Projective versions of the properties in the Scheepers Diagram

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Abstract

Let $\mathcal{P}$ be a topological property. A.V. Arhangel'skii calls X *projectively* $\mathcal{P}$ if every second countable continuous image of X is $\mathcal{P}$. Lj.D.R. Kočinac characterized the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in term of continuous images in $\mathbb{R}^\omega$. In this paper we study the functional characterizations of all projective versions of the selection properties in the Scheepers Diagram.

Keywords:
projectively Rothberger space, projectively Menger space, projectively Hurewicz space, projectively Gerlits-Nagy space, function spaces, selection principles, C_p-theory, Scheepers Diagram
2010 MSC: 54C35, 54C05, 54C65, 54A20, 54D65

1. Introduction

Many topological properties are characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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$U_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

The papers [10, 11, 22, 25, 26, 28, 29, 30, 31] have initiated the simultaneous consideration of these properties in the case where $\mathcal{A}$ and $\mathcal{B}$ are important families of open covers of a topological space X.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is:

• an ω-*cover* if every finite subset of X is contained in a member of $\mathcal{U}$.

• a γ-*cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

For a topological space X we denote:

• $\mathcal{O}$ — the family of all open covers of X;
• $\mathcal{O}^{\omega}_{cz}$ — the family of all countable cozero covers of X;
• Γ — the family of all open γ-covers of X;
• Γ_{cz} — the family of all cozero γ-covers of X;
• Ω — the family of all open ω-covers of X;
• Ω^{ω}_{cz} — the family of countable cozero ω-covers of X;
• Ω^{ω}_{cl} — the family of all countable clopen ω-covers of X;
• $\mathcal{D}$ — the family of all dense subsets of X;
• $\mathcal{S}$ — the family of all sequentially dense subsets of X;
• $\mathcal{D}^{\omega}$ — the family of all countable dense subsets of X;
• $\mathcal{S}^{\omega}$ — the family of all countable sequentially dense subsets of X.

Many equivalences hold among the selection properties, and the surviving ones appear in the following the Scheepers Diagram (where an arrow denotes implication), to which no arrow can be added except perhaps from $U_{fin}(\mathcal{O},\Gamma)$ or $U_{fin}(\mathcal{O},\Omega)$ to $S_{fin}(\Gamma,\Omega)$ [10].

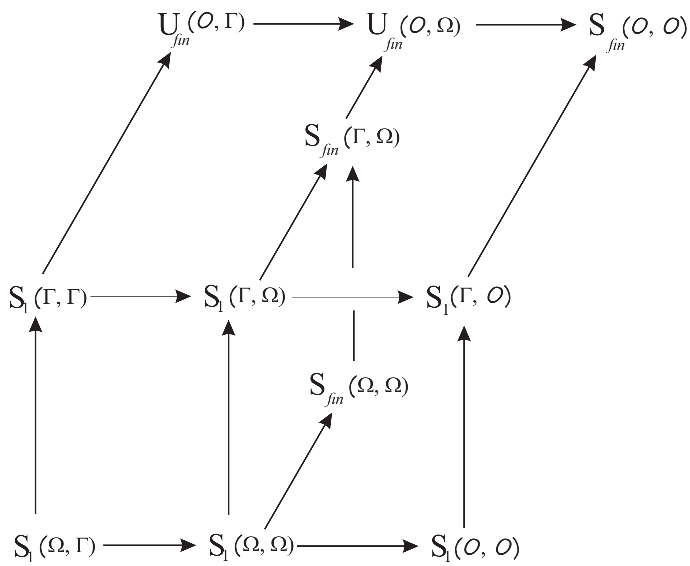

Fig. 1. The Scheepers Diagram for Lindelöf spaces.

Let $\mathcal{P}$ be a topological property. A.V. Arhangel'skii calls X *projectively* $\mathcal{P}$ if every second countable continuous image of X is $\mathcal{P}$ [1].

A.V. Arhangel'skii consider projective $\mathcal{P}$ for $\mathcal{P}=\sigma$-compact, analytic and other properties in [3]. The projective selection principles were introduced and first time considered in [12]. Lj.D.R. Kočinac characterized the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in term of continuous images in $\mathbb{R}^\omega$. Characterizations of the classical covering properties in terms a selection principle restricted to countable covers by cozero sets are given in [4].

In this paper we study the functional characterizations of the projective versions of the properties in the Scheepers Diagram (Fig. 1).

2. Main definitions and notation

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$;
- $\Omega_x^\omega = \{A \subseteq X : |A| = \aleph_0 \text{ and } x \in \overline{A} \setminus A\}$;

- $\Gamma^\omega_x = \{A \subseteq X : |A| = \aleph_0 \text{ and } x = \lim A\}$.

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ (resp., $\Pi(\mathcal{A}, \mathcal{B}_x)$) without specifying x, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$ (resp., $(\forall x)\Pi(\mathcal{A}, \mathcal{B}_x)$).

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by $\mathbb{N}$. Let $\mathbb{R}$ be the real line, we put $\mathbb{I} = [0,1] \subset \mathbb{R}$, and let $\mathbb{Q}$ be the rational numbers. For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. The symbol $\mathbf{0}$ stands for the constant function to 0. Since $C_p(X)$ is homogenous space we may always consider the point $\mathbf{0}$ when studying local properties of this space.

A basic open neighborhood of $\mathbf{0}$ is of the form $[F, (-\epsilon, \epsilon)] = \{f \in C(X) : f(F) \subset (-\epsilon, \epsilon)\}$, where F is a finite subset of X and $\epsilon > 0$.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set.

Recall that the cardinal $\mathfrak{p}$ is the smallest cardinal so that there is a collection of $\mathfrak{p}$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

For $f, g \in \mathbb{N}^{\mathbb{N}}$, let $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. $\mathfrak{b}$ is the minimal cardinality of a $\leq^*$-unbounded subset of $\mathbb{N}^{\mathbb{N}}$. A set $B \subset [\mathbb{N}]^\infty$ is unbounded if the set of all increasing enumerations of elements of B is unbounded in $\mathbb{N}^{\mathbb{N}}$, with respect to $\leq^*$. It follows that $|B| \geq \mathfrak{b}$ (See [7] for more on small cardinals including $\mathfrak{p}$).

Theorem 2.1. *(Noble [14]) A space $C_p(X)$ is separable if and only if X has a coarser second countable topology.*

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. If D is a countable sequentially dense subset of X then X call sequentially separable space.

Call X strongly sequentially separable if X is separable and every countable dense subset of X is sequentially dense. Clearly, every strongly sequentially separable space is sequentially separable, and every sequentially separable space is separable.

Definition 2.2. A space X has the *V-property* ($X \models V$), if there exists a condensation (= a continuous bijection) $f : X \mapsto Y$ from a space X on a

separable metric space Y such that $f(U)$ is an F_σ-set of Y for any cozero-set U of X.

Theorem 2.3. *(Velichko [8]). A space $C_p(X)$ is sequentially separable if and only if $X \models V$.*

3. The projectively Rothberger property

Definition 3.1. ([15]) Let $n \in \mathbb{N}$. A set $A \subseteq C_p(X)$ is called *n-dense* in $C_p(X)$, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ such that $x_i \neq x_j$ for $i \neq j$ and an open sets $W_1, ..., W_n$ in $\mathbb{R}$ there is $f \in A$ such that $f(x_i) \in W_i$ for $i \in \overline{1, n}$.

Obviously, that if A is a n-dense set of $C_p(X)$ for each $n \in \mathbb{N}$ then A is a dense set of $C_p(X)$.

For a space $C_p(X)$ we denote:

$\mathcal{D}[n]$ — the family of all n-dense subsets of $C_p(X)$;

$\mathcal{D}^\omega[n]$ — the family of all countable n-dense subsets of $C_p(X)$.

Definition 3.2. Let $f \in C(X)$ and $n \in \mathbb{N}$. A set $B \subseteq C_p(X)$ is called *n-dense at point* f, if for each n-finite set $\{x_1, ..., x_n\} \subset X$ and $\epsilon > 0$ there is $h \in B$ such that $h(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon)$ for $i \in \overline{1, n}$.

Obviously, that if B is a n-dense at point f for each $n \in \mathbb{N}$ then $f \in \overline{B}$.

For a space $C_p(X)$ and $f \in C_p(X)$ we denote:

$\mathcal{D}_f[n]$ — the family of all n-dense at point f subsets of $C_p(X)$;

$\mathcal{D}^\omega_f[n]$ — the family of all countable n-dense at point f subsets of $C_p(X)$.

By Theorem 11.3 in [17], we proved the following result where the symbol $\mathbf{0}$ stands for the constant function to 0.

Theorem 3.3. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_1(\mathcal{D}[1], \mathcal{D}[1])$;*
2. *X satisfies $S_1(\mathcal{O}, \mathcal{O})$ [Rothberger property];*
3. *$C_p(X)$ satisfies $S_1(\mathcal{D}_{\mathbf{0}}[1], \mathcal{D}_{\mathbf{0}}[1])$;*
4. *$C_p(X)$ satisfies $S_1(\mathcal{D}[1], \mathcal{D}_{\mathbf{0}}[1])$;*
5. *$C_p(X)$ satisfies $S_1(\mathcal{D}, \mathcal{D}[1])$.*

In ([4], Theorem 37), M. Bonanzinga, F. Cammaroto, M. Matveev proved

Theorem 3.4. *The following conditions are equivalent for a space X:*

1. *X is projectively $S_1(\mathcal{O},\mathcal{O})$ [$projectively Rothberger$];*
2. *every Lindelöf continuous image of X is Rothberger;*
3. *for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is Rothberger;*
4. *for every continuous mapping $f : X \mapsto \mathbb{R}$, $f(X)$ is Rothberger;*
5. *X satisfies $S_1(\mathcal{O}^\omega_{cz},\mathcal{O})$.*

Then, we have the next result.

Theorem 3.5. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_1(\mathcal{D}^\omega[1],\mathcal{D}[1])$;*
2. *X is projectively $S_1(\mathcal{O},\mathcal{O})$;*
3. *$C_p(X)$ satisfies $S_1(\mathcal{D}^\omega_{\mathbf{0}}[1],\mathcal{D}_{\mathbf{0}}[1])$;*
4. *$C_p(X)$ satisfies $S_1(\mathcal{D}^\omega[1],\mathcal{D}_{\mathbf{0}}[1])$;*
5. *$C_p(X)$ satisfies $S_1(\mathcal{D}^\omega,\mathcal{D}[1])$.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{O}_n : n \in \mathbb{N})$ be a sequence of countable cozero covers of X. Let $\mathcal{O}_n = \{U^n_i : i \in \mathbb{N}\}$ for every $n \in \mathbb{N}$, $U^n_i = \bigcup\limits_{k\in\mathbb{N}} F^n_{i,k}$ where $F^n_{i,k}$ is a zero-set of X for any $n,i,k \in \mathbb{N}$. Renumber the rational numbers $\mathbb{Q}$ as $\{q_k : k \in \mathbb{N}\}$.

We set $A_n = \{f^n_{i,k} \in C(X) : f^n_{i,k} \upharpoonright (X \setminus U^n_i) = 1$ and $f^n_{i,k} \upharpoonright F^n_{i,k} = q_k$ for $U^n_i \in \mathcal{O}_n$, the zero-set set $F^n_{i,k} \subset U^n_i$ and $q_k \in \mathbb{Q}\}$. It is not difficult to see that each A_n is a countable 1-dense subset of $C_p(X)$ because each $\mathcal{O}_n$ is a cover of X. By the assumption there exists $f^n_{i(n),k(n)} \in A_n$ such that $\{f^n_{i(n),k(n)} : n \in \mathbb{N}\} \in \mathcal{D}^\omega[1]$.

For each $f^n_{i(n),k(n)}$ we take $U^n_{i(n)} \in \mathcal{O}_n$ such that $f^n_{i(n),k(n)} \upharpoonright (X \setminus U^n_{i(n)}) = 1$.

Set $\mathcal{U} = \{U^n_{i(n)} : n \in \mathbb{N}\}$. For $x \in X$ we consider the basic open neighborhood $[x,W]$ of $\mathbf{0}$, where $W = (-\frac{1}{2},\frac{1}{2})$.

Note that there is $m \in \mathbb{N}$ such that $[x,W]$ contains $f^m_{i(n),k(n)} \in \{f^n_{i(n),k(n)} : n \in \mathbb{N}\}$. This means $x \in U^m_{i(m)}$. Consequently $\mathcal{U}$ is a countable cozero cover of X. By Theorem 3.4, X is projectively $S_1(\mathcal{O},\mathcal{O})$.

(2) $\Rightarrow$ (3). Let $B_n \in \mathcal{D}^\omega_f[1]$ for each $n \in \mathbb{N}$. We renumber $\{B_n\}_{n\in\mathbb{N}}$ as $\{B_{i,j}\}_{i,j\in\mathbb{N}}$. Since $C(X)$ is homogeneous, we may think that $f = \mathbf{0}$. We set $\mathcal{U}_{i,j} = \{g^{-1}[(-1/i,1/i)] : g \in B_{i,j}\}$ for each $i,j \in \mathbb{N}$. Since $B_{i,j} \in \mathcal{D}^\omega_{\mathbf{0}}[1]$, $\mathcal{U}_{i,j}$ is a countable cozero cover of X for each $i,j \in \mathbb{N}$. In case the set

$M = \{i \in \mathbb{N} : X \in \mathcal{U}_{i,j}\}$ is infinite, choose $g_m \in B_{m,j}$ $m \in M$ so that $g^{-1}[(-1/m, 1/m)] = X$, then $\{g_m : m \in \mathbb{N}\} \in \mathcal{D}_{\mathbf{0}}[1]$.

So we may assume that there exists $i' \in \mathbb{N}$ such that for each $i \geq i'$ and $g \in B_{i,j}$ we have that $g^{-1}[(-1/i, 1/i)]$ is not X.

For the sequence $\mathcal{V}_i = (\mathcal{U}_{i,j} : j \in \mathbb{N})$ of cozero covers there exists $f_{i,j} \in B_{i,j}$ such that $\mathcal{U}_i = \{f_{i,j}^{-1}[(-1/i, 1/i)] : j \in \mathbb{N}\}$ is a cover of X. Let $[x, W]$ be any basic open neighborhood of $\mathbf{0}$, where $W = (-\epsilon, \epsilon)$, $\epsilon > 0$. There exists $m \geq i'$ and $j \in \mathbb{N}$ such that $1/m < \epsilon$ and $x \in f_{m,j}^{-1}[(-1/m, 1/m)]$. This means $\{f_{i,j} : i, j \in \mathbb{N}\} \in \mathcal{D}_{\mathbf{0}}^{\omega}[1]$.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Let $A_n \in \mathcal{D}^{\omega}[1]$ for each $n \in \mathbb{N}$. We renumber $\{A_n\}_{n\in\mathbb{N}}$ as $\{A_{i,j}\}_{i,j\in\mathbb{N}}$. Renumber the rational numbers $\mathbb{Q}$ as $\{q_i : i \in \mathbb{N}\}$. Fix $i \in \mathbb{N}$. By the assumption there exists $f_{i,j} \in A_{i,j}$ such that $\{f_{i,j} : j \in \mathbb{N}\} \in \mathcal{D}_{q_i}[1]$ where q_i is the constant function to q_i. Then $\{f_{i,j} : i, j \in \mathbb{N}\} \in \mathcal{D}^{\omega}[1]$.

The remaining implications are proved in the same way as in the proof of Theorem 11.3 in [17] by replacing n-dense (dense) subsets of $C_p(X)$ with countable n-dense (dense) subsets of $C_p(X)$.

$\square$

Proposition 3.6. *(Proposition 38 in [4])*

1. *A space is Rothberger iff it is Lindelöf and projectively Rothberger [12].*
2. *Every projectively Rothberger space is zero-dimensional.*
3. *Every space of cardinality less than $cov(\mathcal{M})$ is projectively Rothberger.*
4. *The projectively Rothberger property is preserved by continuous images, by countably unions, by C^*-embedded zero-sets, and by cozero sets.*

Note that for a Tychonoff space X always there exists a countable 1-dense subset in $C_p(X)$. Namely, let $A = \{f_q \in C(X) : \text{where } f_q(x) = q \text{ for } \forall x \in X$ and $q \in \mathbb{Q}\}$.

Theorem 3.7. *A space X is Lindelöf if and only if each* 1*-dense set in $C_p(X)$ contains a countable* 1*-dense subset.*

Proof. ($\Rightarrow$). Let B be a 1-dense set in $C_p(X)$ and let A be a countable 1-dense in $C_p(X)$. Fix $m \in \mathbb{N}$. For each $x \in X$ there is $f_{q,m,x} \in A$ such that $f_{q,m,x}(x) \in (-\frac{1}{m} + q, \frac{1}{m} + q)$ where $q \in \mathbb{Q}$. Fix $q \in \mathbb{Q}$. Consider $\gamma_{q,m} = \{V_{f_{q,m,x}} : x \in X\}$ where $V_{f_{q,m,x}} = f_{q,m,x}^{-1}[(-\frac{1}{m} + q, \frac{1}{m} + q)]$ for each

$x \in X$. Then $\gamma_{q,m}$ is an open cover of X, hence, there is countable subcover $\gamma'_{q,m} = \{V_{f_{q,m,x_i}} : i \in \mathbb{N}\} \subset \gamma_{q,m}$ of X. Consider $\gamma = \bigcup\limits_{m\in\mathbb{N}, q\in\mathbb{Q}} \gamma'_{q,m}$.

Claim that $C_q = \{f_{q,m,x_i} : i \in \mathbb{N}, m \in \mathbb{N}\} \in B_q$, i.e. C is a 1-dense set in the point $f_q(x) = q$. Let $y \in X$ and $\epsilon > 0$. Then there are $m' \in \mathbb{N}$ and $f_{q,m',x_{i'}} \in C$ such that $\frac{1}{m'} < \epsilon$ and $f_{q,m',x_{i'}}(y_j) \in (-\frac{1}{m'} + q, \frac{1}{m'} + q) \subset (-\epsilon + q, \epsilon + q)$. Define $A = \bigcup\limits_{q\in\mathbb{Q}} C_q$. Clearly, that $A \subseteq B$ and A is a countable 1-dense subset of $C_p(X)$.

($\Leftarrow$). Let $\gamma = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X. Consider a set $B = \{f_{x,\lambda} \in C(X) : f_{x,\lambda}(x) = q$ and $f(X \setminus U_\lambda) \subset \{0\}$ where $x \in U_\lambda$ and $q \in \mathbb{Q}\}$. Since the space X is Tychonoff and γ is an open cover of X, B is a 1-dense subset of $C_p(X)$. There is a countable 1-dense subset $A = \{f_{x_i,\lambda_i} \in C(X) : i \in \mathbb{N}\} \subset B$. Then $\beta = \{U_{\lambda_i} : i \in \mathbb{N}\}$ is a countable cover of X. $\square$

By Proposition 3.6 and Theorem 3.7, we have the next

Proposition 3.8. 1. *A space $C_p(X)$ has the property $S_1(\mathcal{D}[1], \mathcal{D}[1])$ if and only if it is has the property $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$ and each 1-dense subset of $C_p(X)$ contains a countable 1-dense subset of $C_p(X)$.*

2. *If a space X has cardinality less than $cov(\mathcal{M})$ then $C_p(X)$ has the property $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$.*
3. *If $f : X \to Y$ is continuous mapping from a Tychonoff space X onto a Tychonoff space Y and $C_p(X)$ has the property $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$, then $C_p(Y)$ has the property $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$.*
4. *If $C_p(X)$ has the property $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$, then $C_p(X)^\omega$ has the property $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$.*
5. *If X has the property projectively Rothberger and Y is C^*-embedded zero-set in X (or cozero set of X), then $C_p(Y)$ has the property $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$.*

By Theorem 40 in [4] and Theorem 3.5, we have the next result.

Proposition 3.9. *If $C_p(X^n)$ has the property $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$ for every $n \in \mathbb{N}$, then all countable subspaces of $C_p(X)$ have countable strong fan tightness.*

4. The projectively Menger property

By Theorem 12.1 in [17], we have the following result.

Theorem 4.1. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}[1], \mathcal{D}[1])$;*
2. *X satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$ [Menger property];*
3. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}_{\mathbf{0}}[1], \mathcal{D}_{\mathbf{0}}[1])$;*
4. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}[1], \mathcal{D}_{\mathbf{0}}[1])$;*
5. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D}[1])$.*

In ([4], Theorem 6), M. Bonanzinga, F. Cammaroto, M. Matveev proved

Theorem 4.2. *The following conditions are equivalent for a space X:*

1. *X is projectively $S_{fin}(\mathcal{O}, \mathcal{O})$ [$projectively Menger$];*
2. *every Lindelöf continuous image of X is Menger;*
3. *for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is Menger;*
4. *for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is not dominating;*
5. *X satisfies $S_{fin}(\mathcal{O}^\omega_{cz}, \mathcal{O})$.*

Theorem 4.3. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}^\omega[1], \mathcal{D}[1])$;*
2. *X is projectively $S_{fin}(\mathcal{O}, \mathcal{O})$;*
3. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}^\omega_{\mathbf{0}}[1], \mathcal{D}_{\mathbf{0}}[1])$;*
4. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}^\omega[1], \mathcal{D}_{\mathbf{0}}[1])$;*
5. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}^\omega, \mathcal{D}[1])$.*

Proof. Similarly to the proofs of Theorem 4.1 and Theorem 12.1 in [17]. $\square$

Proposition 4.4. *(Proposition 8 in [4])*

1. *A space is Menger if and only if it is Lindelöf and projectively Menger [12].*
2. *Every σ-pseudocompact space is projectively Menger.*
3. *Every space of cardinality less than $\mathfrak{d}$ is projectively Menger.*
4. *The projectively Menger property is preserved by continuous images, by countably unions, by C^*-embedded zero-sets (Proposition 14 in [4]), and by cozero sets (Proposition 16 in [4]).*

By Proposition 4.4 and Theorem 4.3, we have the next

Proposition 4.5. 1. *A space $C_p(X)$ has the property $S_{fin}(\mathcal{D}[1],\mathcal{D}[1])$ iff is has the property $S_{fin}(\mathcal{D}^\omega[1],\mathcal{D}[1])$ and each 1-dense set in $C_p(X)$ contains a countable 1-dense set in $C_p(X)$.*

2. *If a space X has cardinality less than $\mathfrak{d}$ then $C_p(X)$ has the property $S_1(\mathcal{D}^\omega[1],\mathcal{D}[1])$.*
3. *If $f: X \mapsto Y$ is continuous mapping from a Tychonoff space X onto a Tychonoff space Y and $C_p(X)$ has the property $S_{fin}(\mathcal{D}^\omega[1],\mathcal{D}[1])$, then $C_p(Y)$ has the property $S_{fin}(\mathcal{D}^\omega[1],\mathcal{D}[1])$.*
4. *If $C_p(X)$ has the property $S_{fin}(\mathcal{D}^\omega[1],\mathcal{D}[1])$, then $C_p(X)^\omega$ has the property $S_{fin}(\mathcal{D}^\omega[1],\mathcal{D}[1])$.*
5. *If X has the projectively Menger property and Y is C^*-embedded zero-set in X (or cozero set of X), then $C_p(Y)$ has the property $S_{fin}(\mathcal{D}^\omega[1],\mathcal{D}[1])$.*

By Theorem 18 in [4] and Theorem 4.3, we have the next proposition.

Proposition 4.6. *If $C_p(X^n)$ has the property $S_{fin}(\mathcal{D}^\omega[1],\mathcal{D}[1])$ for every $n \in \mathbb{N}$, then all countable subspaces of $C_p(X)$ have countable fan tightness.*

5. The projectively Hurewicz property

Definition 5.1. (Sakai) An γ-cover $\mathcal{U}$ of cozero sets (**F**unctionally open sets) of X is *γ_F-shrinkable* if there exists a γ-cover $\{F_U : U \in \mathcal{U}\}$ of zero-sets of X with $F_U \subset U$ for every $U \in \mathcal{U}$.

For a topological space X we denote:

• Γ_F — the family of all γ_F-shrinkable covers of X.

By Theorem 4.1 in [18], X has the Hurewicz property if and only if X satisfies $U_{fin}(\Gamma_F,\Gamma)$ and X is Lindelöf.

Definition 5.2. A countable set $A \subset C(X)$ is called *weakly sequential dense subset of* $C_p(X)$ if $A = \{\mathcal{F}_n : \mathcal{F}_n \in [A]^{<\omega},\ n \in \mathbb{N}\}$ and for each $f \in C(X)$ there is $\{\mathcal{F}_{n_k} : k \in \mathbb{N}\} \subset A$ such that $\{\min\limits_{h \in \mathcal{F}_{n_k}} |h - f| : k \in \mathbb{N}\} \in \Gamma_{\mathbf{0}}$.

Clearly that any countable sequential dense subset of $C_p(X)$ is weakly sequential dense.

For a topological space X and $f \in C(X)$ we denote:

• $w\mathcal{S}$ — the family of all countable weakly sequential dense subset of $C_p(X)$.

• $w\Gamma_{\mathbf{f}} = \{A : A = \{\mathcal{F}_n : \mathcal{F}_n \in [A]^{<\omega}, n \in \mathbb{N}\} \subset C(X)$ such that $\{\min\limits_{h\in\mathcal{F}_n} |h - f| : n \in \mathbb{N}\} \in \Gamma_{\mathbf{0}}\}$.

• $w\Omega_{\mathbf{f}} = \{A\colon A = \{\mathcal{F}_n : \mathcal{F}_n \in [A]^{<\omega}, n \in \mathbb{N}\} \subset C(X)$ such that $\{\min\limits_{h\in\mathcal{F}_n} |h - f| : n \in \mathbb{N}\} \in \Omega_{\mathbf{0}}\}$.

• $w\mathcal{D} = \{A\colon A = \{\mathcal{F}_n : \mathcal{F}_n \in [A]^{<\omega}, n \in \mathbb{N}\} \subset C(X)$ such that $A \in w\Omega_{\mathbf{g}}$ for each $g \in C(X)\}$.

Note that $\mathcal{S} \subset w\mathcal{S}$, $\Gamma_{\mathbf{0}} \subset w\Gamma_{\mathbf{0}}$, $\Omega_{\mathbf{0}} \subset w\Omega_{\mathbf{0}}$ and $\mathcal{D} \subset w\mathcal{D}$.

In ([4], Theorem 30), M. Bonanzinga, F. Cammaroto, M. Matveev proved

Theorem 5.3. *The following conditions are equivalent for a space X:*

1. *X is projectively $U_{fin}(\mathcal{O}, \Gamma)$ [$projectively Hurewicz$];*
2. *Every Lindelöf continuous image of X is Hurewicz;*
3. *for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is Hurewicz;*
4. *for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is bounded;*
5. *X satisfies $U_{fin}(\mathcal{O}^\omega_{cz}, \Gamma)$.*

Theorem 5.4. *A space X is projectively Hurewicz if and only if X has the property $U_{fin}(\Gamma_F, \Gamma)$.*

Proof. Assume that X has the property $U_{fin}(\Gamma_F, \Gamma)$. We claim that X satisfies $U_{fin}(\mathcal{O}^\omega_{cz}, \Gamma)$. Let $(\mathcal{V}_i : i \in \mathbb{N})$ be a sequence of countable cozero covers of X where $\mathcal{V}_i = \{V^n_i : n \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Since V^n_i is a cozero set, we can represent $V^n_i = \bigcup\limits_{j=1}^{\infty} F^n_{i,j}$ where $F^n_{i,j}$ is a zero-set of X for each $i, j, n \in \mathbb{N}$ and $F^n_{i,j} \subset F^n_{i,j+1}$ for $j \in \mathbb{N}$. Consider $\mathcal{S}_i = \{S^n_i := \bigcup\limits_{p=1}^{n} F^p_{i,n} : n \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Note that $\mathcal{S}_i \in \Gamma_F$. Since X has the property $U_{fin}(\Gamma_F, \Gamma)$, there are finite sets $\mathcal{D}_i \subseteq \mathcal{S}_i$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{D}_i : i \in \mathbb{N}\} \in \Gamma$. It follows that X satisfies $U_{fin}(\mathcal{O}^\omega_{cz}, \Gamma)$.

□

By Theorem 4.2 in [18] and Theorem 5.4, we have the next theorem.

Theorem 5.5. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_{fin}(\Gamma_{\mathbf{0}}, w\Gamma_{\mathbf{0}})$;*
2. *X satisfies $U_{fin}(\Gamma_F, \Gamma)$;*
3. *X is projectively Hurewicz.*

By Theorem 4.5 in [18] and Theorem 5.4 we have the next theorem.

Theorem 5.6. *Assume that X has the V-property. Then the following statements are equivalent:*

1. $C_p(X)$ *satisfies* $S_{fin}(\mathcal{S}, w\mathcal{S})$*;*
2. X *satisfies* $U_{fin}(\Gamma_F, \Gamma)$*;*
3. X *is projectively Hurewicz;*
4. $C_p(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, w\Gamma_{\mathbf{0}})$*;*
5. $C_p(X)$ *satisfies* $S_{fin}(\mathcal{S}, w\Gamma_{\mathbf{0}})$*.*

Proposition 5.7. *(Proposition 31 in [4])*

1. *A space is Hurewicz iff it is Lindelöf and projectively Hurewicz [12].*
2. *Every σ-pseudocompact space is projectively Hurewicz.*
3. *Every space of cardinality less than $\mathfrak{b}$ is projectively Hurewicz.*
4. *The projectively Hurewicz property is preserved by continuous images, by countably unions, by C^*-embedded zero-sets, and by cozero sets.*

By Proposition 5.7 and Theorem 5.6, we have the next

Proposition 5.8.

1. *A space $C_p(X)$ has the property $S_{fin}(\mathcal{D}[1], w\mathcal{S})$ iff it has the property $S_{fin}(\mathcal{S}, w\mathcal{S})$ and each 1-dense set in $C_p(X)$ contains a countable 1-dense set in $C_p(X)$.*
2. *If a space X has cardinality less than $\mathfrak{b}$ then $C_p(X)$ has the property $S_{fin}(\mathcal{S}, w\mathcal{S})$.*
3. *If $f : X \mapsto Y$ is a continuous mapping from a Tychonoff space X onto a Tychonoff space Y and $C_p(X)$ has the property $S_{fin}(\mathcal{S}, w\mathcal{S})$, then $C_p(Y)$ has the property $S_{fin}(\mathcal{S}, w\mathcal{S})$.*
4. *If $C_p(X)$ has the property $S_{fin}(\mathcal{S}, w\mathcal{S})$, then $C_p(X)^\omega$ has the property $S_{fin}(\mathcal{S}^\omega, w\mathcal{S})$.*
5. *If X has the projectively Hurewicz property and Y is a C^*-embedded zero-set in X (or cozero set of X), then $C_p(Y)$ has the property $S_{fin}(\mathcal{S}, w\mathcal{S})$.*

6. Projectively Hurewicz + projectively Rothberger properties

Theorem 6.1. *(Theorem 50 in [4]) The following conditions are equivalent for a space X:*

1. *X is both projectively Hurewicz and projectively Rothberger;*

2. *every Lindelöf continuous image of X is both Hurewicz and Rothberger;*
3. *for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is both Hurewicz and Rothberger;*
4. *for every continuous mapping $f : X \mapsto \mathbb{R}$, $f(X)$ is both Hurewicz and Rothberger;*
5. *For every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of countable covers of X by cozero sets, one can pick $U_n \in \mathcal{U}_n$ so that $(U_n : n \in \mathbb{N})$ is groupable, that is there is a strictly increasing function $f : \omega \mapsto \omega$ such that for every $x \in X$, $x \in \bigcup\{U_i : f(n) \leq i < f(n+1)\}$ for all but finitely many n.*

Recall that $add(\mathcal{M}) = \min\{\mathfrak{b}, cov(\mathcal{M})\}$ [13].

Proposition 6.2. *(Proposition 51 in [4])*

1. *A space is both Hurewicz and Rothberger iff it is Lindelöf and it is both projectively Hurewicz and projectively Rothberger [12].*
2. *Every space of cardinality less than $add(\mathcal{M})$ is both projectively Hurewicz and projectively Rothberger.*

By Proposition 6.2, Theorem 5.6 and Theorem 4.1, we have the next result.

Proposition 6.3. 1. *A space $C_p(X)$ has properties $S_{fin}(\mathcal{D}[1], w\mathcal{S})$ and $S_1(\mathcal{D}[1], \mathcal{D}[1])$ iff it has properties $S_{fin}(\mathcal{S}, w\mathcal{S})$ and $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$ and each 1-dense set in $C_p(X)$ contains a countable 1-dense set in $C_p(X)$.*
2. *If a space X has cardinality less than $add(\mathcal{M})$ then $C_p(X)$ has properties $S_{fin}(\mathcal{S}, w\mathcal{S})$ and $S_1(\mathcal{D}^\omega[1], \mathcal{D}[1])$.*

7. The projectively Gerlits-Nagy property

Gerlits and Nagy [9] proved

Theorem 7.1. *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_1(\Omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;*
2. *X satisfies $S_1(\Omega, \Gamma)$.*

By Theorem 5.6 in [16], we have the next theorem.

Theorem 7.2. *Let X be a space with a coarser second countable topology. The following assertions are equivalent:*

1. $C_p(X)$ *satisfies* $S_1(\mathcal{D}, \mathcal{S})$*;*
2. *Each dense subspace of* $C_p(X)$ *contains a countable sequentially dense set in* $C_p(X)$*;*
3. X *satisfies* $S_1(\Omega, \Gamma)$*;*
4. $C_p(X)$ *satisfies* $S_1(\Omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$*;*
5. $C_p(X)$ *satisfies* $S_1(\mathcal{D}, \Gamma_{\mathbf{0}})$*.*

In ([4], Theorem 54), M. Bonanzinga, F. Cammaroto, M. Matveev proved

Theorem 7.3. *The following conditions are equivalent for a space* X*:*

1. X *satisfies projective* $S_1(\Omega, \Gamma)$ *[projectively Gerlits-Nagy];*
2. *every Lindelöf image of* X *has property* (γ)*;*
3. *for every continuous mapping* $f : X \mapsto \mathbb{R}^\omega$*,* $f(X)$ *satisfies* $S_1(\Omega, \Gamma)$*;*
4. *for every continuous mapping* $f : X \mapsto \mathbb{R}$*,* $f(X)$ *satisfies* $S_1(\Omega, \Gamma)$*;*
5. *for every countable* ω*-cover* $\mathcal{U}$ *of* X *by cozero sets, one can pick* $U_n \in \mathcal{U}$ *so that every* $x \in X$ *is contained in all but finitely many* U_n*;*
6. X *satisfies* $S_1(\Omega^\omega_{cz}, \Gamma)$*.*

Theorem 7.4. *The following conditions are equivalent for a space* X*:*

1. X *is projective* $S_1(\Omega, \Gamma)$*;*
2. $C_p(X)$ *satisfies* $S_1(\Omega^\omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$*.*

Proof. (1) $\Rightarrow$ (2). By Theorem 63 in [4].

(2) $\Rightarrow$ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open ω-covers of X. We set $A_n = \{f \in C(X) : f \upharpoonright (X \setminus U) = 0 \text{ for some } U \in \mathcal{U}_n\}$. It is not difficult to see that each A_n is dense in $C(X)$ since each $\mathcal{U}_n$ is an ω-cover of X and X is Tychonoff. Let f be the constant function to 1. By the assumption there exist $f_n \in A_n$ such that $f_n \mapsto f$ $(n \mapsto \infty)$.

For each f_n we take $U_n \in \mathcal{U}_n$ such that $f_n \upharpoonright (X \setminus U_n) = 0$.

Set $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$. For each finite subset $\{x_1, ..., x_k\}$ of X we consider the basic open neighborhood of f $[x_1, ..., x_k; W, ..., W]$, where $W = (0, 2)$.

Note that there is $n' \in \mathbb{N}$ such that $[x_1, ..., x_k; W, ..., W]$ contains f_n for $n > n'$. This means $\{x_1, ..., x_k\} \subset U_n$ for $n > n'$. Consequently $\mathcal{U}$ is an γ-cover of X.

□

Theorem 7.5. *Let* X *be a space with a coarser second countable topology. The following assertions are equivalent:*

1. $C_p(X)$ *satisfies* $S_1(\mathcal{D}^\omega, \mathcal{S})$;
2. $C_p(X)$ *is strongly sequentially separable;*
3. X *satisfies* $S_1(\Omega^\omega_{cz}, \Gamma)$;
4. $C_p(X)$ *satisfies* $S_1(\Omega^\omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;
5. $C_p(X)$ *satisfies* $S_1(\mathcal{D}^\omega, \Gamma_{\mathbf{0}})$;
6. X *is projectively* $S_1(\Omega, \Gamma)$ [*projectively Gerlits-Nagy*].

Recall that $l^*(X) \leq \aleph_0$ (X is called an ϵ-space) if all finite powers of X are Lindelëof (or, by Proposition in [9], if every ω-cover of X contains an at most countable ω-subcover of X).

Proposition 7.6. *(Proposition 55 in [4])*

1. *A space has property* $S_1(\Omega, \Gamma)$ *iff it is an* ϵ*-space and projectively* $S_1(\Omega, \Gamma)$ *[12].*
2. *Every projectively* $S_1(\Omega, \Gamma)$ *space is zero-dimensional.*
3. *Every space of cardinality less than* $\mathfrak{p}$ *is projectively* $S_1(\Omega, \Gamma)$.
4. *The projectively* $S_1(\Omega, \Gamma)$ *property is preserved by continuous images.*

Proposition 7.7. *Let* X *be a space with a coarser second countable topology. A space* X *is an* ϵ*-space iff each dense subset of* $C_p(X)$ *consists a countable dense subset of* $C_p(X)$.

Proof. ($\Rightarrow$). Let X be an ϵ-space, D be a dense subset of $C_p(X)$. By the Noble's Theorem 2.1, there is a countable dense subset $S = \{s_i : i \in \mathbb{N}\}$ of $C_p(X)$. By the Arhangel'skii-Pytkeev Theorem in [3], $t(C_p(X)) \leq \aleph_0$. For every $s \in S$ there exists $D_s \subseteq D$ such that $|D_s| = \aleph_0$ and $s \in \overline{D_s}$. A set $P = \bigcup\limits_{s\in S} D_s$. Then $|P| = \aleph_0$, $P \subseteq D$ and $\overline{P} = C_p(X)$.

($\Leftarrow$). Let $\mathcal{V}$ be a ω-cover of X. Consider a set $A_{V,K} = \{f \in C(X) : f(X \setminus V) \subseteq \{0\}$ and $f(k) = q_k$ where $k \in K$ and $q_k \in \mathbb{Q}\}$ where $V \in \mathcal{V}$, $K \in [X]^{<\omega}$ and $K \subset V$. Then $A = \bigcup\{A_{V,K} : V \in \mathcal{V}, K \in [X]^{<\omega}$ and $K \subset V\}$ is a dense subset of $C_p(X)$.

□

Proposition 7.8.
1. *A space* $C_p(X)$ *is strongly sequentially dense and separable iff it is strongly sequentially separable and each dense subset of* $C_p(X)$ *consists a countable dense subset of* $C_p(X)$.
2. *If* $C_p(X)$ *is strongly sequentially separable, then* X *is zero-dimensional.*

3. *If a space X of cardinality less than $\mathfrak{p}$, then $C_p(X)$ is strongly sequentially separable.*
4. *If $f : X \mapsto Y$ is a continuous mapping from a Tychonoff space X onto a Tychonoff space Y with a coarser second countable topology and $C_p(X)$ is strongly sequentially separable, then $C_p(Y)$ is strongly sequentially separable.*

By Theorem 6.1 in [15], we have

Proposition 7.9. (CH) *There is a consistent example of projectively $S_1(\Omega, \Gamma)$ space X with a coarser second countable topology such that X is not $S_1(\Omega, \Gamma)$.*

Proposition 7.10. *There is a projectively $S_1(\Omega, \Gamma)$ space X such that X^2 is not projectively $S_1(\Omega, \Gamma)$.*

Proof. Example 58 in [4]. □

Note that $S_1(\Omega, \Gamma) = S_{fin}(\Omega, \Gamma)$ (see [10]). It follows that the projectively $S_1(\Omega, \Gamma)$ property coincides with the projectively $S_{fin}(\Omega, \Gamma)$ property.

By Theorem 63 in [4] and Theorem 7.5,

Proposition 7.11. *If $C_p(X)$ is strongly sequentially separable, then all countable subspaces of $C_p(X)$ are strictly Fréchet-Urysohn.*

8. The projectively $S_1(\Omega, \Omega)$ property

In [20] (Lemma, Theorem 1), M. Sakai proved:

Theorem 8.1. $(Sakai)$ *For each space X the following are equivalent.*

1. *$C_p(X)$ satisfies $S_1(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$.*
2. *X^n satisfies $S_1(\mathcal{O}, \mathcal{O})$ (X^n has Rothberger's property C'') for each $n \in \mathbb{N}$.*
3. *X satisfies $S_1(\Omega, \Omega)$.*

In ([27], Theorem 13) M. Scheepers proved the following result.

Theorem 8.2. $(Scheepers)$ *For each separable metric space X, the following are equivalent:*

1. $C_p(X)$ *satisfies* $S_1(\mathcal{D}, \mathcal{D})$;
2. X *satisfies* $S_1(\Omega, \Omega)$.

By Theorem 57 in [5], [20] and Theorem 2.1, we have

Theorem 8.3. *Let X be a space with a coarser second countable topology. The following assertions are equivalent:*

1. $C_p(X)$ *satisfies* $S_1(\mathcal{D}, \mathcal{D})$ [*R-separable*];
2. $C_p(X)$ *satisfies* $S_1(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$;
3. $C_p(X)$ *satisfies* $S_1(\mathcal{D}, \Omega_{\mathbf{0}})$;
4. X *satisfies* $S_1(\Omega, \Omega)$;
5. X^n *satisfies* $S_1(\mathcal{O}, \mathcal{O})$ *for each* $n \in \mathbb{N}$.

Proposition 8.4. *The following conditions are equivalent for a space X:*

1. X *is projectively* $S_1(\Omega, \Omega)$;
2. X *satisfies* $S_1(\Omega^{\omega}_{cz}, \Omega)$;
3. *for every continuous mapping* $f : X \mapsto \mathbb{R}^{\omega}$, $f(X)$ *is* $S_1(\Omega, \Omega)$.
4. $C_p(X)$ *satisfies* $S_1(\Omega^{\omega}_{\mathbf{0}}, \Omega_{\mathbf{0}})$;

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable ω-covers of X by cozero sets. For every $n \in \mathbb{N}$ and every $U \in \mathcal{U}_n$ fix a continuous function $f_U : X \mapsto \mathbb{R}$ such that $U = f_U^{-1}[\mathbb{R} \setminus \{0\}]$. Put $h = \prod\{f_U : U \in \mathcal{U}_n, n \in \mathbb{N}\}$. Then h is a continuous mapping from X onto $h(X) \subset \mathbb{R}^{\omega}$, thus by (1), $h(X)$ satisfies $S_1(\Omega, \Omega)$. Since $(h(\mathcal{U}_n) : n \in \mathbb{N})$ be a sequence of open ω-covers of $h(X)$ we get (2).

(2) $\Rightarrow$ (3). Let f be a continuous mapping $f : X \mapsto \mathbb{R}^{\omega}$, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open ω-covers of $f(X)$. Since $f(X)$ is separable metrizable space, there is a refinement $\mathcal{V}_n$ of $\mathcal{U}_n$ that countable ω-cover of $f(X)$ and consists of cozero sets. Put $\mathcal{O}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$. Then $\mathcal{O}_n$ is a countable ω-cover of X by cozero sets. By (2), there is $H_n \in \mathcal{O}_n$ such that $\{H_n : n \in \mathbb{N}\}$ is a countable ω-cover of X and consists of cozero sets. For every $n \in \mathbb{N}$ pick $U_{H_n} \in \mathcal{U}_n$ such that $U_{H_n} \supset f(H_n)$. Put $\mathcal{F} = \{U_{H_n} : n \in \mathbb{N}\}$. Then $\mathcal{F}$ is an open ω-cover of $f(X)$. This proves that $f(X)$ satisfies $S_1(\Omega, \Omega)$.

(3) $\Rightarrow$ (1) follows from the fact that every second countable space can be embedded into $\mathbb{R}^{\omega}$.

(2) $\Rightarrow$ (4). Let $f \in \bigcap_n \overline{A_n}$, where A_n is a countable subset of $C(X)$. Since $C(X)$ is homogeneous, we may think that f is the constant function to the zero. We set $\mathcal{U}_n = \{g^{-1}[(-1/n, 1/n)] : g \in A_n\}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and each finite subset $\{x_1, ..., x_k\}$ of X a neighborhood $[x_1, ..., x_k; W, ..., W]$ of f, where $W = (-1/n, 1/n)$, contains some $g \in A_n$. This means that each $\mathcal{U}_n$ is a countable cozero ω-cover of X. In case the set $M = \{n \in \mathbb{N} : X \in \mathcal{U}_n\}$ is infinite, choose $g_m \in A_m$ $m \in M$ so that $g^{-1}(-1/m, 1/m) = X$, then $g_m \mapsto f$. So we may assume that there exists $n \in \mathbb{N}$ such that for each $m \geq n$ and $g \in A_m$ $g^{-1}[(-1/m, 1/m)]$ is not X. For the sequence $\{\mathcal{U}_m : m \geq n\}$ of cozero ω-covers there exist $f_m \in A_m$ such that $\mathcal{U} = \{f_m^{-1}[(-1/m, 1/m)] : m > n\}$ is a ω-cover of X. Let $[x_1, ..., x_k; W, ..., W]$ be any basic open neighborhood of f, where $W = (-\epsilon, \epsilon)$, $\epsilon > 0$. There exists $m \geq n$ such that $\{x_1, ..., x_k\} \subset f_m^{-1}[(-1/m, 1/m)]$ and $1/m < \epsilon$. This means $f \in \overline{\{f_m : m \in \mathbb{N}\}}$.

(4) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of countable cozero ω-covers of X. Let $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Since $U_{n,m}$ is cozero set, $U_{n,m} = \bigcup_{i \in \mathbb{N}} F_i^{n,m}$ where $F_i^{n,m}$ is zero set of X and $F_i^{n,m} \subset F_{i+1}^{n,m}$ for each $i \in \mathbb{N}$.

We set $A_n = \{f_i^{n,m} \in C(X) : f_i^{n,m} \upharpoonright (X \setminus U_{n,m}) = 1$ and $f_i^{n,m} \upharpoonright F_i^{n,m} = 0$ for $m, i \in \mathbb{N}\}$. It is not difficult to see that $f_0 \in \overline{A_n}$ (f_0 is the constant function to the zero) for each $n \in \mathbb{N}$ since each $\mathcal{U}_n$ is an ω-cover of X and X is Tychonoff.

By the assumption, there exists $f_{i(n)}^{n,m(n)} \in A_n$ such that $f_0 \in \overline{\{f_{i(n)}^{n,m(n)} : n \in \mathbb{N}\}}$.

For each $f_{i(n)}^{n,m(n)}$ we take $U_{n,m(n)} \in \mathcal{U}_n$. Set $\mathcal{U} = \{U_{n,m(n)} : n \in \mathbb{N}\}$.

For each finite subset $\{x_1, ..., x_k\}$ of X we consider the basic open neighborhood of f_0 $[x_1, ..., x_k; W, ..., W]$, where $W = (-1, 1)$.

Note that there is $n \in \mathbb{N}$ such that $[x_1, ..., x_k; W, ..., W]$ contains $f_{i(n)}^{n,m(n)}$. This means $\{x_1, ..., x_k\} \subset U_{n,m(n)}$. Consequently $\mathcal{U}$ is an ω-cover of X. $\square$

Definition 8.5. A space X is *R_ω-separable* if for every sequence $(D_n : n \in \mathbb{N})$ of countable dense subspaces of X one can pick $p_n \in D_n$ so that $\{p_n : n \in \mathbb{N}\}$ is dense in X, i.e X satisfies $S_1(\mathcal{D}^\omega, \mathcal{D})$.

Theorem 8.6. *For a space X with a coarser second countable topology, the following are equivalent:*

1. *$C_p(X)$ satisfies $S_1(\mathcal{D}^\omega, \mathcal{D})$ [R_ω-separable];*
2. *X satisfies $S_1(\Omega_{cz}^\omega, \Omega)$;*

3. $C_p(X)$ *satisfies* $S_1(\Omega^\omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$*;*
4. $C_p(X)$ *satisfies* $S_1(\mathcal{D}^\omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$*;*
5. X *is projectively* $S_1(\Omega, \Omega)$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of countable cozero ω-covers of X and $\{h_j : j \in \mathbb{N}\}$ be a countable dense subset of $C_p(X)$. Let $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Since $U_{n,m}$ is cozero set, $U_{n,m} = \bigcup\limits_{i \in \mathbb{N}} F_i^{n,m}$ where $F_i^{n,m}$ is zero set of X and $F_i^{n,m} \subset F_{i+1}^{n,m}$ for each $i \in \mathbb{N}$.

We set $A_n = \{f_i^{n,m} \in C(X) : f_i^{n,m} \restriction (X \setminus U_{n,m}) = 1$ and $f_i^{n,m} \restriction F_i^{n,m} = h_i$ for $m, i \in \mathbb{N}\}$. It is not difficult to see that A_n is a countable dense subspace of $C_p(X)$ for each $n \in \mathbb{N}$ since each $\mathcal{U}_n$ is an ω-cover of X and X is Tychonoff.

By the assumption there exists $h_{i(n)}^{n,m(n)} \in A_n$ such that $\{h_{i(n)}^{n,m(n)} : n \in \mathbb{N}\}$ is a dense subset of $C_p(X)$.

For each $h_{i(n)}^{n,m(n)}$ we take $U_{n,m(n)} \in \mathcal{U}_n$. Set $\mathcal{U} = \{U_{n,m(n)} : n \in \mathbb{N}\}$.

For each finite subset $\{x_1, ..., x_k\}$ of X we consider the basic open neighborhood of $\mathbf{0}$ $[x_1, ..., x_k; W, ..., W]$, where $W = (-1, 1)$.

Note that there is $n \in \mathbb{N}$ such that $[x_1, ..., x_k; W, ..., W]$ contains $h_{i(n)}^{n,m(n)}$. This means $\{x_1, ..., x_k\} \subset U_{n,m(n)}$. Consequently $\mathcal{U}$ is an ω-cover of X.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5). By Proposition 8.4 and Noble's Theorem 2.1.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Let $D = \{d_n : n \in \mathbb{N}\}$ be a countable dense subspace of $C_p(X)$. Given a sequence of countable dense subspace of $C_p(X)$, enumerate it as $\{S_{n,m} : n, m \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, pick $d_{n,m} \in S_{n,m}$ so that $d_n \in \overline{\{d_{n,m} : m \in \mathbb{N}\}}$. Then $\{d_{n,m} : m, n \in \mathbb{N}\}$ is dense in $C_p(X)$.

$\square$

By definition of projectively $S_1(\Omega, \Omega)$ space, a separable metrizable projectively $S_1(\Omega, \Omega)$ space has the property $S_1(\Omega, \Omega)$.

Proposition 8.7. (CH) *There is a consistent example of projectively* $S_1(\Omega, \Omega)$ *space* X *with a coarser second countable topology such that* X *is not* $S_1(\Omega, \Omega)$.

Proof. The Brendle's Theorem in [6] shows that there is a set of reals Z of size $\mathfrak{c}$ $(=\aleph_1)$ which has property $S_1(B_\Omega, B_\Gamma)$. We can certainly assume that $Z \subset (0, 1)$. Let $Y = Z \cup (-Z)$. Let X be a set Y with the topology of Sorgenfrey line. Then the space X such that X satisfies $S_1(\Omega^\omega_{cz}, \Gamma)$ and $iw(X) = \aleph_0$, but X^2 is not Lindelöf (Theorem 6.1 in [15]). Since $\Gamma^\omega_{cz} \subset \Omega^\omega_{cz}$,

by Proposition 8.4, X projectively $S_1(\Omega,\Omega)$. Since the property $S_1(\Omega,\Omega)$ is preserved under taking finite powers (Theorem 3.4 in [10]), X has not property $S_1(\Omega,\Omega)$ because X^2 is not Lindelöf.

$\square$

Proposition 8.8. ($\diamondsuit_{\omega_1}$) *There is a consistent example of projectively $S_1(\Omega,\Omega)$ space X with a coarser second countable topology such that X is not $S_1(\Omega,\Omega)$.*

Proof. By Theorem 6.2 in [15].

$\square$

Corollary 8.9. (CH or $\diamondsuit_{\omega_1}$) There is a consistent example of space X with a coarser second countable topology such that X satisfies $S_1(\Omega^{\omega}_{cz},\Omega)$, but X^2 is not $S_1(\mathcal{O}^{\omega}_{cz},\mathcal{O})$.

Clearly, that a countable R_ω-separable space is a R-separable space.

It is interesting to consider the following Question (Question 64, [5]):

Does there exists an X such that $C_p(X)$ is not R-separable but contains a dense R-separable subspace ?

Note that D. Repovš and L. Zdomskyy showed that there exists a Tychonoff space S such that $C_p(S)$ is not M-separable, but $C_p(S)$ contains a dense subset which is GN-separable (hence R-separable) under $\mathfrak{p}=\mathfrak{d}$ [19]. This implies a positive answer to Question under $\mathfrak{p}=\mathfrak{d}$.

By Proposition 8.7, we get a positive answer to this Question under CH or $\diamondsuit_{\omega_1}$.

Corollary 8.10. (CH or $\diamondsuit_{\omega_1}$) There is a consistent example of space X with a coarser second countable topology such that $C_p(X)$ is not R-separable, but for every countable dense subspace $M\subset C_p(X)$, M is R-separable.

Proposition 8.11. *Every space of cardinality less than $cov(\mathcal{M})$ is projectively $S_1(\Omega,\Omega)$.*

9. The projectively $S_{fin}(\Omega,\Omega)$ property

In ([3], Theorem 2.2.2 in [2]) A.V. Arhangel'skii proved the following result

Theorem 9.1. ($A.V.Arhangel'skii$) *For a space X, the following are equivalent:*

1. *$C_p(X)$ satisfies $S_{fin}(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$;*
2. *$(\forall n \in \mathbb{N})$ X^n satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$.*

It is known (see [10]) that X satisfies $S_{fin}(\Omega, \Omega)$ iff $(\forall n \in \mathbb{N})$ X^n satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$.

By Theorem 21 in [5] and Theorem 3.9 in [10], we have a next result.

Theorem 9.2. *For a space X with a coarser second countable topology the following are equivalent:*

1. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$;*
2. *X satisfies $S_{fin}(\Omega, \Omega)$;*
3. *$(\forall n \in \mathbb{N})$ $X^n \in S_{fin}(\mathcal{O}, \mathcal{O})$;*
4. *$C_p(X)$ satisfies $S_{fin}(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$;*
5. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}, \Omega_{\mathbf{0}})$.*

Proposition 9.3. *The following conditions are equivalent for a space X:*

1. *X is projectively $S_{fin}(\Omega, \Omega)$;*
2. *X satisfies $S_{fin}(\Omega^{\omega}_{cz}, \Omega)$;*
3. *for every continuous mapping $f : X \mapsto \mathbb{R}^{\omega}$, $f(X)$ is $S_{fin}(\Omega, \Omega)$;*
4. *$C_p(X)$ satisfies $S_{fin}(\Omega^{\omega}_{\mathbf{0}}, \Omega_{\mathbf{0}})$.*

Proof. By Theorem 4.3 in [23], each of the conditions (1),(2),(4) are equivalent to the condition: for any sequence $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ $(n \in \mathbb{N})$ of countable ω-covers of X consisting of cozero-sets in X, there is some $\varphi \in \omega^{\omega}$ such that $\{U_{n,m} : n \in \mathbb{N}, m \leq \varphi(n)\}$ is an ω-cover of X.

(3) $\Rightarrow$ (1) follows from the fact that every second countable space can be embedded into $\mathbb{R}^{\omega}$. $\square$

Definition 9.4. A space X is *M_{ω}-separable* if for every sequence $(D_n : n \in \mathbb{N})$ of countable dense subspaces of X one can select finite $F_n \subset D_n$ so that $\bigcup\{F_n : n \in \mathbb{N}\}$ is dense in X, i.e X satisfies $S_{fin}(\mathcal{D}^{\omega}, \mathcal{D})$.

Theorem 9.5. *For a space X with a coarser second countable topology, the following are equivalent:*

1. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}^{\omega}, \mathcal{D})$;*
2. *X satisfies $S_{fin}(\Omega^{\omega}_{cz}, \Omega)$;*
3. *$C_p(X)$ satisfies $S_{fin}(\Omega^{\omega}_{\mathbf{0}}, \Omega_{\mathbf{0}})$;*
4. *$C_p(X)$ satisfies $S_{fin}(\mathcal{D}^{\omega}, \Omega_{\mathbf{0}})$;*
5. *X is projectively $S_{fin}(\Omega, \Omega)$.*

Proposition 9.6. *Every space of cardinality less than $\mathfrak{d}$ is projectively $S_{fin}(\Omega, \Omega)$.*

10. The projectively $S_1(\Gamma, \Omega)$ property

Recall that a set A in a space X is a Z_σ-set in X, if $A = \bigcup_{i=1}^{\infty} F_i$ where F_i is a zero-set in X for each $i \in \mathbb{N}$. A set B is a CZ_σ-set in X, if $X \setminus B$ is a Z_σ-set in X.

Definition 10.1. A space X is called a *z-space*, if any Z_σ-set in X is a CZ_σ-set in X.

Note that if a perfectly normal space X is a z-space, then X is a σ-space (every F_σ-set is a G_δ-set).

Theorem 10.2. *For a z-space X, the following statements are equivalent:*

1. *X is projectively $S_1(\Gamma, \Omega)$;*
2. *X satisfies $S_1(\Gamma_{cz}, \Omega)$;*
3. *X satisfies $S_1(\Gamma_F, \Omega)$;*
4. *$C_p(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$.*

Proof. (1) $\Rightarrow$ (2). Assume that X is projectively $S_1(\Gamma, \Omega)$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable covers of X such that $\mathcal{U}_n \in \Gamma_{cz}$ for each $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$, fix a continuous function $f_U : X \mapsto \mathbb{R}$ such that $U = f_U^{-1}(\mathbb{R} \setminus \{0\})$. Put $f = \prod\{f_U : U \in \mathcal{U}_n, n \in \mathbb{N}\}$. Then f is a continuous mapping from X to $\mathbb{R}^\omega$, and thus by (1), $Y = f(X)$ has the property $S_1(\Gamma, \Omega)$. Put $\mathcal{V}_n = \{f(U) : U \in \mathcal{U}_n\}$. Then $\mathcal{V}_n$ is an γ-cover of Y. Since Y has the property $S_1(\Gamma, \Omega)$, there is $H_n \in \mathcal{V}_n$ such that $\{H_n : n \in \mathbb{N}\}$ is ω-cover of Y. Put $F_n = f^{-1}(H_n)$. Then $F_n \in \mathcal{U}_n$, and $\{F_n : n \in \mathbb{N}\}$ is ω-cover of X.

(2) $\Rightarrow$ (1). Let $f : X \mapsto Y$ be continuous mapping from X onto a second countable space Y, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ-covers of Y. Since Y is second countable, there is a countable subcover $\mathcal{W}_n \subset \mathcal{V}_n$. Put $\mathcal{O}_n = \{f^{-1}(W) : W \in \mathcal{W}_n\}$. Then $\mathcal{O}_n$ is a countable γ-cover of X by cozero sets. By (2), there is $H_n \in \mathcal{O}_n$ such that $\{H_n : n \in \mathbb{N}\}$ is ω-cover of X. For every $n \in \mathbb{N}$, pick $U_H \in \mathcal{U}_n$ such that $U_H \supset f(H)$. Put $\mathcal{F} = \{U_{H_n} : n \in \mathbb{N}\}$. Then $\mathcal{F}$ is ω-cover of Y. This proves that Y has the property $S_1(\Gamma, \Omega)$.

(2) $\Leftrightarrow$ (3). By Proposition 3.3 in [21].

(3) $\Leftrightarrow$ (4). By Proposition 6.4 in [16]. ☐

By Theorem 6.6 in [16] and Proposition 10.2, we have the next result.

Theorem 10.3. *For a z-space X with a coarser second countable topology, the following statements are equivalent:*

1. $C_p(X)$ *satisfies* $S_1(\mathcal{S}, \mathcal{D})$*;*
2. X *satisfies* $S_1(\Gamma_F, \Omega)$*;*
3. $C_p(X)$ *satisfies* $S_1(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$*;*
4. $C_p(X)$ *satisfies* $S_1(\mathcal{S}, \Omega_{\mathbf{0}})$*;*
5. X *is projectively* $S_1(\Gamma, \Omega)$*.*

Proposition 10.4. *Every space of cardinality less than $\mathfrak{d}$ is projectively $S_1(\Gamma, \Omega)$.*

11. The projectively $S_{fin}(\Gamma, \Omega)$ property

Proposition 11.1. *For a z-space X, the following statements are equivalent:*

1. X *is projectively* $S_{fin}(\Gamma, \Omega)$*;*
2. X *satisfies* $S_{fin}(\Gamma_{cz}, \Omega)$*;*
3. X *satisfies* $S_{fin}(\Gamma_F, \Omega)$*;*
4. $C_p(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$*.*

Proof. Similar the proof in Proposition 10.2 and by Theorem 7.2 in [16] and Theorem 76 in [4]. □

By Theorem 7.2 in [16] and Proposition 11.1, we have the next result.

Theorem 11.2. *For a z-space X with a coarser second countable topology, the following statements are equivalent:*

1. $C_p(X)$ *satisfies* $S_{fin}(\mathcal{S}, \mathcal{D})$*;*
2. X *satisfies* $S_{fin}(\Gamma_F, \Omega)$*;*
3. $C_p(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$*;*
4. $C_p(X)$ *satisfies* $S_{fin}(\mathcal{S}, \Omega_{\mathbf{0}})$*;*
5. X *is projectively* $S_{fin}(\Gamma, \Omega)$*.*

Proposition 11.3. *Every space of cardinality less than $\mathfrak{d}$ is projectively $S_{fin}(\Gamma, \Omega)$.*

12. The projectively $S_1(\Gamma, \Gamma)$ property

In [21] (Theorem 2.5), M. Sakai proved:

Theorem 12.1. (*Sakai*) *For a space X, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;*
2. *X satisfies $S_1(C_\Gamma, C_\Gamma)$ and it is strongly zero-dimensional.*

Theorem 12.2. *(Theorem 67 in [4]) The following properties are equivalent for a space X:*

1. *$C_p(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;*
2. *X satisfies $S_1(\Gamma_{cz}, \Gamma)$;*
3. *X is projectively $S_1(\Gamma, \Gamma)$.*

By Theorem 8.8 in [16] and Theorem 12.2, we have the next result.

Theorem 12.3. *For a z-space X and $X \models V$, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $S_1(\mathcal{S}, \mathcal{S})$;*
2. *X satisfies $S_1(\Gamma_F, \Gamma)$;*
3. *$C_p(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;*
4. *$C_p(X)$ satisfies $S_1(\mathcal{S}, \Gamma_{\mathbf{0}})$;*
5. *X is projectively $S_1(\Gamma, \Gamma)$;*
6. *X is projectively $S_{fin}(\Gamma, \Gamma)$.*

Proposition 12.4. *Every space of cardinality less than $\mathfrak{b}$ is projectively $S_1(\Gamma, \Gamma)$.*

13. The projectively $U_{fin}(\mathcal{O}, \Omega)$ property

By Theorem 3.4 in [18], X satisfies $U_{fin}(\mathcal{O}, \Omega)$ if and only if X satisfies $U_{fin}(\Gamma_F, \Omega)$ and X is Lindelöf.

Theorem 13.1. *For a space X, the following statements are equivalent:*

1. *X is projectively $U_{fin}(\mathcal{O}, \Omega)$;*
2. *X satisfies $U_{fin}(\mathcal{O}^{\omega}_{cz}, \Omega)$;*

3. X *satisfies* $U_{fin}(\Gamma_F, \Omega)$.

Proof. (1) $\Rightarrow$ (2). Assume that X is projectively $U_{fin}(\mathcal{O}, \Omega)$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable covers of X by cozero sets. For every $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$, fix a continuous function $f_U : X \mapsto \mathbb{R}$ such that $U = f_U^{-1}(\mathbb{R} \setminus \{0\})$. Put $f = \prod\{f_U : U \in \mathcal{U}_n, n \in \mathbb{N}\}$. Then f is a continuous mapping from X to $\mathbb{R}^\omega$, and thus by (1), $Y = f(X)$ has the property $U_{fin}(\mathcal{O}, \Omega)$. Put $\mathcal{V}_n = \{f(U) : U \in \mathcal{U}_n\}$. Then $\mathcal{V}_n$ is an open cover of Y. Since Y has the property $U_{fin}(\mathcal{O}, \Omega)$, there are finite subfamilies $\mathcal{H}_n \subset \mathcal{V}_n$ such that $\bigcup\{\mathcal{H}_n : n \in \mathbb{N}\}$ is ω-cover of Y. Put $\mathcal{F}_n = \{f^{-1}(H) : H \in \mathcal{H}_n\}$. Then $\mathcal{F}_n$ is a finite subfamily of $\mathcal{U}_n$, and $\bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$ is ω-cover of X.

(2) $\Rightarrow$ (1). Let $f : X \mapsto Y$ be continuous mapping from X onto a second countable space Y, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. Since Y is Lindelöf, there is a countable subcover $\mathcal{W}_n \subset \mathcal{V}_n$. Put $\mathcal{O}_n = \{f^{-1}(W) : W \in \mathcal{W}_n\}$. Then $\mathcal{O}_n$ is a countable cover of X by cozero sets. By (2), there are finite subfamilies $\mathcal{H}_n \subset \mathcal{O}_n$ such that $\bigcup\{\mathcal{H}_n : n \in \mathbb{N}\}$ is ω-cover of X. For every $n \in \mathbb{N}$ and every $H \in \mathcal{H}_n$, pick $U_H \in \mathcal{U}_n$ such that $U_H \supset f(H)$. Put $\mathcal{F}_n = \{U_H : H \in \mathcal{H}_n\}$. Then $\mathcal{F}_n$ is a finite subfamily of $\mathcal{U}_n$, and $\bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$ is ω-cover of Y. This proves that Y has the property $U_{fin}(\mathcal{O}, \Omega)$.

(2) $\Leftrightarrow$ (3). Proved analogously to the proof of Theorem 5.4. $\square$

By Theorem 13.1, Theorem 3.1 in [18] we have the next result.

Theorem 13.2. *For a space X, the following statements are equivalent:*

1. $C_p(X)$ *satisfies* $F_{fin}(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$;
2. X *satisfies* $U_{fin}(\Gamma_F, \Omega)$.
3. X *is projectively* $U_{fin}(\mathcal{O}, \Omega)$;
4. X *satisfies* $U_{fin}(\mathcal{O}^\omega_{cz}, \Omega)$.

By Theorem 13.1, Theorem 3.3 in [18] and Theorem 13.2 we have the next result.

Theorem 13.3. *Let X be a space with a coarser second countable topology. Then the following statements are equivalent:*

1. $C_p(X)$ *satisfies* $S_{fin}(\mathcal{S}, w\mathcal{D})$;
2. X *satisfies* $U_{fin}(\Gamma_F, \Omega)$;
3. $C_p(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, w\Omega_{\mathbf{0}})$;

4. $C_p(X)$ *satisfies* $S_{fin}(\mathcal{S}, w\Omega_{\mathbf{0}})$*;*
5. X *is projectively* $U_{fin}(\mathcal{O}, \Omega)$*;*
6. X *satisfies* $U_{fin}(\mathcal{O}^{\omega}_{cz}, \Omega)$*.*

Proposition 13.4. *Every space of cardinality less than* $\mathfrak{d}$ *is projectively* $U_{fin}(\mathcal{O}, \Omega)$*.*

14. The projectively $S_1(\Gamma, \mathcal{O})$ property

Similarly to the proof of Theorem 10.2 we have the next result.

Proposition 14.1. *For a* z*-space* X*, the following statements are equivalent:*

1. X *is projectively* $S_1(\Gamma, \mathcal{O})$*;*
2. X *satisfies* $S_1(\Gamma_{cz}, \mathcal{O})$*;*
3. X *satisfies* $S_1(\Gamma_F, \mathcal{O})$*;*
4. $C_p(X)$ *satisfies* $S_1(\Gamma_{\mathbf{0}}, \mathcal{D}_{\mathbf{0}}[1])$*.*

Theorem 14.2. *For a* z*-space* X *with a coarser second countable topology, the following statements are equivalent:*

1. $C_p(X)$ *satisfies* $S_1(\mathcal{S}, \mathcal{D}[1])$*;*
2. X *satisfies* $S_1(\Gamma_F, \mathcal{O})$*;*
3. $C_p(X)$ *satisfies* $S_1(\Gamma_{\mathbf{0}}, \mathcal{D}_{\mathbf{0}}[1])$*;*
4. $C_p(X)$ *satisfies* $S_1(\mathcal{S}, \mathcal{O}_{\mathbf{0}})$*;*
5. X *is projectively* $S_1(\Gamma, \mathcal{O})$*.*

Proposition 14.3. *Every space of cardinality less than* $\mathfrak{d}$ *is projectively* $S_1(\Gamma, \mathcal{O})$*.*

We can summarize the relationships between considered notions in next diagrams.

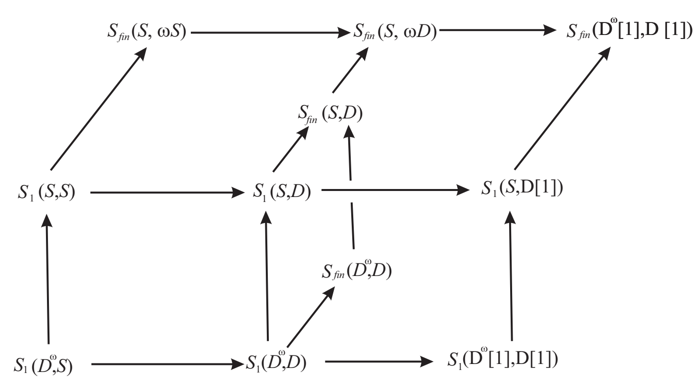

Fig. 2. The Diagram of selectors for sequences of dense (1-dense) sets of $C_p(X)$.

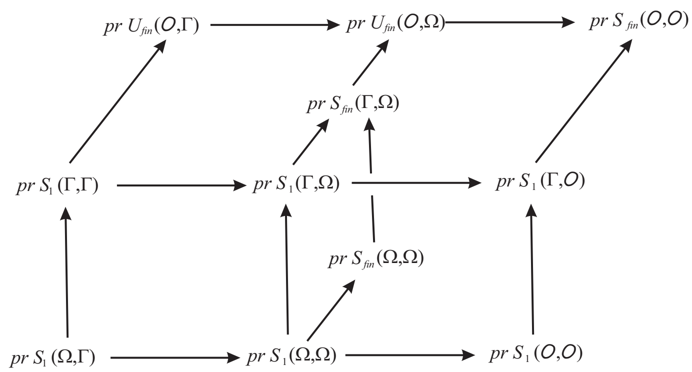

Fig. 3. The Diagram of projective selection principles for a space X (with corresponding conditions) corresponding to selectors for sequences of dense sets of $C_p(X)$.

15. Acknowledgements

The author would like to thank the referee for their careful reading of this paper.

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