and $\mathbf{v}_2''$ such that $\mathbf{v}_2 = \mathbf{v}_2' x \mathbf{v}_2''$, $\mathrm{con}(\mathbf{v}_2'') \subseteq \mathrm{con}(\mathbf{v}_3)$ and $x \notin \mathrm{con}(\mathbf{v}_3)$. Taking into account that $x \in \mathrm{mul}(\mathbf{w})$, we get that $x \in \mathrm{con}(\mathbf{v}_1 a \mathbf{v}_2')$. Then $\mathbf{J}$ satisfies the identities

$$\begin{aligned}
\mathbf{w} &= \mathbf{v}_1 a \mathbf{v}_2' x \mathbf{v}_2'' a \mathbf{v}_3 \\
&\approx \mathbf{v}_1 a \mathbf{v}_2' x^2 \mathbf{v}_2'' a \mathbf{v}_3 && \text{by the identity (2.14)} \\
&\approx \mathbf{v}_1 a \mathbf{v}_2' x \mathbf{v}_2'' x a \mathbf{v}_3 && \text{by the induction assumption} \\
&\approx \mathbf{v}_1 a \mathbf{v}_2' x \mathbf{v}_2'' x^2 a^2 \mathbf{v}_3 && \text{by the identity (2.14)} \\
&\approx \mathbf{v}_1 a \mathbf{v}_2' x \mathbf{v}_2'' a^2 x^2 \mathbf{v}_3 && \text{by the identity (2.10)} \\
&\approx \mathbf{v}_1 a \mathbf{v}_2' x \mathbf{v}_2'' a x \mathbf{v}_3 && \text{by the identity (2.14)} \\
&\approx \mathbf{v}_1 a^2 \mathbf{v}_2' x \mathbf{v}_2'' x \mathbf{v}_3 && \text{by the induction assumption} \\
&\approx \mathbf{v}_1 a^2 \mathbf{v}_2' x^2 \mathbf{v}_2'' \mathbf{v}_3 && \text{by the induction assumption}
\end{aligned}$$

$$\approx \mathbf{v}_1 a^2 \mathbf{v}_2' x \mathbf{v}_2'' \mathbf{v}_3 \qquad \text{by the identity (2.14)}$$
$$= \mathbf{v}_1 a^2 \mathbf{v}_2 \mathbf{v}_3,$$

and we are done. □

Let $\mathbf{w}$ be a word and $x, y \in \mathrm{mul}(\mathbf{w})$. Suppose that $t_0\mathbf{w}_0 t_1 \mathbf{w}_1 \cdots t_m \mathbf{w}_m$ is the decomposition of $\mathbf{w}$ and $t_i = h_2(\mathbf{w}, x)$, $t_{i'} = h_2(\mathbf{w}, y)$, $t_j = t(\mathbf{w}, x)$, $t_{j'} = t(\mathbf{w}, y)$ for some $0 \le i \le j \le m$ and $0 \le i' \le j' \le m$. The letters x and y are said to be *integrated* in the word $\mathbf{w}$ if either $i' \le i \le j'$ or $i \le i' \le j$.

Lemma 3.4. *Let $\mathbf{w} = \mathbf{w}' ab \mathbf{w}''$ where $\mathbf{w}'$ and $\mathbf{w}''$ are possibly empty words, a and b are multiple letters in $\mathbf{w}$. Suppose that one of the following holds:*

(i) *the letters a and b are integrated in $\mathbf{w}$;*
(ii) *$\mathbf{w}' = \mathbf{v}_1 b \mathbf{v}_2$ for some words $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathrm{con}(\mathbf{v}_2) \subseteq \mathrm{mul}(\mathbf{w})$.*

Then $\mathbf{J}$ satisfies the identity $\mathbf{w} \approx \mathbf{w}' ba \mathbf{w}''$.

Proof. (i) If $a, b \in \mathrm{con}(\mathbf{w}')$ then

$$\mathbf{w} = \mathbf{w}' ab \mathbf{w}'' \stackrel{(2.14)}{\approx} \mathbf{w}' a^2 b^2 \mathbf{w}'' \stackrel{(2.10)}{\approx} \mathbf{w}' b^2 a^2 \mathbf{w}'' \stackrel{(2.14)}{\approx} \mathbf{w}' ba \mathbf{w}'', \tag{3.5}$$

and we are done. Thus, we can assume without loss of generality that $a \notin \mathrm{con}(\mathbf{w}')$. Then $a \in \mathrm{con}(\mathbf{w}'')$. If $b \notin \mathrm{con}(\mathbf{w}'')$ then $\mathbf{J}$ satisfies the identities

$$\mathbf{w} = \mathbf{w}' ab \mathbf{w}'' \stackrel{(2.14)}{\approx} \mathbf{w}' ab^2 \mathbf{w}''.$$

This fact allows us to assume that $b \in \mathrm{con}(\mathbf{w}'')$.

If there is some occurrence of b between the second and the latest occurrences of a in $\mathbf{w}$ then $\mathbf{w}'' = \mathbf{v}_1 a \mathbf{v}_2 b \mathbf{v}_3 a \mathbf{v}_4$ for some possibly empty words $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Then the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{w}' ab \mathbf{w}' = \mathbf{w}' ab \mathbf{v}_1 a \mathbf{v}_2 b \mathbf{v}_3 a \mathbf{v}_4 \stackrel{(2.15)}{\approx} \mathbf{w}' ab \mathbf{v}_1 a \mathbf{v}_2 ab \mathbf{v}_3 a \mathbf{v}_4 \\ &\stackrel{(3.1)}{\approx} \mathbf{w}' ba \mathbf{v}_1 a \mathbf{v}_2 ab \mathbf{v}_3 a \mathbf{v}_4 \stackrel{(2.15)}{\approx} \mathbf{w}' ba \mathbf{v}_1 a \mathbf{v}_2 b \mathbf{v}_3 a \mathbf{v}_4 = \mathbf{w}' ba \mathbf{w}'' \end{aligned}$$

hold in $\mathbf{J}$. So, we can assume that there are no occurrences of b between the second and the latest occurrences of a in $\mathbf{w}$. Analogously, the second and the latest occurrences of a in $\mathbf{w}$ do not lie between the second and the latest occurrences of b in $\mathbf{w}$. Then either the latest occurrence of a precedes the second occurrence of b in $\mathbf{w}$ or the latest occurrence of b precedes the second occurrence of a. Since the letters a and b are integrated in the word $\mathbf{w}$, either the latest occurrence of a and the second occurrence of b in $\mathbf{w}$ or the latest occurrence of b and the second occurrence of a in $\mathbf{w}$ lie in the same block.

If the latest occurrence of a and the second occurrence of b in $\mathbf{w}$ lie in the same block then $\mathbf{w}'' = \mathbf{v}_1 a \mathbf{v}_2 b \mathbf{v}_3$ for some possibly empty words $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathrm{con}(\mathbf{v}_2) \subseteq \mathrm{mul}(\mathbf{w})$ and $a, b \notin \mathrm{con}(\mathbf{v}_2)$. Then the identities

$$\begin{aligned} \mathbf{w} &= \mathbf{w}' ab \mathbf{v}_1 a \mathbf{v}_2 b \mathbf{v}_3 \\ &\approx \mathbf{w}' ab \mathbf{v}_1 a^2 \mathbf{v}_2 b \mathbf{v}_3 && \text{by the identity (2.14)} \\ &\approx \mathbf{w}' ab \mathbf{v}_1 a \mathbf{v}_2 ab \mathbf{v}_3 && \text{by Lemma 3.3} \\ &\approx \mathbf{w}' ba \mathbf{v}_1 a \mathbf{v}_2 ab \mathbf{v}_3 && \text{by the identity (3.1)} \\ &\approx \mathbf{w}' ba \mathbf{v}_1 a^2 \mathbf{v}_2 b \mathbf{v}_3 && \text{by Lemma 3.3} \end{aligned}$$

$$\approx \mathbf{w}'ba\mathbf{v}_1a\mathbf{v}_2b\mathbf{v}_3 \qquad \text{by the identity (2.14)}$$
$$= \mathbf{w}'ba\mathbf{w}''$$

hold in $\mathbf{J}$. It follows that $\mathbf{J}$ satisfies the identity $\mathbf{w} \approx \mathbf{w}'ba\mathbf{w}''$.

The case when the latest occurrence of b and the second occurrence of a in $\mathbf{w}$ lie in the same block is considered similarly.

(ii) Since $a \in \mathrm{mul}(\mathbf{w})$, Lemma 3.3 implies that $\mathbf{J}$ satisfies the identities

$$\mathbf{w} = \mathbf{v}_1b\mathbf{v}_2ab\mathbf{w}'' \approx \mathbf{v}_1b^2\mathbf{v}_2a\mathbf{w}'' \approx \mathbf{v}_1b\mathbf{v}_2ba\mathbf{w}'' = \mathbf{w}'ba\mathbf{w}'',$$

i.e., the identity $\mathbf{w} \approx \mathbf{w}'ba\mathbf{w}''$. □

The following assertion is evident.

Remark 3.5. *Let $\mathbf{v} \in \{xzyx^pty^q,\ yx^pty^q,\ xyzx^pty^q \mid p,q \in \mathbb{N}\}$. If $a, b \in \mathrm{con}(\mathbf{v})$ and $a \neq b$ then the subword ab of the word $\mathbf{v}$ has exactly one occurrence in this word.* □

Lemma 3.6. *Let p_1 and p_2 be natural numbers.*

(i) *If the variety $\mathbf{J}$ satisfies an identity $xzyx^{p_1}ty^{p_2} \approx \mathbf{w}$ then $\mathbf{w} = xzyx^{q_1}ty^{q_2}$ for some natural numbers q_1 and q_2.*
(ii) *If the variety $\mathbf{I}$ satisfies an identity $yx^{p_1}ty^{p_2} \approx \mathbf{w}$ then $\mathbf{w} = yx^{q_1}ty^{q_2}$ for some natural numbers q_1 and q_2.*
(iii) *If the variety $\mathbf{H}$ satisfies an identity $xyzx^{p_1}ty^{p_2} \approx \mathbf{w}$ then $\mathbf{w} = xyzx^{q_1}ty^{q_2}$ for some natural numbers q_1 and q_2.*

Proof. Put

$$\Psi = \{\Phi,\ xyxztx \approx xyxzxtx,\ \mathbf{w}_n[\pi] \approx \mathbf{w}_n'[\pi] \mid n \in \mathbb{N},\ \pi \in S_n\}.$$

(i) Put $\mathbf{v} = xzyx^{p_1}ty^{p_2}$. By Lemma 2.2 and induction, we can reduce our considerations to the case when either $\mathbf{v} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$, $\mathbf{w} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ or $\mathbf{v} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$, $\mathbf{w} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in F^1$, $\xi \in \mathrm{End}(F^1)$ and $\mathbf{s} \approx \mathbf{t} \in \Psi$. We can assume without loss of generality that the words $\mathbf{v}$ and $\mathbf{w}$ are different.

If $\xi(x)$ is the empty word then $\xi(\mathbf{s}) = \xi(\mathbf{t})$, but this is impossible because $\mathbf{v} \neq \mathbf{w}$. Thus, $\xi(x) \neq \lambda$. Then, since $\mathrm{con}(\xi(x)) \subseteq \mathrm{mul}(\xi(\mathbf{s})) \subseteq \mathrm{mul}(\mathbf{v})$, Remark 3.5 implies that $\xi(x) = c^k$ for some $k \in \mathbb{N}$ and $c \in \{x, y\}$.

The identity (2.15) allows us to add and delete the occurrences of the letter x between the second and the latest occurrences of this letter, while the identity (2.14) allows us to add and delete the occurrences of the letter x next to the non-first occurrence of this letter. This implies that if $\mathbf{s} \approx \mathbf{t}$ coincides with one of the identities (2.14) or (2.15) then $\mathbf{w} = xzyx^{q_1}ty^{q_2}$ for some $q_1, q_2 \in \mathbb{N}$.

Suppose now that $\mathbf{s} \approx \mathbf{t} \in \{x^2y^2 \approx y^2x^2,\ xyzxy \approx yxzxy\}$. Note that $\xi(y) \neq \lambda$ because the identity $\mathbf{v} \approx \mathbf{w}$ is non-trivial. Then $\xi(y) = d^{k'}$ for some letter d and some natural number k' by Remark 3.5. Since the words $\mathbf{v}$ and $\mathbf{w}$ are different, $c \neq d$. Corollary 2.5 implies that $\mathbf{w} = xzx^ryx^{q_1}ty^{q_2}$ for some numbers q_1, q_2 and r. The case when the identity $\mathbf{s} \approx \mathbf{t}$ coincides with the identity (2.10) is impossible because the word $\mathbf{v}$ does not contain any subword of the form $c^{2k}d^{2k'}$ and $d^{2k'}c^{2k}$. The identity $\mathbf{s} \approx \mathbf{t}$ can not also coincide with the identity $xyzxy \approx yxzxy$ because the words $\mathbf{v}$ and $\mathbf{w}$ may contain at most one occurrence of the word $c^kd^{k'}$.

So, it remains to consider the case when the identity $\mathbf{s} \approx \mathbf{t}$ coincides with the identity $\mathbf{w}_n[\pi] \approx \mathbf{w}_n'[\pi]$ for some $n \in \mathbb{N}$ and $\pi \in S_n$. If $\mathbf{v} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$

then

$$xzyx^{p_1}ty^{p_2} = \mathbf{a}\,c^k\xi(z_{1\pi}z_{2\pi}\cdots z_{n\pi})c^k\xi\Bigg(\prod_{i=1}^{n} t_i z_i^{\ell_i}\Bigg)\,\mathbf{b},$$

whence $\xi(z_{1\pi}z_{2\pi}\cdots z_{n\pi}) = c^h$ for some $h \geq 0$. This contradicts the fact that the identity $\mathbf{v} \approx \mathbf{w}$ is non-trivial. If $\mathbf{v} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ then

$$xzyx^{p_1}ty^{p_2} = \mathbf{a}\,c^{2k}\xi(z_{1\pi}z_{2\pi}\cdots z_{n\pi})\xi\Bigg(\prod_{i=1}^{n} t_i z_i^{\ell_i}\Bigg)\,\mathbf{b},$$

whence $\xi(z_{1\pi}z_{2\pi}\cdots z_{n\pi}) = c^h$ for some $h \geq 0$. We obtain a contradiction with the inequality $\mathbf{v} \neq \mathbf{w}$ again.

(ii) Put $\mathbf{v} = yx^{p_1}ty^{p_2}$. By Lemma 2.2 and induction, we can reduce our considerations to the case when either $\mathbf{v} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$, $\mathbf{w} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ or $\mathbf{v} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$, $\mathbf{w} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in F^1$, $\xi \in \mathrm{End}(F^1)$ and $\mathbf{s} \approx \mathbf{t} \in \{\Psi,\, xzxyty \approx xzyxty\}$. We can assume without loss of generality that the words $\mathbf{v}$ and $\mathbf{w}$ are different.

Suppose that the $\mathbf{s} \approx \mathbf{t} \in \Psi$. Corollary 2.5 implies that $\mathbf{w} = x^r yx^{q_1}ty^{q_2}$ for some numbers q_1, q_2 and r. If $r > 0$ then we multiply the identity $\mathbf{v} \approx \mathbf{w}$ by xz on the left and obtain a contradiction with the claim (i).

So, we can assume that $\mathbf{s} \approx \mathbf{t}$ equals the identity (2.13). Since the identity $\mathbf{v} \approx \mathbf{w}$ is non-trivial, we have that $\xi(x) \neq \lambda$, $\xi(y) \neq \lambda$ and $\xi(x) \neq \xi(y)$. Then $\xi(x) = c^k$ and $\xi(y) = d^k$ for some $k, k' \in \mathbb{N}$ by Remark 3.5. If $\mathbf{v} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ then

$$\mathbf{v} = yx^{p_1}ty^{p_2} = \mathbf{a}c^k\xi(z)c^k d^{k'}\xi(t)d^{k'}\mathbf{b}.$$

Then $\{x, y\} = \{c, d\}$. The case when $y = c$ and $x = d$ is impossible because the second occurrence of y is preceded the latest occurrence of x in $\mathbf{v}$. So, $x = c$ and $y = d$. This implies that $\xi(z) = \lambda$ and, therefore, $yx^{p_1}ty^{p_2} = \mathbf{a}x^{2k}y^{k'}\xi(t)y^{k'}\mathbf{b}$, a contradiction. If $\mathbf{v} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ then

$$\mathbf{v} = yx^{p_1}ty^{p_2} = \mathbf{a}c^k\xi(z)d^{k'}c^k\xi(t)d^{k'}\mathbf{b}.$$

But this equality is impossible too. The claim (ii) is proved.

(iii) Put $\mathbf{v} = xyzx^{p_1}ty^{p_2}$. As in the proof of the claims (i) and (ii), by Lemma 2.2 and induction, we can reduce our considerations to the case when either $\mathbf{v} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$, $\mathbf{w} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ or $\mathbf{v} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$, $\mathbf{w} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in F^1$, $\xi \in \mathrm{End}(F^1)$ and $\mathbf{s} \approx \mathbf{t} \in \{\Psi,\, xyxty \approx yx^2ty\}$. We can assume without loss of generality that the words $\mathbf{v}$ and $\mathbf{w}$ are different.

Suppose that $\mathbf{s} \approx \mathbf{t} \in \Psi$. Corollary 2.5 implies that

$$\mathbf{w} \in \{xyzx^{q_1}ty^{q_2},\, yxzx^{q_1}ty^{q_2} \mid q_1, q_2 \in \mathbb{N}\}.$$

If $\mathbf{w} = yxzx^{q_1}ty^{q_2}$ for some $q_1, q_2 \in \mathbb{N}$ then we substitute 1 for z in the identity $\mathbf{v} \approx \mathbf{w}$ and obtain a contradiction with the claim (ii).

So, we can assume that $\mathbf{s} \approx \mathbf{t}$ coincides with the identity (3.8). Since the identity $\mathbf{v} \approx \mathbf{w}$ is non-trivial, we have that $\xi(x) \neq \lambda$, $\xi(y) \neq \lambda$ and $\xi(x) \neq \xi(y)$. Then $\xi(x) = c^k$ and $\xi(y) = d^k$ for some $k, k' \in \mathbb{N}$ by Remark 3.5. If $\mathbf{v} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ then

$$\mathbf{v} = xyzx^{p_1}ty^{p_2} = \mathbf{a}c^k d^{k'}c^k\xi(t)d^{k'}\mathbf{b}.$$

It is easy to see that this equality is impossible. If $\mathbf{v} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ then

$$\mathbf{v} = xyzx^{p_1}ty^{p_2} = \mathbf{a}d^{k'}c^{2k}\xi(t)d^{k'}\mathbf{b}.$$

But this equality is impossible too. The claim (iii) is proved. □

If $x \in \mathrm{con}(\mathbf{w})$ and $i \le \mathrm{occ}_x(\mathbf{w})$ then $\ell_i(\mathbf{w}, x)$ denotes the length of the minimal prefix $\mathbf{p}$ of $\mathbf{w}$ with $\mathrm{occ}_x(\mathbf{p}) = i$.

Proof of Proposition 3.1. In view of Lemma 3.2, it remains to verify that the interval $[\mathbf{F} \vee \overleftarrow{\mathbf{E}}, \mathbf{J}]$ is the chain $\mathbf{F} \vee \overleftarrow{\mathbf{E}} \subset \mathbf{H} \subset \mathbf{I} \subset \mathbf{J}$. Lemmas 2.8 and 3.6 imply that $\mathbf{F} \vee \overleftarrow{\mathbf{E}} \subset \mathbf{H} \subset \mathbf{I} \subset \mathbf{J}$. So, it remains to verify that if $\mathbf{V}$ is a monoid variety such that $\mathbf{F} \vee \overleftarrow{\mathbf{E}} \subseteq \mathbf{V} \subset \mathbf{J}$ then $\mathbf{V}$ coincides with one of the varieties $\mathbf{F} \vee \overleftarrow{\mathbf{E}}$, $\mathbf{H}$ or $\mathbf{I}$. Since $\mathbf{V} \subset \mathbf{J}$, there is a non-trivial identity $\mathbf{u} \approx \mathbf{v}$ that holds in $\mathbf{V}$ but does not hold in $\mathbf{J}$. The identities (2.14) and (2.15) allow us to assume that the claim (2.16) is true. In view of Corollary 2.6 and inclusion $\mathbf{E} \subseteq \mathbf{V}$, if (2.7) is the decomposition of $\mathbf{u}$ then the decomposition of $\mathbf{v}$ has the form (2.8). This fact, Corollary 2.5 and the claim (2.16) imply that the claim (2.17) is true. Since the identity $\mathbf{u} \approx \mathbf{v}$ is non-trivial, there is $0 \le i \le m$ such that $\mathbf{u}_i \ne \mathbf{v}_i$. Let $\mathbf{p}$ be the greatest common prefix of the words $\mathbf{u}_i$ and $\mathbf{v}_i$. Suppose that $\mathbf{u}_i = \mathbf{p}x\mathbf{u}_i'$ for some letter x and some word $\mathbf{u}_i'$. The claim (2.17) implies that there are words $\mathbf{a}, \mathbf{b}$ and the letter y such that $\mathbf{v}_i = \mathbf{pa}yx\mathbf{b}$ and $x \notin \mathrm{con}(\mathbf{a}y)$. We note also that $y \in \mathrm{con}(\mathbf{u}_i')$ by the claim (2.17).

By induction we can assume without loss of generality that $\mathbf{J}$ violates the identity

$$\mathbf{v} \approx \mathbf{v}'\,\mathbf{pa}\,xy\,\mathbf{b}\,\mathbf{v}'', \tag{3.6}$$

where $\mathbf{v}' = \mathbf{v}_0 t_1 \mathbf{v}_1 t_2 \cdots \mathbf{v}_{i-1} t_i$ and $\mathbf{v}'' = t_{i+1}\mathbf{v}_{i+1} \cdots t_m \mathbf{v}_m$. Then the letters x and y are non-integrated in the word $\mathbf{v}$ by Lemma 3.4(i). Then either the third occurrence of x precedes the second occurrence of y in $\mathbf{v}$ or the third occurrence of y precedes the second occurrence of x.

Case 1: the third occurrence of x precedes the second occurrence of y in $\mathbf{v}$. Then there is j such that $\ell_3(\mathbf{v}, x) < \ell_1(\mathbf{v}, t_j) < \ell_2(\mathbf{v}, y)$. In view of Corollary 2.5, $\ell_3(\mathbf{u}, x) < \ell_1(\mathbf{u}, t_j) < \ell_2(\mathbf{u}, y)$. It follows that $y \notin \mathrm{con}(\mathbf{v}'\mathbf{pa})$.

First, we are going to verify that $\mathbf{V} \subseteq \mathbf{I}$. Suppose that $x \notin \mathrm{con}(\mathbf{v}'\mathbf{p})$. Then $\mathbf{V}$ satisfies the identities

$$x^s y x^r t_j y^2 = \mathbf{u}(x, y, t_j) \approx \mathbf{v}(x, y, t_j) = y x^3 t y^2$$

for some $s \ge 1$ and $r \ge 0$. Now we substitute xt for t_j in these identities and obtain the identity $x^s y x^{r+1} t y^2 \approx y x^4 t y^2$. Then, since the identity

$$xyxty \approx x^2 yty \tag{3.7}$$

holds in the variety $\mathbf{V}$, this variety satisfies

$$xyxty \stackrel{(2.14)}{\approx} xyx^{s+r}ty^2 \stackrel{(3.7)}{\approx} x^s y x^{r+1} t y^2 \approx y x^4 t y^2 \stackrel{(2.14)}{\approx} yx^2ty,$$

i.e., the identity

$$xyxty \approx yx^2ty. \tag{3.8}$$

Clearly, the identity (2.13) follows from the identities (2.14) and (3.8), whence $\mathbf{V} \subseteq \mathbf{I}$.

Suppose now that $x \in \mathrm{sim}(\mathbf{v}'\mathbf{p})$. Then Lemma 3.4(ii) and the fact that $\mathbf{J}$ violates the identity (3.6) imply that the first and the second occurrences of x in $\mathbf{v}$ lie in different blocks in $\mathbf{v}$, whence $\ell_1(\mathbf{v}, x) < \ell_1(\mathbf{v}, t_i)$. Taking into account Corollary 2.5, we have that $\ell_1(\mathbf{u}, x) < \ell_1(\mathbf{u}, t_i)$. If the third occurrence of x precedes the first occurrence of y in $\mathbf{u}$ then $\mathbf{V}$ satisfies the identities

$$xt_ixyt_jy \stackrel{(2.14)}{\approx} xt_ix^2yt_jy^2 = \mathbf{u}(x, y, t_i, t_j) \approx \mathbf{v}(x, y, t_i, t_j) = xt_iyx^2t_jy^2 \stackrel{(2.14)}{\approx} xt_iyxt_jy.$$

We rename the letters in these identities and obtain that the identity (2.13) holds in $\mathbf{V}$. If the third occurrence of x is preceded the first occurrence of y in $\mathbf{u}$ then $\mathbf{V}$ satisfies

$$\begin{aligned} xt_ixyt_jy &\stackrel{(2.14)}{\approx} xt_ix^2yt_jy^2 \stackrel{(3.7)}{\approx} xt_ixyxt_jy^2 = \mathbf{u}(x, y, t_i, t_j) \\ &\approx \mathbf{v}(x, y, t_i, t_j) = xt_iyx^2t_jy^2 \stackrel{(2.14)}{\approx} xt_iyxt_jy, \end{aligned}$$

i.e., the identity (2.13).

Finally, suppose that $x \in \mathrm{con}_2(\mathbf{v}'\mathbf{p})$. Then the identities

$$\mathbf{v}'\,\mathbf{pa}\,yx\,\mathbf{b}\,\mathbf{v}'' \stackrel{(2.15)}{\approx} \mathbf{v}'\,\mathbf{pa}\,xyx\,\mathbf{b}\,\mathbf{v}'' \stackrel{(3.7)}{\approx} \mathbf{v}'\,\mathbf{pa}\,x^2y\,\mathbf{b}\,\mathbf{v}'' \stackrel{(2.14)}{\approx} \mathbf{v}'\,\mathbf{pa}\,xy\,\mathbf{b}\,\mathbf{v}'' = \mathbf{v}$$

hold in the variety $\mathbf{J}$. We obtain a contradiction with the fact that this variety violates the identity (3.6). So, we have proved that $\mathbf{V} \subseteq \mathbf{I}$.

Suppose now that $\mathbf{V} \subset \mathbf{I}$. Then we can believe that the identity (3.6) does not hold in the variety $\mathbf{I}$. We will prove that $\mathbf{V} \subseteq \mathbf{H}$. Note that the identity (2.13) allows us to swap a non-latest occurrence and a non-first occurrence of two multiple letters whenever these occurrences are adjacent to each other. Hence $x \notin \mathrm{con}(\mathbf{v}'\mathbf{p})$. Then we can verify that $\mathbf{V}$ satisfies the identity (3.8) (see the second paragraph of Case 1), and therefore, is contained in $\mathbf{H}$.

Suppose now that $\mathbf{V} \subset \mathbf{H}$. Then we can believe that the identity (3.6) does not hold in the variety $\mathbf{H}$. We will prove that $\mathbf{V} = \mathbf{F} \vee \overleftarrow{\mathbf{E}}$. Arguments similar to those from the previous paragraph imply that $x \notin \mathrm{con}(\mathbf{v}'\mathbf{p})$. If the first and the second occurrence of x in $\mathbf{v}$ lie in the same block then $\mathbf{b} = \mathbf{b}'x\mathbf{b}''$ for some words $\mathbf{b}'$ and $\mathbf{b}''$. Then we have:

$$\begin{aligned} \mathbf{v}'\mathbf{pa}yx\mathbf{b}'x\mathbf{b}''\mathbf{v}'' &\approx \mathbf{v}'\mathbf{pa}yx^2\mathbf{b}'\mathbf{b}''\mathbf{v}'' && \text{by Lemma 3.3} \\ &\approx \mathbf{v}'\mathbf{pa}xyx\mathbf{b}'\mathbf{b}''\mathbf{v}'' && \text{by the identity (3.8)} \\ &\approx \mathbf{v}'\mathbf{pa}x^2y\mathbf{b}'\mathbf{b}''\mathbf{v}'' && \text{by the identity (3.7)} \\ &\approx \mathbf{v}'\mathbf{pa}xy\mathbf{b}'x\mathbf{b}''\mathbf{v}''. && \text{by Lemma 3.3.} \end{aligned}$$

This contradicts the fact that the variety $\mathbf{H}$ violates the identity (3.6). Therefore, the first and the second occurrence of x in $\mathbf{v}$ lie in different blocks. Then there is $i < k < j$ such that $\ell_1(\mathbf{v}, t_k) < \ell_2(\mathbf{v}, x)$. In view of Corollary 2.5, $\ell_1(\mathbf{u}, t_k) < \ell_2(\mathbf{u}, x)$. Then $\mathbf{V}$ satisfies the identities

$$xyt_kxt_jy \stackrel{(2.14)}{\approx} xyt_kx^2t_jy^2 = \mathbf{u}(x, y, t_i, t_j) \approx \mathbf{v}(x, y, t_i, t_j) = yxt_kx^2t_jy^2 \stackrel{(2.14)}{\approx} yxt_kxt_jy.$$

So, the identity (2.12) holds in $\mathbf{V}$. Hence $\mathbf{V} = \mathbf{F} \vee \overleftarrow{\mathbf{E}}$ by Lemma 2.8.

Case 2: the third occurrence of y precedes the second occurrence of x in $\mathbf{v}$. This case is considered similarly to the previous one. Then there is j such that $\ell_3(\mathbf{v}, y) < \ell_1(\mathbf{v}, t_j) < \ell_2(\mathbf{v}, x)$. In view of Corollary 2.5, $\ell_3(\mathbf{u}, y) < \ell_1(\mathbf{u}, t_j) < \ell_2(\mathbf{u}, x)$.

First, we are going to verify that $\mathbf{V} \subseteq \mathbf{I}$. Suppose that $y \notin \mathrm{con}(\mathbf{v}'\mathbf{pa})$. Then $\mathbf{V}$ satisfies the identities

$$xy^2t_jx \stackrel{(2.14)}{\approx} xy^3t_jx^2 = \mathbf{u}(x,y,t_j) \approx \mathbf{v}(x,y,t_j) = yxy^2tx^2 \stackrel{(2.14)}{\approx} yxytx.$$

We rename the letters in these identities and obtain that the identity (3.8) holds in $\mathbf{V}$. Clearly, the identity (2.13) follows from the identity (3.8), whence $\mathbf{V} \subseteq \mathbf{I}$.

Suppose now that $y \in \mathrm{sim}(\mathbf{v}'\mathbf{pa})$. Then Lemma 3.4(i) and the fact that $\mathbf{J}$ violates the identity (3.6) imply that $x \notin \mathrm{con}(\mathbf{v}'\mathbf{pab})$. Besides that, Lemma 3.4(ii) and the fact that $\mathbf{J}$ violates the identity (3.6) imply that the first and the second occurrences of y in $\mathbf{v}$ lie in different blocks in $\mathbf{v}$, whence $\ell_1(\mathbf{v},y) < \ell_1(\mathbf{v},t_i)$. Taking into account Corollary 2.5, we have that $\ell_1(\mathbf{u},y) < \ell_1(\mathbf{u},t_i)$. Then $\mathbf{V}$ satisfies the identities

$$\begin{aligned} yt_ixyt_jx &\stackrel{(2.14)}{\approx} yt_ixy^2t_jx^2 = \mathbf{u}(x,y,t_i,t_j) \approx \mathbf{v}(x,y,t_i,t_j) \\ &= yt_iyxyt_jx^2 \stackrel{(3.7)}{\approx} xt_iy^2xt_jy^2 \approx yt_iyxt_jx. \end{aligned}$$

We rename the letters in these identities and obtain that the identity (2.13) holds in $\mathbf{V}$, whence $\mathbf{V} \subseteq \mathbf{I}$.

Finally, suppose that $y \in \mathrm{con}_2(\mathbf{v}'\mathbf{pa})$. Then the identities

$$\mathbf{v}'\,\mathbf{pa}\,yx\,\mathbf{b}\,\mathbf{v}'' \stackrel{(2.14)}{\approx} \mathbf{v}'\,\mathbf{pa}\,y^2x\,\mathbf{b}\,\mathbf{v}'' \stackrel{(3.7)}{\approx} \mathbf{v}'\,\mathbf{pa}\,yxy\,\mathbf{b}\,\mathbf{v}'' \stackrel{(2.15)}{\approx} \mathbf{v}'\,\mathbf{pa}\,xy\,\mathbf{b}\,\mathbf{v}'' = \mathbf{v}$$

hold in the variety $\mathbf{J}$. We obtain a contradiction with the fact that this variety violates the identity (3.6). So, we have proved $\mathbf{V} \subseteq \mathbf{I}$.

Suppose now that $\mathbf{V} \subset \mathbf{I}$. Then we can believe that the identity (3.6) does not hold in the variety $\mathbf{I}$. We will prove that $\mathbf{V} \subseteq \mathbf{H}$. Since the identity (2.13) allows us to swap a non-latest occurrence and a non-first occurrence of two multiple letters whenever these occurrences are adjacent to each other, we have $y \notin \mathrm{con}(\mathbf{v}'\mathbf{pa})$. Then we can verify that $\mathbf{V}$ satisfies the identity (3.8) (see the second paragraph of Case 2), and therefore, is contained in $\mathbf{H}$.

Suppose now that $\mathbf{V} \subset \mathbf{H}$. Then we can believe that the identity (3.6) does not hold in the variety $\mathbf{H}$. We will prove that $\mathbf{V} = \mathbf{F} \vee \overleftarrow{\mathbf{E}}$. Arguments similar to those from the previous paragraph imply that $y \notin \mathrm{con}(\mathbf{v}'\mathbf{pa})$. If the first and the second occurrence of y in $\mathbf{v}$ lie in the same block then $\mathbf{b} = \mathbf{b}'y\mathbf{b}''$ for some words $\mathbf{b}'$ and $\mathbf{b}''$. Then we have:

$$\begin{aligned} \mathbf{v}'\mathbf{pa}yx\mathbf{b}'y\mathbf{b}''\mathbf{v}'' &\approx \mathbf{v}'\mathbf{pa}y^2x\mathbf{b}'\mathbf{b}''\mathbf{v}'' && \text{by Lemma 3.3} \\ &\approx \mathbf{v}'\mathbf{pa}yxy\mathbf{b}'\mathbf{b}''\mathbf{v}'' && \text{by the identity (3.7)} \\ &\approx \mathbf{v}'\mathbf{pa}xy^2\mathbf{b}'\mathbf{b}''\mathbf{v}'' && \text{by the identity (3.8)} \\ &\approx \mathbf{v}'\mathbf{pa}xy\mathbf{b}'y\mathbf{b}''\mathbf{v}''. && \text{by Lemma 3.3.} \end{aligned}$$

This contradicts the fact that the variety $\mathbf{H}$ violates the identity (3.6). Therefore, the first and the second occurrences of y in $\mathbf{v}$ lie in different blocks. Then there is $i < k < j$ such that $\ell_1(\mathbf{v},t_k) < \ell_2(\mathbf{v},y)$. In view of Corollary 2.5, $\ell_1(\mathbf{u},t_k) < \ell_2(\mathbf{u},y)$. Then $\mathbf{V}$ satisfies the identities

$$xyt_kyt_jx \stackrel{(2.14)}{\approx} xyt_ky^2t_jx^2 = \mathbf{u}(x,y,t_i,t_j) \approx \mathbf{v}(x,y,t_i,t_j) = yxt_ky^2t_jx^2 \stackrel{(2.14)}{\approx} yxt_kyt_jx.$$

It follows that the identity (2.12) holds in $\mathbf{V}$. Hence $\mathbf{V} = \mathbf{F} \vee \overleftarrow{\mathbf{E}}$ by Lemma 2.8.

Proposition 3.1 is proved. □

4. Proof of Theorem 1.1

Put

$$\mathbf{u}_n[p, \ell_1, \ell_2, \dots, \ell_n] = xz_1z_2\cdots z_nx^p\left(\prod_{i=1}^{n} t_iz_i^{\ell_i}\right)$$

for every positive integers n, p, $\ell_1, \ell_2, \dots, \ell_n$. We note that $\mathbf{u}_n[1, 1, \dots, 1] = \mathbf{w}_n[\varepsilon]$ where ε denotes the identity element of S_n.

The following statement is evident.

Remark 4.1. *Let* $\mathbf{v} = \mathbf{u}_n[p, \ell_1, \ell_2, \dots, \ell_n]$ *for some natural numbers* n, p, $\ell_1, \ell_2, \dots, \ell_n$. *If* $a, b \in \mathrm{con}(\mathbf{v})$ *and* $a \ne b$ *then the subword* ab *of the word* $\mathbf{v}$ *has exactly one occurrence in this word.* □

Proof of Theorem 1.1. It follows from [10, Proposition 3.1] that the variety $\mathbf{J}$ is locally finite. Since the variety $\mathbf{J}$ is small by Proposition 3.1, Lemma 2.1 implies that this variety is finitely generated.

In view of Proposition 3.1, the varieties $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\overleftarrow{\mathbf{E}}$, $\mathbf{E}\vee\overleftarrow{\mathbf{E}}$, $\mathbf{F}$, $\mathbf{F}\vee\overleftarrow{\mathbf{E}}$, $\mathbf{H}$, $\mathbf{I}$ are the only proper subvarieties of $\mathbf{J}$. The varieties $\mathbf{E}\vee\overleftarrow{\mathbf{E}}$ and $\mathbf{F}\vee\overleftarrow{\mathbf{E}}$ are finitely based by Lemmas 2.7 and 2.8 respectively. The remaining of the listed varieties are finitely based by their definitions.

We note that to prove that every proper subvariety of $\mathbf{J}$ is finitely based, one does not need to describe the whole lattice $L(\mathbf{J})$. It is sufficient to establish only the fact that $\mathbf{I}$ is the unique maximal subvariety of $\mathbf{J}$. Indeed, it is proved in [8, Theorem 1.1(i)] that every monoid variety that satisfies the identities (2.11) and (2.12) is finitely based. Obviously, the identities (2.11) and (2.12) hold in $\mathbf{I}$. Taking into account the fact that $\mathbf{I}$ is the unique maximal subvariety of $\mathbf{J}$, we obtain that every proper subvariety of $\mathbf{J}$ is finitely based. We have decided to describe the lattice $L(\mathbf{J})$ to prove the fact that the variety $\mathbf{J}$ is limit, because this description is of certain independent interest and may be useful in further research.

So, it remains to verify that $\mathbf{J}$ is non-finitely based. Arguing by contradiction, we suppose that $\mathbf{J}$ has a finite basis of identities Σ. Let k be a maximum of length of left-hand or right-hand sides of the identities from Σ. We are going to verify that if $n > k$ then the identity system Σ does not imply the identity $\mathbf{w}_n[\varepsilon] \approx \mathbf{w}'_n[\varepsilon]$. To establish this fact, it suffices to prove that if an identity $\mathbf{u}_n[p, k_1, k_2, \dots, k_n] \approx \mathbf{w}$ follows from the identity system Σ then $\mathbf{w} = \mathbf{u}_n[q, \ell_1, \ell_2, \dots, \ell_n]$ for some natural numbers $q, \ell_1, \ell_2, \dots, \ell_n$. Put $\mathbf{v} = \mathbf{u}_n[p, k_1, k_2, \dots, k_n]$. By Lemma 2.2 and induction, we can reduce our considerations to the case when $\mathbf{v} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ and $\mathbf{w} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in F^1$, $\xi \in \mathrm{End}(F^1)$ and $\mathbf{s} \approx \mathbf{t} \in \Sigma$. We can assume without loss of generality that the words $\mathbf{v}$ and $\mathbf{w}$ are different.

In view of Corollary 2.5,

$$\mathbf{w} = \mathbf{a}\xi(\mathbf{t})\mathbf{b} = \mathbf{u}\left(\prod_{i=1}^{n} t_iz_i^{\ell_i}\right)$$

for some natural numbers $\ell_1, \ell_2, \dots, \ell_n$ and some word $\mathbf{u}$ such that $\mathrm{sim}(\mathbf{u}) = \{z_1, z_2, \dots, z_n\}$ and $\mathrm{mul}(\mathbf{u}) = \{x\}$. Lemma 3.6(iii) implies that $\mathbf{u}(z_1, z_2, \dots, z_n) = z_1z_2\cdots z_n$. Besides that, the first occurrence of x in $\mathbf{u}$ precedes the first occurrence of z_1 in $\mathbf{u}$ by Lemma 3.6(ii).

If the word $xz_1z_2\cdots z_n$ is a prefix of the word $\mathbf{a}$ then the required conclusion is evident. Suppose now that $\mathbf{a} = xz_1z_2\cdots z_i$ for some $0 \le i < n$ (if $i = 0$ then we mean that $\mathbf{a} = x$). Then

$$\xi(\mathbf{s})\mathbf{b} = z_{i+1}z_{i+2}\cdots z_n x^p \left(\prod_{i=1}^{n} t_i z_i^{k_i}\right) \text{ and } \xi(\mathbf{t})\mathbf{b} = \mathbf{u}'\left(\prod_{i=1}^{n} t_i z_i^{\ell_i}\right)$$

for some suffix $\mathbf{u}'$ of the word $\mathbf{u}$. Since the identity $\xi(\mathbf{s})\mathbf{b} \approx \xi(\mathbf{t})\mathbf{b}$ holds in $\mathbf{J}$, the inclusion $\mathbf{F} \vee \overleftarrow{\mathbf{E}} \subset \mathbf{J}$ and Corollary 2.5 imply that $\mathrm{con}(\mathbf{u}') = \{x, z_{i+1}, z_{i+2}, \ldots, z_n\}$. In view of Lemma 3.6(iii), $\mathbf{u}'(z_{i+1}, z_{i+2}, \ldots, z_n) = z_{i+1}z_{i+2}\cdots z_n$. If $p = 1$ then $\mathbf{u}'(x, z_n) = z_n x$ by Corollary 2.5, and we obtain a contradiction with the inequality $\mathbf{v} \neq \mathbf{w}$. If $p > 1$ then Lemma 3.6(ii) implies that $\mathbf{u}'(x, z_n) = z_n x^q$ for some $q > 1$, whence $\mathbf{w} = \mathbf{u}_n[q, \ell_1, \ell_2, \ldots, \ell_n]$.

Finally, suppose that $\mathbf{a} = \lambda$. If $\xi(\mathbf{s}) = xz_1z_2\cdots z_j$ for some $0 \leq j \leq n$ then the letters $x, z_1, z_2, \ldots, z_j$ are the images of simple letters of the word $\mathbf{s}$. Then $\xi(\mathbf{t}) = xz_1z_2\cdots z_j$ by Corollary 2.5. Therefore, we can assume that the word $xz_1z_2\cdots z_n x$ is a prefix of $\xi(\mathbf{s})$.

Let $e_0\mathbf{s}_0e_1\mathbf{s}_1\cdots e_m\mathbf{s}_m$ be the decomposition of the word $\mathbf{s}$. Then the decomposition of the word $\mathbf{t}$ has the form $e_0\mathbf{t}_0e_1\mathbf{t}_1\cdots e_m\mathbf{t}_m$ by the inclusion $\mathbf{E} \subset \mathbf{J}$ and Corollary 2.6. Remark 4.1 and the fact that the length of the word $\mathbf{s}$ is less than n imply that there is $j \in \{1, 2, \ldots n\}$ such that $z_j \in \mathrm{con}(\xi(e_i))$ for some $i \in \{1, 2, \ldots, m\}$. We can assume that j is the greatest number with this property. Let $j' \leq j$ be the least number with $z_{j'} \in \mathrm{con}(\xi(e_i))$. Then $xz_1z_2\cdots z_{j'-1} = \xi(e_0\mathbf{s}_0e_1\mathbf{s}_1\cdots e_{i-1}\mathbf{s}_{i-1})$ and the word $z_{j'}z_{j'+1}\cdots z_j$ is a prefix of $\xi(e_i)$. In view of Corollary 2.5, we have that $xz_1z_2\cdots z_{j'-1} = \xi(e_0\mathbf{t}_0e_1\mathbf{t}_1\cdots e_{i-1}\mathbf{t}_{i-1})$. If $j = n$ then we obtain the required conclusion. So, we can assume that $j < n$. Then $z_n \in \mathrm{con}(\xi(a))$ for some $a \in \mathrm{mul}(\mathbf{s})$. It follows that there is r such that $i < r$, $\ell_1(\mathbf{s}, e_r) < \ell_2(\mathbf{s}, a)$ and $t_1 \in \mathrm{con}(\xi(e_r))$. In view of Remark 4.1, $\xi(a) = z_n$.

Suppose that $\ell_2(\mathbf{u}, x) < \ell_1(\mathbf{u}, z_n)$. Clearly, $\ell_1(\mathbf{u}, z_j) < \ell_2(\mathbf{u}, x)$. Then there is $b \in \mathrm{mul}(\mathbf{t}) = \mathrm{mul}(\mathbf{s})$ such that $x \in \mathrm{con}(\xi(b))$. Remark 4.1 implies that $\xi(b) = x^h$ for some $h \in \mathbb{N}$. Clearly, b does not occur in the words $\mathbf{s}_r, \mathbf{t}_r, \mathbf{s}_{r+1}, \mathbf{t}_{r+1}, \ldots, \mathbf{s}_m, \mathbf{t}_m$. If $\ell_1(\mathbf{s}, a) < \ell_1(\mathbf{s}, b)$ then $\mathbf{J}$ satisfies the identities

$$ab^{f_1}e_ra^{f_2} = \mathbf{s}(a, b, e_r) \approx \mathbf{t}(a, b, e_r) = b^{g_1}ab^{g_2}e_ra^{g_3}$$

for some $f_2, g_1, g_3 \in \mathbb{N}$, $f_1 \geq 2$ and $g_2 \geq 0$. This contradicts Lemma 3.6(ii). If $\ell_1(\mathbf{s}, b) < \ell_1(\mathbf{s}, a)$ then $b \in \mathrm{sim}(\mathbf{s}_0\mathbf{s}_1\cdots\mathbf{s}_{i-1})$. Taking into account Corollary 2.5, we obtain that $b \in \mathrm{sim}(\mathbf{t}_0\mathbf{t}_1\cdots\mathbf{t}_{i-1})$. Then $\mathbf{J}$ satisfies the identities

$$be_iab^{f_1}e_ra^{f_2} = \mathbf{s}(a, b, e_i, e_r) \approx \mathbf{t}(a, b, e_i, e_r) = be_ib^{g_1}ab^{g_2}e_ra^{g_3}$$

for some $f_1, f_2, g_1, g_3 \in \mathbb{N}$ and $g_2 \geq 0$. A contradiction with Lemma 3.6(i). Therefore, $\ell_1(\mathbf{u}, z_n) < \ell_2(\mathbf{u}, x)$. This implies that $\mathbf{u} = xz_1z_2\cdots z_nx^q$ for some q. □

Proof of Corollary 1.2. The variety $\mathbf{J}$ is non-Cross by Theorem 1.1. Let $\mathbf{X}$ be a proper subvariety of $\mathbf{J}$. In view of Theorem 1.1, $\mathbf{X}$ is finitely based. According to Proposition 3.1, $\mathbf{X}$ is small. Then $\mathbf{X}$ is finitely generated by Lemma 2.1. So, $\mathbf{X}$ is a Cross variety. □

Acknowledgments. The author is sincerely grateful to Professor Boris Vernikov for his attention and assistance in the writing of the article, to Dr. Edmond W.H. Lee for offering of a shorter proof of Theorem 1.1 and his numerous valuable suggestions for improving the manuscript and to the anonymous referee for several useful remarks.

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