Selectors for sequences of subsets of hyperspaces

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Abstract

For a Hausdorff space X we denote by 2^X the family of all closed subsets of X. In this paper we continue to research relationships between closure-type properties of hyperspaces over a space X and covering properties of X. We investigate selectors for sequence of subsets of the space 2^X with the $\mathbf{Z}^+$-topology and the upper Fell topology ($\mathbf{F}^+$-topology). Also we consider the selection properties of the bitopological space $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$.

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1. Introduction

Given a Hausdorff space X we denote by 2^X the family of all closed subsets of X. If A is a subset of X and $\mathcal{A}$ a family of subsets of X, then

$A^c = X \setminus A$ and $\mathcal{A}^c = \{A^c : A \in \mathcal{A}\}$,

$A^- = \{F \in 2^X : F \cap A \neq \emptyset\}$,

$A^+ = \{F \in 2^X : F \subset A\}$.

Let Δ be a subset of 2^X closed for finite unions and containing all singletons. We consider the next important cases:

- Δ is the collection $CL(X) = 2^X \setminus \{\emptyset\}$;
- Δ is the family $\mathbb{K}(X)$ of all non-empty compact subsets of X;
- Δ is the family $\mathbb{F}(X)$ of all non-empty finite subsets of X.

For $\Delta \subset 2^X$, the *upper* Δ*-topology*, denoted by Δ^+, is the topology whose base is the collection $\{(D^c)^+ : D \in \Delta\} \cup \{2^X\}$.

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When $\Delta = CL(X)$ we have the well-known *upper Vietoris topology* $\mathbf{V}^+$, when $\Delta = \mathbb{K}(X)$ we have the *upper Fell topology* (known also as the co-compact topology) $\mathbf{F}^+$, and when $\Delta = \mathbb{F}(X)$ we have the $\mathbf{Z}^+$-topology.

Many topological properties are defined or characterized in terms of the following classical selection principles ([4, 17, 19]). Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover of a space is *large* if each element of the space belongs to infinitely many elements of the cover.

An open cover $\mathcal{U}$ of a space X is called:

• an ω-cover (a k-cover) if each finite (compact) subset C of X is contained in an element of $\mathcal{U}$;

• a γ-cover (a γ_k-cover) if $\mathcal{U}$ is infinite and for each finite (compact) subset C of X the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

Because of these definitions all spaces are assumed to be *Hausdorff non-compact*, unless otherwise stated.

Let us mention that any ω-cover (k-cover) is infinite and large, and that any infinite subfamily of a γ-cover (γ_k-cover) is also a γ-cover (γ_k-cover).

For a topological space X we denote:

• $\mathcal{O}$ — the family of all open covers of X;
• Γ — the family of all open γ-covers of X;
• Γ_k — the family of all open γ_k-covers of X;
• Ω — the family of all open ω-covers of X;
• $\mathcal{K}$ — the family of all open k-covers of X.

Different Δ-covers (k-covers, ω-covers, k_F-covers, c_F-covers,...) exposed many dualities in hyperspace topologies such as co-compact topology $\mathbf{F}^+$, co-finite topology $\mathbf{Z}^+$, Pixley-Roy topology, Fell topology and Vietoris topology. They also play important roles in selection principles ([1,4-9,12-16]).

In this paper we continue to research relationships between closure-type properties of hyperspaces over a space X and covering properties of X. We investigate selectors for sequence of subsets of the space 2^X with the $\mathbf{Z}^+$-topology and the upper Fell topology ($\mathbf{F}^+$-topology). Also we consider the selection properties of the bitopological space $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$.

2. Main definitions and notation

The following lammas will be often used throughout the paper, sometimes without explicit reference.

Lemma 2.1. *(Lemma 1 in [1]) Let Y be an open subset of a space X and $\mathcal{U}$ an open cover of Y. Then the following holds:*

(1) $\mathcal{U}$ is a k-cover of $Y \Leftrightarrow Y^c \in Cl_{F^+}(\mathcal{U}^c)$;

(2) $\mathcal{U}$ is an ω-cover of $Y \Leftrightarrow Y^c \in Cl_{Z^+}(\mathcal{U}^c)$.

Lemma 2.2. *(Lemma 2 in [1]) Let X be a topological space, Y an open subsets of X and $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ an open cover of Y. Then*

(1) $\mathcal{U}$ is a γ_k-cover of $Y \Leftrightarrow$ the sequence $(U_n^c : n \in \mathbb{N})$ converges to Y^c in $(2^X, \mathbf{F}^+)$;

(2) $\mathcal{U}$ is an γ-cover of $Y \Leftrightarrow$ the sequence $(U_n^c : n \in \mathbb{N})$ converges to Y^c in $(2^X, \mathbf{Z}^+)$.

Lemma 2.3. *(Lemma 3 in [1]) Given a space X and an open cover $\mathcal{U}$ of X the following holds:*

(1) $\mathcal{U}$ is a k-cover of $X \Leftrightarrow \mathcal{U}^c$ is a dense subset of $(2^X, \mathbf{F}^+)$;

(2) $\mathcal{U}$ is an ω-cover of $X \Leftrightarrow \mathcal{U}^c$ is a dense subset of $(2^X, \mathbf{Z}^+)$.

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to x. So, simply Γ_x may be the set of non-trivial convergent sequences to x.

Definition 2.4. Let X be a space and let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then $\mathcal{U}^c = \{U_\alpha^c : \alpha \in \Lambda\}$ *converges to* $\{\emptyset\}$ in $(2^X, \tau)$ where τ is a topology on 2^X, if for every $F \in 2^X$ the $\mathcal{U}^c$ converges to F, i.e. for each neighborhood W of F in the space $(2^X, \tau)$, $|\{\alpha : U_\alpha^c \notin W, \alpha \in \Lambda\}| < \aleph_0$.

Lemma 2.5. *Let X be a space and let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then the following are equivalent:*

1. *$\mathcal{U}$ is an γ-cover of X;*
2. *$\mathcal{U}^c$ converges to $\{\emptyset\}$ in $(2^X, \mathbf{Z}^+)$.*

For a topological space $(2^X, \tau)$ we denote:
• $\mathcal{D}_\Omega$ — the family of dense subsets of $(2^X, \tau)$;
• $\mathcal{D}_\Gamma$ — the family of converging to $\{\emptyset\}$ subsets of $(2^X, \tau)$.

Since every γ-cover contains a countably γ-cover, then each converging to $\{\emptyset\}$ subset of $(2^X, \mathbf{Z}^+)$ contains a countable converging to $\{\emptyset\}$ subset of $(2^X, \mathbf{Z}^+)$.

3. Hyperspace $(2^X, \mathbf{Z}^+)$

Theorem 3.1. *Assume that $\Phi, \Psi \in \{\Gamma, \Omega\}$, $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

1. *X satisfies $S_\star(\Phi, \Psi)$;*
2. *$(2^X, \mathbf{Z}^+)$ satisfies $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$.*

Proof. We prove the theorem for $\star = fin$, the other proofs being similar.

(1) $\Rightarrow$ (2). Let $(D_i : i \in \mathbb{N})$ be a sequence of dense subsets of $(2^X, \mathbf{Z}^+)$ such that $D_i \in \mathcal{D}_\Phi$ for each $i \in \mathbb{N}$. Then $(D_i^c : i \in \mathbb{N})$ is a sequence of open covers of X such that $D_i^c \in \Phi$ for each $i \in \mathbb{N}$. Since X satisfies $S_{fin}(\Phi, \Psi)$, there is a sequence $(A_i : i \in \mathbb{N})$ of finite sets such that for each i, $A_i \subseteq D_i^c$, and $\bigcup_{i\in\mathbb{N}} A_i \in \Psi$. It follows that $\bigcup_{i\in\mathbb{N}} A_i^c \in \mathcal{D}_\Psi$.

(2) $\Rightarrow$ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X such that $\mathcal{U}_n \in \Phi$. For each n, $\mathcal{A}_n := \mathcal{U}_n^c$ is a dense subset of $(2^X, \mathbf{Z}^+)$ such that $\mathcal{A}_n \in \mathcal{D}_\Phi$. Applying that $(2^X, \mathbf{Z}^+)$ satisfies $S_{fin}(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$, there is a sequence $(A_n : n \in \mathbb{N})$ of finite sets such that for each n, $A_n \subseteq \mathcal{A}_n$, and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}_\Psi$. Then $\bigcup_{n\in\mathbb{N}} U_n$ is an open cover of X where $U_n = A_n^c$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} U_n \in \Psi$.

□

Corollary 3.2. (Theorem 5 in [8]) For a space X the following are equivalent:

1. X satisfies $S_1(\Omega, \Omega)$;
2. $(2^X, \mathbf{Z}^+)$ satisfies $S_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

Corollary 3.3. (Theorem 13 in [8]) For a space X the following are equivalent:

1. X satisfies $S_{fin}(\Omega, \Omega)$;
2. $(2^X, \mathbf{Z}^+)$ satisfies $S_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

Theorem 3.4. *Assume that $\Phi, \Psi \in \{\Gamma, \Omega, \}$, $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

1. *Each open set $Y \subset X$ has the property $S_\star(\Phi, \Psi)$;*
2. *For each $E \in 2^X$, $(2^X, \mathbf{Z}^+)$ satisfies $S_\star(\Phi_E, \Psi_E)$.*

Proof. We prove the theorem for $\star = 1$, the other proofs being similar.

(1) $\Rightarrow$ (2). Let $E \in 2^X$ and let $(\mathcal{A}_n : n \in \mathbb{N})$ be a sequence such that $\mathcal{A}_n \in \Phi_E$ for each $n \in \mathbb{N}$. Then $(\mathcal{A}_n^c : n \in \mathbb{N})$ is a sequence of open covers of E^c such that $\mathcal{A}_n^c \in \Phi$ for each $n \in \mathbb{N}$. Since E^c has the property $S_\star(\Phi, \Psi)$, there is a sequence $(A_n^c : n \in \mathbb{N})$ such that $A_n^c \in \mathcal{A}_n^c$ for each $n \in \mathbb{N}$ and $\{A_n^c : n \in \mathbb{N}\}$ is open cover of E^c such that $\{A_n^c : n \in \mathbb{N}\} \in \Psi$. It follows that $\{A_n : n \in \mathbb{N}\} \in \Psi_E$.

(2) $\Rightarrow$ (1). Let Y be an open subset of X and let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of open covers of Y such that $\mathcal{F}_n \in \Phi_Y$ (where Φ_Y is the Φ family of covers of Y). Let $E = X \setminus Y$. Put $\mathcal{A}_n = \mathcal{F}_n^c$ for each $n \in \mathbb{N}$. Then $\mathcal{A}_n \subset 2^X$ and $\mathcal{A}_n \in \Phi_E$ for each $n \in \mathbb{N}$. Since, by (2), $(2^X, \mathbf{Z}^+)$ satisfies $S_1(\Phi_E, \Psi_E)$, there is a sequence $(A_n : n \in \mathbb{N})$ such that $A_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}$ and $\{A_n : n \in \mathbb{N}\} \in \Psi_E$. It follows that $\{F_n : F_n = A_n^c, n \in \mathbb{N}\} \in \Psi$.

□

Corollary 3.5. (Theorem 3 in [5]) For a space X the following statements are equivalent:

1. Each open set $Y \subset X$ has the property $S_1(\Omega, \Gamma)$;
2. $(2^X, \mathbf{Z}^+)$ is Fréchet-Urysohn;
3. $(2^X, \mathbf{Z}^+)$ is strongly Fréchet-Urysohn.

Corollary 3.6. (Theorem 1 in [8]) For a space X the following are equivalent:

1. Each open set $Y \subset X$ satisfies $S_1(\Omega, \Omega)$;
2. $(2^X, \mathbf{Z}^+)$ has countable strong fan tightness (For each $E \in 2^X$, $(2^X, \mathbf{Z}^+)$ satisfies $S_1(\Omega_E, \Omega_E)$).

Corollary 3.7. (Theorem 9 in [8]) For a space X the following are equivalent:

1. Each open set $Y \subset X$ satisfies $S_{fin}(\Omega, \Omega)$;

2. $(2^X, \mathbf{Z}^+)$ has countable fan tightness (For each $E \in 2^X$, $(2^X, \mathbf{Z}^+)$ satisfies $S_{fin}(\Omega_E, \Omega_E)$).

Recall that a space is *perfect* if every open subset is an F_σ-subset [10]. Clearly every semi-stratifiable space is perfect.

Note that all properties in the Scheepers Diagram ([3, 18]) are hereditary for F_σ-subsets (Corollary 2.4 in [11]).

Proposition 3.8. *Assume that* $\Phi, \Psi \in \{\Gamma, \Omega, \}$, $\star \in \{1, fin\}$. *Then for a perfect topological space* X *the following statements are equivalent:*

1. X *satisfies* $S_\star(\Phi, \Psi)$*;*
2. *Each open set* $Y \subset X$ *has the property* $S_\star(\Phi, \Psi)$.

Theorem 3.9. *Assume that* $\Phi, \Psi \in \{\Gamma, \Omega\}$, $\star \in \{1, fin\}$. *Then for a perfect space* X *the following statements are equivalent:*

1. X *satisfies* $S_\star(\Phi, \Psi)$*;*
2. $(2^X, \mathbf{Z}^+)$ *satisfies* $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$*;*
3. *For each* $E \in 2^X$, $(2^X, \mathbf{Z}^+)$ *satisfies* $S_\star(\Phi_E, \Psi_E)$.

Clearly that every perfectly normal space is perfect.

Corollary 3.10. Assume that $\Phi, \Psi \in \{\Gamma, \Omega\}$, $\star \in \{1, fin\}$. Then for a perfectly normal space X the following statements are equivalent:

1. X satisfies $S_\star(\Phi, \Psi)$;
2. $(2^X, \mathbf{Z}^+)$ satisfies $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$;
3. For each $E \in 2^X$, $(2^X, \mathbf{Z}^+)$ satisfies $S_\star(\Phi_E, \Psi_E)$.

4. Hyperspace $(\mathbf{2^X}, \mathbf{F^+})$

Note that $(2^X, \mathbf{F}^+)$ satisfies the selection principle $S_{fin}(\mathcal{O}, \mathcal{O})$ for any space X, because $(2^X, \mathbf{F}^+)$ is always compact (see [2]).

$S_{fin}(\mathcal{O}, \mathcal{O})$ property is called the *Menger property* (see [3, 18]).

Lemma 4.1. *Let* X *be a space and let* $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ *be an open cover of* X. *Then the following are equivalent:*

1. $\mathcal{U}$ *is an* γ_k*-cover of* X*;*
2. $\mathcal{U}^c$ *converges to* $\{\emptyset\}$ *in* $(2^X, \mathbf{F}^+)$.

Theorem 4.2. *Assume that* $\Phi, \Psi \in \{\Gamma_k, \mathcal{K}\}$, $\star \in \{1, fin\}$. *Then for a space* X *the following statements are equivalent:*

1. X *satisfies* $S_\star(\Phi, \Psi)$*;*
2. $(2^X, \mathbf{F}^+)$ *satisfies* $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$.

Proof. The proof is similar to the proof of Theorem 3.1.

□

Corollary 4.3. (Theorem 4 in [8]) For a space X the following are equivalent:

1. X satisfies $S_1(\mathcal{K}, \mathcal{K})$;
2. $(2^X, \mathbf{F}^+)$ satisfies $S_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

Corollary 4.4. (Theorem 12 in [8]) For a space X the following are equivalent:

1. X satisfies $S_{fin}(\mathcal{K}, \mathcal{K})$;
2. $(2^X, \mathbf{F}^+)$ satisfies $S_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

Theorem 4.5. *Assume that* $\Phi, \Psi \in \{\Gamma_k, \mathcal{K}\}$, $\star \in \{1, fin\}$. *Then for a space* X *the following statements are equivalent:*

1. *Each open set* $Y \subset X$ *has the property* $S_\star(\Phi, \Psi)$*;*
2. *For each* $E \in 2^X$, $(2^X, \mathbf{F}^+)$ *satisfies* $S_\star(\Phi_E, \Psi_E)$.

Proof. The proof is similar to the proof of Theorem 3.4.

□

Corollary 4.6. (Theorem 23 in [1]) For a space X the following are equivalent:

1. For each $E \in 2^X$, $(2^X, \mathbf{F}^+)$ satisfies $S_1(\Gamma_E, \Omega_E)$;
2. Each open set $Y \subset X$ has the property $S_1(\Gamma_k, \mathcal{K})$.

Corollary 4.7. (Theorem 32 in [1]) For a space X the following are equivalent:

1. For each $E \in 2^X$, $(2^X, \mathbf{F}^+)$ satisfies $S_1(\Gamma_E, \Gamma_E)$;
2. Each open set $Y \subset X$ has the property $S_1(\Gamma_k, \Gamma_k)$.

Definition 4.8. A subset A of a space X is called an *k-F_σ-set* if A can be represented as $A = \bigcup_{i=1}^{\infty} F_i$ where F_i is a closed set in X for each $i \in \mathbb{N}$ and for any compact set $B \subseteq A$ there exists $i' \in \mathbb{N}$ such that $B \subseteq F_{i'}$.

Definition 4.9. A space X is called *k-perfect* if every open subset is an k-F_σ-subset of X.

Clearly that every k-perfect space is perfect.

Proposition 4.10. *Every perfectly normal space is k-perfect.*

Proof. Let U be an open subset of a perfectly normal space X. Then U may be represented as $U = \bigcup_{i=1}^{\infty} F_i$ where F_i is closed set of X for each $i \in \mathbb{N}$ and $F_i \subset F_{i+1}$. Since X is a perfectly normal space, there exists a sequence $(U_i : i \in \mathbb{N})$ of open sets of X such that $F_i \subseteq U_i \subseteq \overline{U_i} \subseteq U$ and $\overline{U_i} \subseteq U_{i+1}$ for each $i \in \mathbb{N}$. It follow that $U = \bigcup_{i=1}^{\infty} \overline{U_i}$ and for any compact set $B \subseteq U$ there exists $i' \in \mathbb{N}$ such that $B \subseteq \overline{U_{i'}}$.

□

Note that, by Proposition 4.10, each cozero set is an k-F_σ-set.

Question. Is there a k-perfect which is not (perfectly) normal space?

Denote by $\mathbb{S}$ the Sorgenfrey line.

Proposition 4.11. *The space $\mathbb{S}^2$ is perfect, but not k-perfect.*

Proof. By Lemma 2.3 in [10], $\mathbb{S}^2$ is perfect.

Consider the open set $U = \{(x,y) \in \mathbb{S}^2 : -x < y\} \bigcup \{(x,-x) : x \in \mathbb{P}\}$ in the space $\mathbb{S}^2$.

Assume that U is a k-F_σ-set of $\mathbb{S}^2$. Then $U = \bigcup_{i=1}^{\infty} F_i$ where F_i is closed set in $\mathbb{S}^2$ for each $i \in \mathbb{N}$ and for any compact set $B \subseteq U$ there exists $i' \in \mathbb{N}$ such that $B \subseteq F_{i'}$. For each point $p = (y,-y) \in \{(x,-x) : x \in \mathbb{P}\}$ we fix the compact set $Z_p = \{p\} \bigcup \{(y + \frac{1}{n}, -y + \frac{1}{n}) : n \in \mathbb{N}\}$. Note that $Z_p \subset U$ for each $p \in \{(x,-x) : x \in \mathbb{P}\}$. Since $U = \bigcup_{i=1}^{\infty} F_i$ there exists $k \in \mathbb{N}$ such that $|\{p : Z_p \subset F_k\}| > \aleph_0$. Since the set $\{p : Z_p \subset F_k\}$ is uncountable subset of $\{(x,-x): x \in \mathbb{P}\}$ there is a accumulation point z in subspace $\{(x,-x): x \in \mathbb{R}\}$ of the space $\mathbb{R}^2$. Clearly that for any neighborhood $O(z)$ of z in the space $\mathbb{S}^2$ we have that $O(z) \bigcap \bigcup\limits_{Z_p \subset F_k} Z_p \neq \emptyset$. It follows that $O(z) \bigcap F_k \neq \emptyset$. Hence F_k is not closed set in $\mathbb{S}^2$, a contradiction.

□

Theorem 4.12. *Assume that $\Phi, \Psi \in \{\Gamma_k, \mathcal{K}\}$, $\star \in \{1, fin\}$, X has the property $S_\star(\Phi, \Psi)$ and A is an k-F_σ-set. Then A has the property $S_\star(\Phi, \Psi)$.*

Proof. We prove the theorem for $\star = fin$, the other proofs being similar.

Assume that X has the property $S_{fin}(\Phi, \Psi)$ and A is an k-F_σ-set. Consider a sequence $(\mathcal{U}_i : i \in \mathbb{N})$ of covers A such that $\mathcal{U}_i \in \Phi_A$ (where Φ_A is the Φ family of covers of A) for each $i \in \mathbb{N}$. Let $A = \bigcup\limits_{i=1}^{\infty} F_i$ where F_i is a closed set in X for each $i \in \mathbb{N}$ and for any compact set $B \subseteq A$ there exists $i' \in \mathbb{N}$ such that $B \subseteq F_{i'}$. Consider $\mathcal{V}_i = \{(X \setminus F_i) \bigcup U : U \in \mathcal{U}_i\}$ for each $i \in \mathbb{N}$.

We claim that $\mathcal{V}_i \in \Phi$ for each $i \in \mathbb{N}$. Let S be a compact subset of X. Then $S \bigcap F_i$ is a compact subset of A and hence there is $U \in \mathcal{U}_i$ such that $S \bigcap F_i \subset U$. It follows that $S \subset (X \setminus F_i) \bigcup U$ for $(X \setminus F_i) \bigcup U \in \mathcal{V}_i$.

Since X has the property $S_{fin}(\Phi, \Psi)$, there is a sequence $(B_i : i \in \mathbb{N})$ of finite sets such that for each i, $B_i \subset \mathcal{V}_i$, and $\bigcup_{i \in \mathbb{N}} B_i \in \Psi$.

We claim that $\bigcup_{i \in \mathbb{N}} \{B_i \bigcap F_i : B_i \bigcap F_i \subset \mathcal{U}_i, \ i \in \mathbb{N}\} \in \Psi_A$. Let B be a compact subset of A then there exists $i' \in \mathbb{N}$ such that $B \subseteq F_{i'}$. Since $\bigcup_{i \in \mathbb{N}} B_i$ is a large cover of X there is $k \in \mathbb{N}$ and $V_k \in B_k \subset \mathcal{V}_k$ such that $k > i'$ and $B \subset V_k$. But $V_k = (X \setminus F_k) \bigcup U_k$ for $U_k \in \mathcal{U}_k$. Since $k > i'$, $B \subset U_k$. It follows that A has the property $S_{fin}(\Phi, \Psi)$.

□

Proposition 4.13. *Assume that $\Phi, \Psi \in \{\Gamma_k, \mathcal{K}\}$, $\star \in \{1, fin\}$. Then for a k-perfect space X the following statements are equivalent:*

1. *X satisfies $S_\star(\Phi, \Psi)$;*
2. *Each open set $Y \subset X$ has the property $S_\star(\Phi, \Psi)$.*

By Proposition 4.10, we have the following result.

Proposition 4.14. *Assume that $\Phi, \Psi \in \{\Gamma_k, \mathcal{K}\}$, $\star \in \{1, fin\}$. Then for a perfectly normal space X the following statements are equivalent:*

1. *X satisfies $S_\star(\Phi, \Psi)$;*

2. *Each open set* $Y \subset X$ *has the property* $S_\star(\Phi, \Psi)$.

Theorem 4.15. *Assume that* $\Phi, \Psi \in \{\Gamma_k, \mathcal{K}\}$, $\star \in \{1, fin\}$. *Then for a* k*-perfect space* X *the following statements are equivalent:*

1. X *satisfies* $S_\star(\Phi, \Psi)$*;*
2. $(2^X, \mathbf{F}^+)$ *satisfies* $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$*;*
3. *For each* $E \in 2^X$, $(2^X, \mathbf{F}^+)$ *satisfies* $S_\star(\Phi_E, \Psi_E)$.

By Proposition 4.10, we have the following result.

Corollary 4.16. Assume that $\Phi, \Psi \in \{\Gamma_k, \mathcal{K}\}$, $\star \in \{1, fin\}$. Then for a perfectly normal space X the following statements are equivalent:

1. X satisfies $S_\star(\Phi, \Psi)$;
2. $(2^X, \mathbf{F}^+)$ satisfies $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$;
3. For each $E \in 2^X$, $(2^X, \mathbf{F}^+)$ satisfies $S_\star(\Phi_E, \Psi_E)$.

5. Bitopological space $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$

Theorem 5.1. *Assume that* $\Phi \in \{\Gamma_k, \mathcal{K}\}$, $\Psi \in \{\Gamma, \Omega\}$, $\star \in \{1, fin\}$. *Then for a space* X *the following statements are equivalent:*

1. X *satisfies* $S_\star(\Phi, \Psi)$*;*
2. $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ *satisfies* $S_\star(\mathcal{D}_\Phi^{\mathbf{F}^+}, \mathcal{D}_\Psi^{\mathbf{Z}^+})$.

Proof. The proof is similar to the proof of Theorem 3.1.

□

Corollary 5.2. (Theorem 6 in [8]) For a space X the following are equivalent:

1. X satisfies $S_1(\mathcal{K}, \Omega)$;
2. $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ satisfies $S_1(\mathcal{D}_\Omega^{\mathbf{F}^+}, \mathcal{D}_\Omega^{\mathbf{Z}^+})$.

Corollary 5.3. (Theorem 14 in [8]) For a space X the following are equivalent:

1. X satisfies $S_{fin}(\mathcal{K}, \Omega)$;
2. $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ satisfies $S_{fin}(\mathcal{D}_\Omega^{\mathbf{F}^+}, \mathcal{D}_\Omega^{\mathbf{Z}^+})$.

Theorem 5.4. *Assume that* $\Phi \in \{\Gamma_k, \mathcal{K}\}$, $\Psi \in \{\Gamma, \Omega\}$, $\star \in \{1, fin\}$. *Then for a space* X *the following statements are equivalent:*

1. *Each open set* $Y \subset X$ *has the property* $S_\star(\Phi, \Psi)$*;*
2. *For each* $E \in 2^X$*,* $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ *satisfies* $S_\star(\Phi_E^{\mathbf{F}^+}, \Psi_E^{\mathbf{Z}^+})$*.*

Proof. The proof is similar to the proof of Theorem 3.4.

□

Corollary 5.5. (Theorem 31 in [1]) For a space X the following are equivalent:

1. Each open set $Y \subset X$ has the property $S_1(\Gamma_k, \Omega)$;
2. For each $E \in 2^X$, $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ satisfies $S_1(\Gamma_E^{\mathbf{F}^+}, \Omega_E^{\mathbf{Z}^+})$.

Corollary 5.6. (Theorem 33 in [1]) For a space X the following are equivalent:

1. Each open set $Y \subset X$ has the property $S_1(\Gamma_k, \Gamma)$;
2. For each $E \in 2^X$, $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ satisfies $S_1(\Gamma_E^{\mathbf{F}^+}, \Gamma_E^{\mathbf{Z}^+})$.

Theorem 5.7. *Assume that* $\Phi \in \{\Gamma_k, \mathcal{K}\}$*,* $\Psi \in \{\Gamma, \Omega\}$*,* $\star \in \{1, fin\}$*,* X *has the property* $S_\star(\Phi, \Psi)$ *and* A *is an* k*-*F_σ*-set. Then* A *has the property* $S_\star(\Phi, \Psi)$*.*

Proof. The proof is similar to the proof of Theorem 4.12. □

Proposition 5.8. *Assume that* $\Phi \in \{\Gamma_k, \mathcal{K}\}$*,* $\Psi \in \{\Gamma, \Omega\}$*,* $\star \in \{1, fin\}$*. Then for a* k*-perfect space* X *the following statements are equivalent:*

1. X *satisfies* $S_\star(\Phi, \Psi)$*;*
2. *Each open set* $Y \subset X$ *has the property* $S_\star(\Phi, \Psi)$*.*

Theorem 5.9. *Assume that* $\Phi \in \{\Gamma_k, \mathcal{K}\}$*,* $\Psi \in \{\Gamma, \Omega\}$*,* $\star \in \{1, fin\}$*. Then for a* k*-perfect space* X *the following statements are equivalent:*

1. X *satisfies* $S_\star(\Phi, \Psi)$*;*
2. $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ *satisfies* $S_\star(\mathcal{D}_\Phi, \mathcal{D}_\Psi)$*;*
3. *For each* $E \in 2^X$*,* $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$ *satisfies* $S_\star(\Phi_E, \Psi_E)$*.*

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