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Research Article

On selective sequential separability of function spaces with the compact-open topology

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Abstract

For a Tychonoff space X, we denote by $C_k(X)$ the space of all real-valued continuous functions on X with the compact-open topology. A subset $A \subset X$ is said to be sequentially dense in X if every point of X is the limit of a convergent sequence in A. A space $C_k(X)$ is selectively sequentially separable (in Scheepers' terminology: $C_k(X)$ satisfies $S_{fin}(\mathcal{S},\mathcal{S})$) if whenever $(S_n : n \in \mathbb{N})$ is a sequence of sequentially dense subsets of $C_k(X)$, one can pick finite $F_n \subset S_n$ $(n \in \mathbb{N})$ such that $\bigcup\{F_n : n \in \mathbb{N}\}$ is sequentially dense in $C_k(X)$. In this paper, we give a characterization for $C_k(X)$ to satisfy $S_{fin}(\mathcal{S},\mathcal{S})$.

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1. Introduction

If X is a topological space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space X is called sequentially separable if it has a countable sequentially dense set [26, 27].

Let X be a topological space, and $x \in X$. Consider the following collections:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to x. So, simply Γ_x may be the set of non-trivial convergent sequences to x.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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In this paper, by a cover we mean a cover $\mathcal{U}$ with $X \notin \mathcal{U}$.

A cover $\mathcal{U}$ of a space X is called:

• a k-cover if each compact subset C of X is contained in an element of $\mathcal{U}$;

• a γ_k-cover if $\mathcal{U}$ is infinite and for each compact subset C of X the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

Note that a γ_k-cover is a k-cover, and a k-cover is infinite. A compact space has no k-covers.

For a Tychonoff space X, we denote by $C_k(X)$ the space of all real-valued continuous functions on X with the compact-open topology. Subbase open sets of $C_k(X)$ are of the form $[A, U] = \{f \in C(X) : f(A) \subset U\}$, where A is a compact subset of X and U is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of a point $f \in C_k(X)$ as $\langle f, A, \epsilon\rangle$ where $\langle f, A, \epsilon\rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in A\}$, A is a compact subset of X and $\epsilon > 0$.

For a topological space X we denote:

- Γ_k — the family of open γ_k-covers of X;
- $\mathcal{K}$ — the family of open k-covers of X;
- $\mathcal{K}^{\omega}_{cz}$ — the family of countable co-zero k-covers of X;
- $\mathcal{D}$ — the family of dense subsets of $C_k(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_k(X)$;
- $\mathbb{K}(X)$ — the family of all non-empty compact subsets of X.

A space X is said to be a γ_k-set if each k-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a γ_k-cover of X [9].

2. Main definitions and notation

- A space X is R-separable, if X satisfies $S_1(\mathcal{D}, \mathcal{D})$ ([2, Definition 47]).
- A space X is selectively separable (M-separable), if X satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$.
- A space X is selectively sequentially separable (M-sequentially separable), if X satisfies $S_{fin}(\mathcal{S}, \mathcal{S})$ ([4, Definition 1.2]).

For a topological space X we have the next relations of selectors for sequences of dense sets of X.

$$\begin{array}{ccccccc} S_1(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{D}) & \Leftarrow & S_1(\mathcal{S},\mathcal{D}) \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ S_1(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{D}) & \Leftarrow & S_1(\mathcal{D},\mathcal{D}) \end{array}$$

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

- A space X has *property* α_2 (α_2 in the sense of Arhangel'skii), if X satisfies $S_1(\Gamma_x, \Gamma_x)$ [1].
- A space X has *property* α_4 (α_4 in the sense of Arhangel'skii), if X satisfies $S_{fin}(\Gamma_x, \Gamma_x)$ [1].

So we have three types of topological properties described through the selection principles:

- local properties of the form $S_*(\Phi_x, \Psi_x)$;
- global properties of the form $S_*(\Phi, \Psi)$;
- semi-local properties of the form $S_*(\Phi, \Psi_x)$.

In a series of papers it was demonstrated that γ-covers, Borel covers, k-covers play a key role in function spaces ([5],[10]-[8], [13]-[15], [18]-[25] and many others). We continue to investigate applications of k-covers in function spaces with the compact-open topology.

A great attention has recently received the notions of selective separability and selective sequential separability ($S_{fin}(\mathcal{S}, \mathcal{S})$) [2, 3, 6, 7]. In this paper, we give characterizations for $C_k(X)$ to satisfy $S_{fin}(\mathcal{S}, \mathcal{S})$, $S_{fin}(\mathcal{S}, \Gamma_x)$, and $S_{fin}(\Gamma_x, \Gamma_x)$.

3. Main results

Definition 3.1. A γ_k-cover $\mathcal{U}$ of co-zero sets of X is γ_k**-shrinkable** if there exists a γ_k-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subset U$ for every $U \in \mathcal{U}$.

Note that every γ_k-shrinkable cover contains a countable γ_k-shrinkable cover.

For a topological space X we denote:

- Γ_k^{sh} — the family of γ_k-shrinkable covers of X.

-Similar to the proof that $S_1(\mathcal{K}, \Gamma_k) = S_{fin}(\mathcal{K}, \Gamma_k)$ ([9, Theorem 5]), we prove the following.

Lemma 3.2. *For a space X the following are equivalent:*

(1) *X satisfies $S_{fin}(\Gamma_k^{sh}, \Gamma_k)$;*
(2) *X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of (countable) γ_k-shrinkable covers of X; suppose that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Let $V_{n,m} = U_{1,m} \cap ... \cap U_{n,m}$ and let $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of γ_k-shrinkable covers of X. Since X satisfies $S_{fin}(\Gamma_k^{sh}, \Gamma_k)$ choose for each $n \in \mathbb{N}$ a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a γ_k-cover of X. (Note that some $\mathcal{W}_n$'s can be empty.)

As $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is infinite and all $\mathcal{W}_n$'s are finite, there exists a sequence $m_1 < m_2 < ... < m_p < ...$ in $\mathbb{N}$ such that for each $i \in \mathbb{N}$ we have $\mathcal{W}_{m_i} \setminus \bigcup_{j<i} \mathcal{W}_{m_j} \neq \emptyset$. Choose an element $W_{m_i} \in \mathcal{W}_{m_i} \setminus \bigcup_{j<i} \mathcal{W}_{m_j}$, $i \in \mathbb{N}$, and fix its representation $W_{m_i} = U_{1,k_{m_i}} \cap U_{2,k_{m_i}} \cap ... \cap U_{m_i,k_{m_i}}$ as above.

Since each infinite subset of a γ_k-cover is also a γ_k-cover, we have that the set $\{W_{m_i} : i \in \mathbb{N}\}$ is a γ_k-cover of X. For each $n \leq m_1$ let $U_n \in \mathcal{U}_n$ be the n-th coordinate of W_{m_1} in the chosen representation of W_{m_1}, and for each $n \in (m_i, m_{i+1}]$, $i \geq 1$, let $U_n \in \mathcal{U}_n$ be the n-th coordinate of $W_{m_{i+1}}$ in the above representation of $W_{m_{i+1}}$. Observe that each $U_n \supset W_{m_{i+1}}$. Therefore, we obtain a sequence $(U_n : n \in \mathbb{N})$ of elements, one from each $\mathcal{U}_n$, which form a γ_k-cover of X and show that X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$. □

The symbol $\mathbf{0}$ denotes the constantly zero function in $C_k(X)$. Because $C_k(X)$ is homogeneous we can work with $\mathbf{0}$ to study local and semi-local properties of $C_k(X)$.

Theorem 3.3. *For a Tychonoff space X the following statements are equivalent:*

(1) *$C_k(X)$ satisfies $S_1(\Gamma_\mathbf{0}, \Gamma_\mathbf{0})$ [property α_2];*
(2) *X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of (countable) γ_k-shrinkable covers of X; suppose that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ and $\{F(U_{n,m}) : U_{n,m} \in \mathcal{U}_n\}$ is a γ_k-cover of zero-sets of X with $F(U_{n,m}) \subset U_{n,m}$ for every $U_{n,m} \in \mathcal{U}_n$. For each $n, m \in \mathbb{N}$ we fix $f_{n,m} \in C(X)$ such that $f_{n,m} \upharpoonright F(U_{n,m}) \equiv 0$, $f_{n,m} \upharpoonright (X \setminus U_{n,m}) \equiv 1$. Consider $S_n = \{f_{n,m} : m \in \mathbb{N}\}$. Since $\{F(U_{n,m}) : U_{n,m} \in \mathcal{U}_n\}$ is a γ_k-cover of X, then $S_n \in \Gamma_\mathbf{0}$ for each $n \in \mathbb{N}$. By (1), there is $\{f_{n,m(n)} : n \in \mathbb{N}\}$ such that $f_{n,m(n)} \in S_n$ and $\{f_{n,m(n)} : n \in \mathbb{N}\} \in \Gamma_\mathbf{0}$. We show that $\{U_{n,m(n)} : n \in \mathbb{N}\} \in \Gamma_k$. Suppose $A \in \mathbb{K}(X)$ and $W = [A, (-\frac{1}{2}, \frac{1}{2})]$ is a base neighborhood of $\mathbf{0}$ then there exists $n' \in \mathbb{N}$ such that $f_{n,m(n)} \in W$ for every $n > n'$. It follows that $A \subset U_{n,m(n)}$ for every $n > n'$.

(2) $\Rightarrow$ (1). Let $S_n \in \Gamma_\mathbf{0}$ for every $n \in \mathbb{N}$; suppose that for each $n \in \mathbb{N}$, $S_n = \{f_{n,j} : j \in \mathbb{N}\}$. Consider $\mathcal{V}_n = \{f_{n,j}^{-1}((-\frac{1}{n}, \frac{1}{n})) : f_{n,j} \in S_n\}$ for each $n \in \mathbb{N}$.

Let $J = \{n \in \mathbb{N} : f_{n,j}^{-1}((-\frac{1}{n}, \frac{1}{n})) = X$ for some $j \in \mathbb{N}\}$. If J is finite, then we can ignore such finitely many n. If J is infinite, then for some j_n $(n \in J)$, $f_{n,j_n} \to \mathbf{0}$ uniformly. Thus, without loss of generality, we may assume $f_{n,j}^{-1}((-\frac{1}{n}, \frac{1}{n})) \neq X$ for each $n, j \in \mathbb{N}$.

Note that $\mathcal{W}_n = \{f_{n,j}^{-1}([-\frac{1}{n+1}, \frac{1}{n+1}]) : f_{n,j} \in S_n\}$ is a γ_k-cover of zero-sets of X. Hence, $\mathcal{V}_n \in \Gamma_k^{sh}$ for each $n \in \mathbb{N}$. By (2), there is $\{f_{n,j(n)} : n \in \mathbb{N}\}$ such that $\{f_{n,j(n)}^{-1}((-\frac{1}{n}, \frac{1}{n})) :$

$n \in \mathbb{N}\} \in \Gamma_k$. We show that $\{f_{n,j(n)} : n \in \mathbb{N}\} \in \Gamma_{\mathbf{0}}$. Let $[A, (-\epsilon, \epsilon)]$ be a base neighborhood of $\mathbf{0}$ where $A \in \mathbb{K}(X)$ and $\epsilon > 0$. There is $n' \in \mathbb{N}$ such that $A \subset f^{-1}_{n,j(n)}((-\frac{1}{n}, \frac{1}{n}))$ for each $n > n'$. There is $n'' > n'$ such that $\frac{1}{n''} < \epsilon$, hence, $f_{n,j(n)} \in [A, (-\frac{1}{n''}, \frac{1}{n''})] \subset [A, (-\epsilon, \epsilon)]$ for each $n > n''$. □

Proposition 3.4 ([3, Proposition 4.2]). *Every selectively sequentially separable space is sequentially separable.*

We shall prove the following theorem under the condition that the space $C_k(X)$ is sequentially separable.

Theorem 3.5. *For a Tychonoff space X such that $C_k(X)$ is sequentially separable the following statements are equivalent:*

(1) *$C_k(X)$ satisfies $S_1(\mathcal{S}, \mathcal{S})$;*
(2) *$C_k(X)$ satisfies $S_1(\mathcal{S}, \Gamma_{\mathbf{0}})$;*
(3) *$C_k(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ [property α_2];*
(4) *X satisfies $S_1(\Gamma^{sh}_k, \Gamma_k)$;*
(5) *$C_k(X)$ satisfies $S_{fin}(\mathcal{S}, \mathcal{S})$ [selectively sequentially separable];*
(6) *$C_k(X)$ satisfies $S_{fin}(\mathcal{S}, \Gamma_{\mathbf{0}})$;*
(7) *$C_k(X)$ satisfies $S_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ [property α_4];*
(8) *X satisfies $S_{fin}(\Gamma^{sh}_k, \Gamma_k)$.*

Proof. (1) $\Rightarrow$ (4). Let $\{\mathcal{U}_i\} \subset \Gamma^{sh}_k$, $\mathcal{U}_i = \{U^m_i : m \in \mathbb{N}\}$ for each $i \in \mathbb{N}$ and let $S = \{h_m : m \in \mathbb{N}\}$ be a countable sequentially dense subset of $C_k(X)$.

For each $i, m \in \mathbb{N}$ we fix $f^m_i \in C(X)$ such that $f^m_i \upharpoonright F(U^m_i) = h_m$ and $f^m_i \upharpoonright (X \setminus U^m_i) = 1$. Let $S_i = \{f^m_i : m \in \mathbb{N}\}$. Since S is a countable sequentially dense subset of $C_k(X)$, we have that S_i is a countable sequentially dense subset of $C_k(X)$ for each $i \in \mathbb{N}$. Let $h \in C(X)$, there is a set $\{h_{m_s} : s \in \mathbb{N}\} \subset S$ such that $\{h_{m_s}\}_{s\in\mathbb{N}}$ converges to h. Let K be a compact subset of X, $\epsilon > 0$ and let $W = \langle h, K, \epsilon\rangle$ be a base neighborhood of h, then there is a number m_0 such that $K \subset F(U^m_i)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f^{m_s}_i \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f^{m_s}_i \in W$ for each $m_s > m_0$. It follows that a sequence $\{f^{m_s}_i\}_{s\in\mathbb{N}}$ converges to h.

Since $C_k(X)$ satisfies $S_1(\mathcal{S}, \mathcal{S})$, there is a sequence $\{f^{m(i)}_i\}_{i\in\mathbb{N}}$ such that for each i, $f^{m(i)}_i \in S_i$, and $\{f^{m(i)}_i : i \in \mathbb{N}\}$ is an element of $\mathcal{S}$.

We show that $\{U^{m(i)}_i : i \in \mathbb{N}\}$ is a γ_k-cover of X.

There is a sequence $\{f^{m(i_j)}_{i_j}\}$ converges to $\mathbf{0}$. Let K be a compact subset of X and let $U = \langle\mathbf{0}, K, (-1, 1)\rangle$ be a base neighborhood of $\mathbf{0}$. Then there exists $j_0 \in \mathbb{N}$ such that $f^{m(i_j)}_{i_j} \in U$ for each $j > j_0$. It follows that $K \subset U^{m(i_j)}_{i_j}$ for $j > j_0$. By Lemma 3.2, $S_{fin}(\Gamma^{sh}_k, \Gamma_k) = S_1(\Gamma^{sh}_k, \Gamma_k)$.

(4) $\Leftrightarrow$ (3). By Theorem 3.3.

(3) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (1). For each $n \in \mathbb{N}$, let S_n be a sequentially dense subset of $C_k(X)$ and let $\{h_n : n \in \mathbb{N}\}$ be sequentially dense in $C_k(X)$. Take a sequence $\{f^m_n : m \in \mathbb{N}\} \subset S_n$ such that $f^m_n \mapsto h_n$ $(m \mapsto \infty)$. Then $f^m_n - h_n \mapsto \mathbf{0}$ $(m \mapsto \infty)$. Hence, there exists $f^{m_n}_n$ such that $f^{m_n}_n - h_n \mapsto \mathbf{0}$ $(n \mapsto \infty)$. We see that $\{f^{m_n}_n : n \in \mathbb{N}\}$ is sequentially dense. Let $h \in C_k(X)$ and take a sequence $\{h_{n_j} : j \in \mathbb{N}\} \subset \{h_n : n \in \mathbb{N}\}$ converging to h. Then, $f^{m_{n_j}}_{n_j} = (f^{m_{n_j}}_{n_j} - h_{n_j}) + h_{n_j} \mapsto h$ $(j \mapsto \infty)$.

(4) $\Leftrightarrow$ (8). By Lemma 3.2.

The proofs of the remaining implications are similar to those proved above. □

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.

It is well known that if X is hemicompact then $C_k(X)$ is metrizable. It follows that $C_k(X)$ is sequential separable for a hemicompact space X with $iw(X) = \aleph_0$. But, for general case, the author does not know the answer to the next question.

Question 1. Characterize a Tychonoff space X such that a space $C_k(X)$ is sequential separable ?

Proposition 3.6 ([3, Corollary 4.8 (Dow-Barman)])**.** *Every Fréchet-Urysohn separable T_2 space is selectively separable (hence, selectively sequentially separable).*

It is well known that a Tychonoff space X the space $C_k(X)$ is Fréchet-Urysohn if and only if X satisfies $S_1(\mathcal{K}, \Gamma_k)$ ([11]).

A Tychonoff space X the space $C_k(X)$ is separable if and only if $iw(X) = \aleph_0$ [16].

Question 2. Is there a Tychonoff space X with $iw(X) = \aleph_0$ such that $C_k(X)$ satisfies $S_1(\mathcal{S}, \mathcal{S})$, but $C_k(X)$ is not Fréchet-Urysohn (i.e. X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$, but it has not property $S_1(\mathcal{K}, \Gamma_k)$) ?

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