

Topological properties of some function spaces

Saak Gabrielyan

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, P.O. 653, Israel

Alexander V. Osipov

*Krasovskii Institute of Mathematics and Mechanics, Ural Federal University,
Ural State University of Economics, Yekaterinburg, Russia*

Abstract

Let Y be a metrizable space containing at least two points, and let X be a $Y_{\mathcal{I}}$ -Tychonoff space for some ideal \mathcal{I} of compact sets of X . Denote by $C_{\mathcal{I}}(X, Y)$ the space of continuous functions from X to Y endowed with the \mathcal{I} -open topology. We prove that $C_{\mathcal{I}}(X, Y)$ is Fréchet–Urysohn iff X has the property $\gamma_{\mathcal{I}}$. We characterize zero-dimensional Tychonoff spaces X for which the space $C_{\mathcal{I}}(X, \mathbf{2})$ is sequential. Extending the classical theorems of Gerlits, Nagy and Pytkeev we show that if Y is not compact, then $C_p(X, Y)$ is Fréchet–Urysohn iff it is sequential iff it is a k -space iff X has the property γ . An analogous result is obtained for the space of bounded continuous functions taking values in a metrizable locally convex space. Denote by $B_1(X, Y)$ and $B(X, Y)$ the space of Baire one functions and the space of all Baire functions from X to Y , respectively. If H is a subspace of $B(X, Y)$ containing $B_1(X, Y)$, then H is metrizable iff it is a σ -space iff it has countable cs^* -character iff X is countable. If additionally Y is not compact, then H is Fréchet–Urysohn iff it is sequential iff it is a k -space iff it has countable tightness iff X_{\aleph_0} has the property γ , where X_{\aleph_0} is the space X with the Baire topology. We show that if X is a Polish space, then the space $B_1(X, \mathbb{R})$ is normal iff X is countable.

Keywords: function space, $C_p(X, Y)$, Baire function, metric space, Fréchet–Urysohn, sequential, k -space, normal, cs^* -character, σ -space, ideal of compact sets

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1. Introduction

For Tychonoff spaces X and Y , we denote by $C_p(X, Y)$ and $C_k(X, Y)$ the family $C(X, Y)$ of all continuous functions from X to Y endowed with the topology of pointwise convergence or the compact-open topology, respectively. If $Y = \mathbb{R}$, we shall write $C_p(X)$ and $C_k(X)$.

The study of topological properties of spaces of continuous functions is quite an active area of research attracting specialists both from General Topology and Functional Analysis and has a long history. Moreover, the study of topological properties of the function spaces $C_p(X)$ and $C_k(X)$ is one of the main topics in General Topology. For numerous results and historical remarks we refer the reader to the classical texts [1, 28] or to the recent monograph [41] and references therein. Let us recall some of the most famous results (all relevant definitions are given below or can be found for example in the classical book of Engelking [10]). We start from the following two remarkable theorems proved by Pol in his seminal paper [33], where $\mathbb{I} = [0, 1]$.

Email addresses: saak@math.bgu.ac.il (Saak Gabrielyan), OAB@list.ru (Alexander V. Osipov)

Theorem 1.1 (Pol). *If X is a metric space, then the following assertions are equivalent: (a) $C_p(X, \mathbb{I})$ is normal, (b) $C_p(X, \mathbb{I})$ is Lindelöf, (c) the set of all non-isolated points of X is separable.*

Theorem 1.2 (Pol). (i) *If X is a first countable paracompact space, the following assertions are equivalent: (a) the space $C_k(X, \mathbb{I})$ is a k -space, (b) for any compact metrizable space K , the space $C_k(X, K)$ is paracompact and Čech-complete, (c) $X = L \cup D$ is the topological sum of a locally compact Lindelöf space L and a discrete space D .*
(ii) *If X is a metric space, the following assertions are equivalent: (a) $C_k(X, \mathbb{I})$ is normal, (b) $C_k(X, \mathbb{I})$ is Lindelöf, (c) for any compact metrizable space K , the space $C_k(X, K)$ is Lindelöf, (d) the set of all non-isolated points of X is separable.*

The following fundamental result was proved independently by Gerlits and Nagy [14, 15] and Pytkeev [35].

Theorem 1.3 (Gerlits–Nagy–Pytkeev). *For a Tychonoff space X the following assertions are equivalent:*

- (i) $C_p(X)$ is a Fréchet–Urysohn space;
- (ii) $C_p(X)$ is a sequential space;
- (iii) $C_p(X)$ is a k -space;
- (iv) X has the property γ .

For a Tychonoff space X we denote by $C^b(X)$ the subspace of $C(X)$ containing all bounded functions. In [36], Pytkeev proved the following important result which we shall use below (for the definition of an ideal \mathcal{I} of compact sets in X and the \mathcal{I} -open topology see Section 2). Although this theorem is proved in [36] only for $C_{\mathcal{I}}(X)$, the proof for the space $C_{\mathcal{I}}^b(X)$ is exactly the same because all functions used in the proof can be chosen bounded.

Theorem 1.4 (Pytkeev). *Let X be a Tychonoff space, \mathcal{I} be an ideal of compact subsets of X and let $E = C_{\mathcal{I}}(X)$ or $E = C_{\mathcal{I}}^b(X)$. Then the following assertions are equivalent:*

- (i) E is a Fréchet–Urysohn space;
- (ii) E is a sequential space;
- (iii) E is a k -space.

Denote by $\mathbf{2}$ a discrete two element space. For a zero-dimensional Polish space X , a characterization of sequentiality of $C_k(X, \mathbf{2})$ was obtained in [19]. The following more general theorem was proved by the first author.

Theorem 1.5 ([11]). *Let X be a zero-dimensional metric space X . Then:*

- (i) $C_k(X, \mathbf{2})$ is a k -space if and only if either $X = L \cup D$ is a topological sum of a separable metrizable locally compact space L and a discrete space D or X is not locally compact but the set X' of non-isolated points of X is compact;
- (ii) $C_k(X, \mathbf{2})$ is a sequential space if and only if X is a Polish space and either X is locally compact or X is not locally compact but the set X' of non-isolated points of X is compact;
- (iii) $C_k(X, \mathbf{2})$ is a Fréchet–Urysohn space if and only if $C_k(X, \mathbf{2})$ is a Polish space if and only if X is a Polish locally compact space.

Let us also mention the following interesting result of Pol and Smentek [34]: *If X is a zero-dimensional realcompact k -space, then the group $C_k(X, \mathbf{2})$ is reflexive.* For other results concerning topological properties of $C_k(X, \mathbf{2})$ see [3, 11].

Let X be a Tychonoff space and let E be a locally convex space. In the theory of locally convex spaces there is a long tradition to investigate locally convex properties of the space $C(X, E)$ endowed with the pointwise topology or the compact-open topology by means of topological properties of X and locally convex properties of E . The most important case is the case when E is a Banach space (so if additionally X is compact, we obtain the widely studied class of Banach spaces). For numerous results obtained in the eighties of the last century and historical remarks we refer the reader to the well known lecture notes of Schmets [39].

The aforementioned results motivate the following general problem.

Problem 1.6. *Let Y be a metric space containing at least two points, \mathcal{I} be an ideal of compact sets in a topological space X , and let \mathcal{P} be a topological property. Characterize (in terms of X and Y) the spaces $C_{\mathcal{I}}(X, Y)$ with \mathcal{P} .*

In this paper we concentrate mainly on the properties of being a Fréchet–Urysohn, sequential, normal space or a k -space.

If X is connected and Y is discrete, then the function space $C(X, Y)$ contains only constant functions. So to avoid such unpleasant cases and to have the space $C(X, Y)$ sufficiently rich, we have to consider several separation axioms by analogy with the classical notion of Tychonoff spaces (notice that exactly by this reason the space X in [19, 34] and Theorem 1.5 is assumed to be zero-dimensional). For the topology of pointwise convergence such axioms were considered in [4]. In Section 2, we define and study necessary separation axioms and covering properties depending on an ideal of compact subsets of X which are essential for our main results.

In Section 3, in terms of covers of the space X we obtain the following results: (a) a characterization of spaces X for which the space $C_{\mathcal{I}}(X, Y)$ is Fréchet–Urysohn (Theorem 3.4), and (b) a characterization of zero-dimensional T_1 -spaces X for which the space $C_{\mathcal{I}}(X, \mathbf{2})$ is sequential (Theorem 3.8). In Section 4, for a non-compact metric space Y we show that the conclusion of the Gerlits–Nagy–Pytkeev Theorem 1.3 still remains true: $C_p(X, Y)$ is a Fréchet–Urysohn space if and only if it is a sequential space if and only if it is a k -space if and only if X has the property γ (see Theorem 4.3). Applying Pytkeev’s Theorem 1.4, we show in Theorem 4.5 that its conclusion is true also for the spaces $C_{\mathcal{I}}^b(X, Y)$ and $C_{\mathcal{I}}^{rc}(X, Y)$, where Y is a metrizable locally convex space and $C_{\mathcal{I}}^{rc}(X, Y)$ denotes the space of all continuous functions f whose image $f(X)$ is relatively compact in Y .

Although the classes of continuous functions are the most important, there are other classes of (noncontinuous) functions which are of significant importance and widely studied in General Topology and Analysis; for example, the classes of Baire type functions introduced and studied by Baire [2].

Let X and Y be topological spaces. For $\alpha = 0$, we put $B_0(X, Y) := C_p(X, Y)$. For every nonzero countable ordinal α , let $B_\alpha(X, Y)$ be the family of all functions $f : X \rightarrow Y$ that are pointwise limits of sequences $\{f_n\}_{n \in \omega} \subseteq \bigcup_{\beta < \alpha} B_\beta(X, Y)$ and $B(X, Y) := \bigcup_{\alpha < \omega_1} B_\alpha(X, Y)$. All the spaces $B_\alpha(X, Y)$ and $B(X, Y)$ are endowed with the topology of pointwise convergence, inherited from the Tychonoff product Y^X . If $Y = \mathbb{R}$, set $B_\alpha(X) := B_\alpha(X, \mathbb{R})$ and $B(X) := B(X, \mathbb{R})$.

The most important case is the case when X is a Polish space. This is explained not only by the classical results due to René Baire, Henri Lebesgue and others, but also by some deep results in the Banach space theory. If X is a Polish space, the compact subsets of $B_1(X)$ (called *Rosenthal compact*) have been studied intensively also by Godefroy [17], Todorčević [42] and many others.

For a Banach space Y , the compact subsets of $B_1(X, Y)$ are called *Rosenthal–Banach compacts*; they were introduced and studied in [29]. The study of topological properties of spaces of Baire functions is also motivated by the following fundamental theorem which is proved by Bourgain, Fremlin and Talagrand [6] and is a strengthening of a result of Rosenthal [38]:

Theorem 1.7 (Bourgain–Fremlin–Talagrand). *If X is a Polish space, then $B_1(X)$ is angelic.*

In a more general setting when X is a Tychonoff space, some of topological properties of the spaces $B_\alpha(X)$, $\alpha \in (0, \omega_1]$, were studied by Pestryakov in his thesis [32]. Among the others, he proved the following results.

Theorem 1.8 (Pestryakov). *Let X be a Tychonoff space and $0 < \alpha \leq \omega_1$.*

- (A) *The following assertions are equivalent:*
 - (i) $B_\alpha(X)$ *is a Fréchet–Urysohn space;*
 - (ii) $B_\alpha(X)$ *is a sequential space;*
 - (iii) $B_\alpha(X)$ *is a k -space;*
 - (iv) $B_\alpha(X)$ *has countable tightness;*
 - (v) X_{\aleph_0} *is a Lindelöf space;*
 - (vi) X_{\aleph_0} *satisfies the property γ .*
- (B) $t(B_\alpha(X)) = \sup\{l(X_{\aleph_0}^n) : n \in \omega\}$.

The next theorem was proved recently by the first author.

Theorem 1.9 ([12]). *Let G be a non-precompact abelian metrizable group, X a G -Tychonoff first countable space and let H be a subgroup of G^X containing $B_1(X, G)$. Then the following assertions are equivalent:*

- (i) X *is countable;*
- (ii) H *is a metrizable space and $H = G^X$;*
- (iii) H *has countable tightness;*
- (iv) H *has countable cs^* -character;*
- (v) H *is a σ -space;*
- (vi) H *is a k -space.*

If in addition $B_2(X, G) \subseteq H$, then (i)–(vi) are equivalent to

- (vii) H *is a normal space.*

In Section 5 we essentially extend Theorems 1.8 and 1.9, see Corollaries 5.22 and 5.23, respectively (see also Theorem 5.24).

To get our main results of Sections 3–5, we actively use the next two ideas. The first one was successfully applied in [12] to prove Theorem 1.9, and it is that instead of the whole function space $C(X, Y)$ we consider only some of its sufficiently rich and saturated subspaces, see Definition 3.1. For example, if Y is a locally convex space, then the spaces $C_{\mathcal{I}}^b(X, Y)$ and $C_{\mathcal{I}}^{rc}(X, Y)$ satisfy Definition 3.1 (Proposition 4.4). The second general idea used in Section 5 is standard in the theory of spaces of Baire function (see for example [22, 27]), and it is that we reduce the study of the spaces of Baire functions to the study of subspaces of the space $C_p(X_{\aleph_0}, Y)$, where X_{\aleph_0} is the set X endowed with the Baire topology (for details see Section 2). It turns out (Proposition 5.15) that already the space $B_1(X, Y)$ considered as a subspace of $C_p(X_{\aleph_0}, Y)$ is sufficiently rich in the sense

of Definition 3.1 and, therefore, we can apply our main results of Sections 3 and 4 to get essential generalizations of Theorems 1.8 and 1.9.

By Theorem 1.9, if $\alpha > 1$, then the space $B_\alpha(X, G)$ is normal if and only if X is countable. However, the question of whether the same holds true also for the space $B_1(X, G)$ of functions of the first Baire class remains open. In the last Section 6 we obtain a partial answer to this question in the most important case of Polish spaces X proving the following result (Corollary 6.10): *If X is a Polish space, then $B_1(X)$ is normal if and only if X is countable.*

2. Some covering properties and separation axioms

In this section we consider several covering properties and separation axioms which are essentially used in the paper.

2.1. Some covering properties

We start from some necessary definitions and notations. Let X be an arbitrary set and let γ be a family of subsets of X . Set

$$\underline{\lim} \gamma := \{x \in X : \text{the set } \{U \in \gamma : x \notin U\} \text{ is finite}\}.$$

If $\gamma = \{U_n\}_{n \in \omega}$, then $\underline{\lim} \gamma = \underline{\lim} U_n = \bigcup_{n \in \omega} \bigcap_{i \geq n} U_i$, and we also write $U_n \rightarrow \underline{\lim} \gamma$; if additionally $\underline{\lim} \gamma = X$ the sequence γ is called a *p-sequence*.

If \mathcal{U} and \mathcal{U}' are families of subsets of X , say that $\mathcal{U} \leq \mathcal{U}'$ if, for any $U \in \mathcal{U}$, there is $U' \in \mathcal{U}'$ such that $U \subseteq U'$. For families $\mathcal{U}_1, \dots, \mathcal{U}_n$ of subsets of X , set

$$\mathcal{U}_1 \wedge \dots \wedge \mathcal{U}_n := \{U_1 \cap \dots \cap U_n : U_i \in \mathcal{U}_i \text{ for all } i = 1, \dots, n\}.$$

It is clear that $\mathcal{U}_1 \wedge \dots \wedge \mathcal{U}_n \leq \mathcal{U}_i$ for every $i = 1, \dots, n$.

A sequence $\{\mathcal{U}_n\}_{n \in \omega}$ of subsets of X is called *decreasing* if $\mathcal{U}_{n+1} \leq \mathcal{U}_n$ for each $n \in \omega$.

Let X be a topological space. A family γ of subsets of X is called an ω -cover of X if for any finite $A \subseteq X$, there is $U \in \gamma$ such that $A \subseteq U$. Recall (see [1]) that the space X is said to have

- the *property* γ if every open ω -cover contains a *p-sequence*;
- the *property* φ if for any open ω -cover $\eta = \{\eta_n : n \in \omega\}$ of X with $\eta_n \leq \eta_{n+1}$ there exists a *p-sequence* $\xi = \{X_n : n \in \omega\}$ such that X_n is ω -covered by η_n for every $n \in \omega$;
- the *property* ε if any open ω -cover η of X has a countable ω -subcover.

The following proposition is Theorem 1 in [15] and Theorem II.3.2 of [1].

Proposition 2.1. *For a topological space X the following assertions are equivalent:*

- (i) *X has the property γ ;*
- (ii) *X has the properties φ and ε ;*
- (iii) *for any sequence $\{\gamma_n\}_{n \in \omega}$ of open ω -covers of X , one can choose $U_n \in \gamma_n$ for each $n \in \omega$, such that $\underline{\lim} U_n = X$.*

Below we generalize the above-mentioned notions to any ideal of compact sets in X . Recall that a family \mathcal{I} of compact subsets of a topological space X is called an *ideal of compact sets* if $\bigcup \mathcal{I} = X$ and for any sets $A, B \in \mathcal{I}$ and any compact subset $K \subseteq X$ we get $A \cup B \in \mathcal{I}$ and $A \cap K \in \mathcal{I}$, i.e. if \mathcal{I} covers X and is closed under taking finite unions and closed subspaces. The most important

cases are the ideal $\mathcal{I} = \mathcal{F}(X) = [X]^{<\omega}$ of all finite subsets of X and the ideal $\mathcal{I} = \mathcal{K}(X)$ of all compact subsets of X .

Let X be a topological space and let \mathcal{I} be an ideal of compact sets of X . A family γ of subsets of X is called an \mathcal{I} -cover of X if for any $A \in \mathcal{I}$, there is $U \in \gamma$ such that $A \subseteq U$, see [28, § 4.4]. An \mathcal{I} -sequence in X is any sequence $\{C_n\}_{n \in \omega}$ of subsets of X with the property that if $A \in \mathcal{I}$, then there exists an $m \in \omega$ such that $A \subseteq C_n$ for all $n \geq m$. So, if $\mathcal{I} = \mathcal{F}(X)$ or $\mathcal{I} = \mathcal{K}(X)$ and following [28], we shall say that an \mathcal{I} -cover is a p -cover (=an ω -cover) or a k -cover of X and an \mathcal{I} -sequence is a p -sequence or a k -sequence, respectively. Analogously to the properties γ , φ and ε , we say that the space X has

- the property $\gamma_{\mathcal{I}}$ if every open \mathcal{I} -cover contains an \mathcal{I} -sequence;
- the property $\varphi_{\mathcal{I}}$ if for any open \mathcal{I} -cover $\eta = \{\eta_n : n \in \omega\}$ of X with $\eta_n \leq \eta_{n+1}$ there exists an \mathcal{I} -sequence $\xi = \{X_n : n \in \omega\}$ such that X_n is \mathcal{I} -covered by η_n for every $n \in \omega$;
- the property $\varepsilon_{\mathcal{I}}$ if any open \mathcal{I} -cover η of X has a countable \mathcal{I} -subcover.

Remark 2.2. Let $\eta = \{\eta_n : n \in \omega\}$ be an open \mathcal{I} -cover of a Tychonoff space X such that $\eta_n \leq \eta_{n+1}$ for $n \in \omega$, and let $\xi = \{X_n : n \in \omega\}$ be an \mathcal{I} -sequence in X . For the property $\varphi_{\mathcal{I}}$, it is not necessary that X_n is \mathcal{I} -covered exactly by η_n , it suffices to assume that X_n is \mathcal{I} -covered by some η_{k_n} . Indeed, we can assume that $0 \leq k_0 < k_1 < \dots$. If $0 \leq k < k_1$, set $X'_k := \emptyset$. For every $n \geq 1$ and $k_n \leq k < k_{n+1}$, put $X'_k := X_n$. Then the sequence $\xi' = \{X'_k : k \in \omega\}$ \mathcal{I} -covers X and, for every $k \in \omega$, X'_k is \mathcal{I} -covered by η_k . \square

For the property $\gamma_{\mathcal{I}}$ we have the following analogue of Proposition 2.1 (we omit its proof because it is very similar to the proof of Proposition 2.1).

Proposition 2.3. *Let X be a topological space and let \mathcal{I} be an ideal of compact sets of X . Then X has the property $\gamma_{\mathcal{I}}$ if and only if it has the properties $\varphi_{\mathcal{I}}$ and $\varepsilon_{\mathcal{I}}$.*

Let X be a Tychonoff space and let $\{X_n\}_{n \in \omega}$ be an increasing cover of X (i.e., $X = \bigcup_{n \in \omega} X_n$ and $X_n \subseteq X_{n+1}$ for all $n \in \omega$). For every $n \in \omega$, let \mathcal{I}_n be an ideal of compact subsets in X_n such that $\mathcal{I}_n \subseteq \mathcal{I}_{n+1}$. It is easy to see that the family $\mathcal{I} := \bigcup_{n \in \omega} \mathcal{I}_n$ is an ideal of compact subsets of X . Conversely, if \mathcal{I} is an ideal of compact subsets of X , then for every $n \in \omega$, the family $\mathcal{I}_n = \mathcal{I} \cap X_n := \{K \in \mathcal{I} : K \subseteq X_n\}$ is an ideal of compact subsets in X_n . We shall say that $\{X_n\}_{n \in \omega}$ is \mathcal{I} -regular if $\mathcal{I} := \bigcup_{n \in \omega} \mathcal{I}_n$. It is clear that if $\mathcal{I} = \mathcal{F}(X)$, then any increasing cover $\{X_n\}_{n \in \omega}$ of X is $\mathcal{F}(X)$ -regular.

Let X be a Tychonoff space and let \mathcal{I} be an ideal of compact subsets of X . For every $n \in \omega$, denote by X_n the space X and let $Y := \bigoplus_{n \in \omega} X_n$ be the direct topological sum of the spaces X_n . Define the ideal $\mathcal{I}(Y)$ of compact sets in Y as the direct sum of \mathcal{I} , i.e., a compact subset F of Y belongs to $\mathcal{I}(Y)$ if and only if $F \cap X_n \in \mathcal{I}$ for every $n \in \omega$. Then, by the definition of $\mathcal{I}(Y)$, the cover $\{\bigoplus_{i=0}^n X_i\}_{n \in \omega}$ of Y is $\mathcal{I}(Y)$ -regular.

We say that the property $\gamma_{\mathcal{I}}$ is σ -additive on a Tychonoff space X if for every \mathcal{I} -regular increasing cover $\{X_n\}_{n \in \omega}$ of X such that all X_n have the property $\gamma_{\mathcal{I}_n}$ with $\mathcal{I}_n = \mathcal{I} \cap X_n$, it follows that the space X has the property $\gamma_{\mathcal{I}}$. In [23, Corollary 14], Jordan proved that the property γ ($=\gamma_p$) is σ -additive for every Tychonoff space X . We do not know whether for every Tychonoff space X and each ideal \mathcal{I} of compact sets in X , the property $\gamma_{\mathcal{I}}$ is σ -additive. Below we consider only one special case used in what follows.

Proposition 2.4. *Let X be a topological space and let \mathcal{I} be an ideal of compact sets of X . Then X has the property $\gamma_{\mathcal{I}}$ if and only if the direct topological sum $Y = \bigoplus\{X_i : X_i = X \text{ for each } i \in \omega\}$ has the property $\gamma_{\mathcal{I}(Y)}$.*

PROOF. Assume that X has the property $\gamma_{\mathcal{I}}$. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open $\mathcal{I}(Y)$ -cover of Y . We have to show that \mathcal{U} contains an $\mathcal{I}(Y)$ -sequence. For every $n \in \omega$, we identify X_n with X and set

$$\mathcal{V}_n := \{(U_\alpha \cap X_0) \cap \cdots \cap (U_\alpha \cap X_n) : \alpha \in \Lambda\},$$

and observe that \mathcal{V}_n is an \mathcal{I} -cover of X (indeed, if $F \in \mathcal{I}$ take $\alpha \in \Lambda$ such that $\bigoplus_{i=0}^n F \subseteq U_\alpha$). Note that $X \in \mathcal{V}_n$ if and only if $\bigoplus_{i=0}^n X_i \subseteq U_\alpha$ for some $\alpha \in \Lambda$. We distinguish between two cases.

Case 1. There exist a sequence $\{\alpha_k\}_{k \in \omega} \subseteq \Lambda$ and a strictly increasing sequence $\{n_k\}_{k \in \omega} \subseteq \omega$ such that $\bigoplus_{i=0}^{n_k} X_i \subseteq U_{\alpha_k}$. Then the sequence $\{U_{\alpha_k}\}_{k \in \omega}$ is an \mathcal{I} -sequence in \mathcal{U} , and we are done.

Case 2. There is an $m_0 \in \omega$ such that

$$X_0 \oplus \cdots \oplus X_n \not\subseteq U_\alpha \text{ for every } \alpha \in \Lambda \text{ and } n \geq m_0. \quad (2.1)$$

Since X has the property $\gamma_{\mathcal{I}}$ and all \mathcal{V}_n are \mathcal{I} -covers of X , for every $n \geq m_0$ there is an \mathcal{I} -sequence $\mathcal{S}_n = \{V_{\alpha(i,n)} : i \in \omega\} \subseteq \mathcal{V}_n$, where $\alpha(i,n) \in \Lambda$ and

$$V_{\alpha(i,n)} := (U_{\alpha(i,n)} \cap X_0) \cap \cdots \cap (U_{\alpha(i,n)} \cap X_n) \text{ for all } i \in \omega.$$

It follows from (2.1) that

$$V_{\alpha(i,n)} \neq X \text{ for all } i \in \omega \text{ and } n \geq m_0. \quad (2.2)$$

Denote by Ω the set of all finite sequences $\mathbf{i} = (i_0, \dots, i_s) \in \omega^{<\omega}$ such that $s = i_0$ and $i_0 < \cdots < i_s$. For every $\mathbf{i} = (i_0, \dots, i_s) \in \Omega$, set

$$W_{\mathbf{i}} := V_{\alpha(i_0, m_0)} \cap \cdots \cap V_{\alpha(i_s, m_0 + s)}. \quad (2.3)$$

We claim that the family $\mathcal{W} := \{W_{\mathbf{i}} : \mathbf{i} \in \Omega\}$ is an open \mathcal{I} -cover of X . Indeed, let $F \in \mathcal{I}$. Since \mathcal{S}_{m_0} is an \mathcal{I} -sequence, there is $i_0 > 0$ such that $F \subseteq V_{\alpha(i_0, m_0)}$. As all \mathcal{S}_n are \mathcal{I} -sequences, there are $i_1, \dots, i_{i_0} \in \omega$ such that $i_0 < \cdots < i_{i_0}$ and $F \subseteq V_{\alpha(i_t, m_0 + t)}$ for all $t = 1, \dots, i_0$. Then, putting $\mathbf{i} := (i_0, \dots, i_{i_0})$, we obtain $F \subseteq W_{\mathbf{i}}$.

Since X has the property $\gamma_{\mathcal{I}}$, there is a sequence

$$\{\mathbf{i}_k = (i_0^k, \dots, i_{s_k}^k) : k \in \omega\} \subseteq \Omega$$

such that $\{W_{\mathbf{i}_k} : k \in \omega\}$ is an \mathcal{I} -sequence for X . We claim that

$$i_0^k \rightarrow \infty \text{ as } k \rightarrow \infty. \quad (2.4)$$

Indeed, suppose for a contradiction that some subsequence $\{i_0^{k_j} : j \in \omega\}$ is contained in a finite set $\{0, \dots, T\}$. Then, by (2.3), the sets $W_{\mathbf{i}_{k_j}}$ are contained in one of the sets $V_{\alpha(0, m_0)}, \dots, V_{\alpha(T, m_0)}$. But since $\{W_{\mathbf{i}_{k_j}} : j \in \omega\}$ is also an \mathcal{I} -sequence for X , we obtain that the family $\{V_{\alpha(0, m_0)}, \dots, V_{\alpha(T, m_0)}\}$ is an \mathcal{I} -cover of X , and hence $X = V_{\alpha(i, m_0)}$ for some $i \in \{0, \dots, T\}$. However this contradicts (2.2). Thus (2.4) holds true.

To finish the proof of the necessity it is sufficient to show that the sequence $\mathcal{S} := \{U_{\alpha(i_{s_k}^k, m_0 + i_0^k)} : k \in \omega\}$ is an $\mathcal{I}(Y)$ -sequence. To this end, fix an arbitrary $\tilde{F} \in \mathcal{I}(Y)$. As \tilde{F} is compact, there is a $q_0 \in \omega$ such that $\tilde{F} \cap X_n = \emptyset$ for every $n > q_0$. Set $F := \bigcup_{i=0}^{q_0} (\tilde{F} \cap X_i)$. Then $F \in \mathcal{I}$. By (2.4) and since $\{W_{\mathbf{i}_k} : k \in \omega\}$ is an \mathcal{I} -sequence, we can choose an $r \in \omega$ such that

$$m_0 + i_0^k > q_0 \text{ and } F \subseteq W_{\mathbf{i}_k} \text{ for every } k \geq r.$$

Now (2.3) implies that for every $k \geq r$ (recall that $s_k = i_0^k$)

$$F \subseteq W_{i_k} \subseteq V_{\alpha(i_{s_k}^k, m_0 + i_0^k)} = \left(U_{\alpha(i_{s_k}^k, m_0 + i_0^k)} \cap X_0 \right) \cap \cdots \cap \left(U_{\alpha(i_0^k, m_0 + i_0^k)} \cap X_{m_0 + i_0^k} \right). \quad (2.5)$$

Since $m_0 + i_0^k > q_0$, (2.5) implies that $\tilde{F} \subseteq U_{\alpha(i_{s_k}^k, m_0 + i_0^k)}$ for every $k \geq r$. Thus \mathcal{S} is an $\mathcal{I}(Y)$ -sequence.

Assume now that Y has the property $\gamma_{\mathcal{I}(Y)}$. Let \mathcal{U} be an open \mathcal{I} -cover of X . We have to find an \mathcal{I} -sequence in \mathcal{U} . Consider the family $\mathcal{V} := \{U \oplus \bigoplus_{i=1}^{\infty} X_i : U \in \mathcal{U}\}$. Clearly, \mathcal{V} is an open $\mathcal{I}(Y)$ -cover of Y . By the property $\gamma_{\mathcal{I}(Y)}$, choose an $\mathcal{I}(Y)$ -sequence $\{U_n \oplus \bigoplus_{i=1}^{\infty} X_i : n \in \omega\}$ in \mathcal{V} . It is easy to see that the sequence $\{U_n : n \in \omega\} \subseteq \mathcal{U}$ is an \mathcal{I} -sequence, as desired. \square

2.2. Separation axioms

Let X and Y be topological spaces. For every $y \in Y$, define the constant function $\mathbf{y} \in Y^X$ by $\mathbf{y}(x) := y$ for every $x \in X$. Each ideal \mathcal{I} of compact subsets of X determines the \mathcal{I} -open topology $\tau_{\mathcal{I}}$ on the power space Y^X . A subbase of this topology consists of the sets

$$[K; U] = \{f \in Y^X : f(K) \subseteq U\},$$

where $K \in \mathcal{I}$ and U is an open subset of Y . In particular, for every $g \in Y^X$, finite subfamily $\mathcal{F} = \{F_1, \dots, F_n\} \subseteq \mathcal{I}$ and each family $\mathcal{U} = \{U_1, \dots, U_n\}$ of open subsets of Y such that $g(F_i) \subseteq U_i$ for every $i = 1, \dots, n$, the sets of the form

$$W[g; \mathcal{F}, \mathcal{U}] := \{f \in Y^X : f(F_i) \subseteq U_i \text{ for all } i = 1, \dots, n\}.$$

form a base of the \mathcal{I} -open topology $\tau_{\mathcal{I}}$ at the function g . The space $C(X, Y)$ endowed with the \mathcal{I} -open topology induced from $(Y^X, \tau_{\mathcal{I}})$ will be denoted by $C_{\mathcal{I}}(X, Y)$. So, $C_k(X, Y) = C_{\mathcal{I}}(X, Y)$ for the ideal $\mathcal{I} = \mathcal{K}(X)$ of compact subsets of X . For the ideal $\mathcal{I} = \mathcal{F}(X)$ of all finite subsets of X , the function space $C_{\mathcal{I}}(X, Y)$ is denoted by $C_p(X, Y)$. If $Y = G$ is an abelian topological group, then also $(G^X, \tau_{\mathcal{I}})$ is an abelian topological group and we put $[F; U] := \bigcap_{x \in F} W[\mathbf{0}; \{x\}, U]$. So the sets of the form $[F; U]$, where $F \in \mathcal{I}$ and $U \subseteq G$ is an open neighborhood of $0 \in G$, form an open base of the \mathcal{I} -open topology at zero function $\mathbf{0} \in G^X$.

Now we consider some separation axioms.

Definition 2.5 ([4]). Let Y be a topological space. A topological space X is called

- *Y -Tychonoff* if for every closed subset A of X , point $y_0 \in Y$ and function $f : F \rightarrow Y$ defined on a finite subset F of $X \setminus A$, there exists a continuous function $\bar{f} : X \rightarrow Y$ such that $\bar{f}|_F = f$ and $\bar{f}(A) \subseteq \{y_0\}$;
- *Y -normal* if X is a T_1 -space and for any closed set $F \subseteq X$ and each continuous function $f : F \rightarrow Y$ with finite image $f(F)$ there exists a continuous function $\bar{f} : X \rightarrow Y$ such that $\bar{f}|_F = f$.
- *Y -dimensional* if X is a T_1 -space and for any closed set $F \subseteq X$ and continuous function $f : F \rightarrow Y$, there exists a continuous function $\bar{f} : X \rightarrow Y$ such that $\bar{f}|_F = f$. \square

It is easy to see that every Y -normal space is Y -Tychonoff. Observe also that a topological space is \mathbb{R} -Tychonoff or \mathbb{R} -normal if and only if it is Tychonoff or normal in the standard sense, respectively. We will use repeatedly the following assertion.

Proposition 2.6 ([4]). Let Y be a topological space admitting a non-constant continuous function $\chi : Y \rightarrow \mathbb{R}$.

- (i) If X is Y -Hausdorff, then X is functionally Hausdorff.
- (ii) If X is Y -Tychonoff, then X is Tychonoff.
- (iii) If X is Y -normal, then X is normal.

We need the following generalization of Y -Tychonoffness.

Definition 2.7. Let X and Y be topological spaces and let \mathcal{I} be an ideal of compact sets of X . Then X is called $Y_{\mathcal{I}}$ -Tychonoff if for every closed subset A of X , point $y_0 \in Y$, compact set $F \in \mathcal{I}$ with $F \subseteq X \setminus A$, and each continuous function $f : F \rightarrow Y$ with finite image $f(F)$ there exists a continuous function $\bar{f} : X \rightarrow Y$ such that $\bar{f}|_F = f$ and $\bar{f}(A) \subseteq \{y_0\}$. \square

If $\mathcal{I} = \mathcal{F}(X)$ or $\mathcal{I} = \mathcal{K}(X)$, we shall say simply that X is Y_p -Tychonoff ($=Y$ -Tychonoff) or Y_k -Tychonoff, respectively. It is clear that

$$Y_k\text{-Tychonoff} \implies Y_{\mathcal{I}}\text{-Tychonoff} \implies Y_p\text{-Tychonoff}$$

for every ideal \mathcal{I} of compact sets.

The next lemma is a slight extension of Propositions 2.6.9, 2.6.10, 3.4.4 and 3.4.5 of [10], its proof is straightforward and hence omitted.

Lemma 2.8. Let Y be a topological space, X be a $Y_{\mathcal{I}}$ -Tychonoff space for an ideal \mathcal{I} of compact sets of X , and let $Z = \bigoplus \{X_i : X_i = X \text{ for each } i \in \omega\}$. Then the spaces $C_{\mathcal{I}(Z)}(Z, Y)$, $C_{\mathcal{I}}(X, Y^\omega)$ and the product $C_{\mathcal{I}}(X, Y)^\omega$ are homeomorphic.

Proposition 2.9. Let Y be a topological space admitting a non-constant continuous function $\chi : Y \rightarrow \mathbb{R}$, X be a topological space, and let \mathcal{I} be an ideal of compact sets of X . Then:

- (i) If X is $Y_{\mathcal{I}}$ -Tychonoff, then X is a Tychonoff space.
- (ii) If in addition Y is path-connected, then X is Tychonoff if and only if it is $Y_{\mathcal{I}}$ -Tychonoff.

PROOF. (i) Since X is also Y -Tychonoff, X is Tychonoff by Proposition 2.6.

(ii) Taking into account (i) it suffices to prove that if X is Tychonoff, then it is a Y_k -Tychonoff space. Fix a closed subset A of X , point $y_0 \in Y$, compact set $F \subseteq X \setminus A$, and continuous function $f : F \rightarrow Y$ with finite image $f(F) = \{y_1, \dots, y_n\}$, where y_1, \dots, y_n are pairwise distinct. Since Y is path-connected, there is a continuous function $h : [0, n] \rightarrow Y$ such that $h(i) = y_i$ for every $0 \leq i \leq n$. For every $i = 1, \dots, n$, set $K_i := f^{-1}(y_i)$, so K_i is a compact subset of F and hence of X . Now [16, 3.11(a)] implies that there is a continuous function $g : X \rightarrow [0, n]$ such that $g(A) \subseteq \{0\}$ and $g(K_i) = \{i\}$ for $1 \leq i \leq n$. Then the function $h \circ g : X \rightarrow Y$ is a desired continuous extension of f . \square

The next proposition extends Proposition 2.7 of [4].

Proposition 2.10. Let X be a T_1 -space, and let \mathcal{I} be an ideal of compact sets of X . Then the following conditions are equivalent:

- (i) X is $Y_{\mathcal{I}}$ -Tychonoff for any nonempty T_1 space Y .
- (ii) X is Y -Tychonoff for any nonempty T_1 space Y .
- (iii) X is $\mathbf{2}_{\mathcal{I}}$ -Tychonoff.
- (iv) X is $\mathbf{2}$ -Tychonoff.
- (v) X is a zero-dimensional space.

PROOF. The implications (i) \Rightarrow (ii), (ii) \Rightarrow (iv), (i) \Rightarrow (iii) and (iii) \Rightarrow (iv) are trivial. The implication (iv) \Rightarrow (v) is proved in Proposition 2.7 of [4]. To prove (v) \Rightarrow (i), assume that X is zero-dimensional. Being also a T_1 space, X is Tychonoff. Given any nonempty T_1 -space Y , fix a closed subset A of X , point $y_0 \in Y$ and continuous function $f : F \rightarrow Y$ with finite image $\{y_1, \dots, y_n\}$ defined on a subset $F \in \mathcal{I}$ of $X \setminus A$. For every $i = 1, \dots, n$, let $F_i := f^{-1}(y_i)$, so $F_i \in \mathcal{I}$ and all (compact) sets F_1, \dots, F_n and A are disjoint. Since X is zero-dimensional and Tychonoff, for every $i = 1, \dots, n$, choose a clopen neighborhood U_i of F_i such that the sets U_1, \dots, U_n and A are disjoint. Define $\bar{f} : X \rightarrow Y$ as follows: $\bar{f}(x) = y_i$ if $x \in U_i$, and $\bar{f}(x) = y_0$ if $x \in X \setminus \bigcup_{i=1}^n U_i$. It is clear that \bar{f} is a continuous extension of f such that $\bar{f}(A) \subseteq \{y_0\}$. \square

Now we recall the definition of the Baire topology of a topological space X which will be used essentially in the proof of the main results concerning spaces of Baire functions, see Section 5.

Let (X, τ) be a topological space. For a function $f \in C(X)$, denote by $Z(f) := f^{-1}(0)$ and $CZ(f) := X \setminus Z(f)$ the *zero-set* and the *cozero-set* of f . It is easy to see that the zero-set $Z(f)$ is a closed G_δ -set of X . If X is normal, then every closed G_δ -set of X is the zero-set of some real-valued continuous function on X , see [16, 3D.3]. The *Baire topology* τ_b on X is the topology on the underlying set X having for a basis the family of all zero-sets of X . Since the countable intersection of zero-sets is also a zero-set, it follows that the space X endowed with the Baire topology and denoted by X_{\aleph_0} is a P -space. Recall that a topological space is called a P -space if the intersection of a countable family of open sets is open. Let us recall also that the family of G_δ -sets in X forms a base of the topology τ_δ on X , and the space X with the topology τ_δ is called the *P -modification of X* and is denoted by PX or X_δ . Clearly, PX is a P -space and τ_δ is finer than the Baire topology τ_b . If X is a Tychonoff space, then $X_{\aleph_0} = PX$ and X_{\aleph_0} is a Tychonoff space.

Let X and Y be topological spaces. Analogously to zero-sets and cozero-sets we define

$$\begin{aligned} Z_Y &:= \{f^{-1}(F) : F \text{ is a closed set in } Y \text{ and } f \in C(X, Y)\}, \\ CZ_Y &:= \{f^{-1}(W) : W \text{ is an open set in } Y \text{ and } f \in C(X, Y)\}. \end{aligned}$$

Definition 2.11. Let Y be a topological space containing at least two points, and let (X, τ) be a Y -Tychonoff space. The family of all countable intersections of elements of Z_Y forms a base of the topology τ_Y on X finer than τ . The space $X_Y := (X, \tau_Y)$ is called the *Y_δ -modification of X* .

Observe that if Y is a normal space, then $\tau_Y \leq \tau_b$ (see the proof of (i) of Proposition 5.8 below). To obtain the equality $\tau_Y = \tau_b$ we introduce the following class of topological spaces.

Definition 2.12. Let Y be a topological space. A topological space X is called

- *Y - z -Tychonoff* if X is a Tychonoff space and for every closed subset $A \subseteq X$, disjoint zero-sets F_1, \dots, F_n in X such that $\bigcup_{i=1}^n F_i \subseteq X \setminus A$, and each points $y_0, \dots, y_n \in Y$ there exists a continuous function $f : X \rightarrow Y$ such that $f(A) \subseteq \{y_0\}$ and $f(F_i) = \{y_i\}$ for every $i = 1, \dots, n$;
- *Y - z -normal* if X is a Tychonoff space and for each zero-set $F \subseteq X$ and every continuous function $f : F \rightarrow Y$ with finite image $f(F)$ there exists a continuous function $\bar{f} : X \rightarrow Y$ such that $\bar{f}|_F = f$.

Proposition 2.13. Let X be a Tychonoff space. Then:

- (i) If X is Y -normal, then X is Y - z -normal.
- (ii) If Y is a path-connected topological space containing at least two points and X is a normal space, then X is Y - z -normal.

- (iii) If X is zero-dimensional, then X is Y - z -normal for every topological space Y .
- (iv) X is Tychonoff if and only if it is \mathbb{R} - z -Tychonoff.
- (v) If X is Y - z -normal, then X is Y - z -Tychonoff.
- (vi) If X is Y - z -Tychonoff, then X is Y -Tychonoff.

PROOF. (i) and (v) immediately follow from the corresponding definitions.

(ii) By Proposition 2.6 and Proposition 2.4 of [4], a topological space X is normal if and only if it is Y -normal. Now (i) applies.

(iii) Since X is zero-dimensional, Proposition 2.8 of [4] states that X is Y -normal for any nonempty topological space Y , and (i) applies.

(iv) follows from Proposition 1.5.13 of [10].

(vi) Fix a closed subset A of X , point $y_0 \in Y$ and function $f : F \rightarrow Y$ defined on a finite subset $F = \{x_1, \dots, x_n\}$ of $X \setminus A$. Since X is Tychonoff, there is a continuous function $g : X \rightarrow [0, n]$ such that $g(A) \subseteq \{0\}$ and $g(x_i) = i$ for every $i = 1, \dots, n$. Then the sets $F_i := g^{-1}([i - \frac{1}{3}, i + \frac{1}{3}])$ are disjoint zero-sets in X such that $\bigcup_{i=1}^n F_i \subseteq X \setminus A$. Define a continuous function $f' : \bigcup_{i=1}^n F_i \rightarrow Y$ by $f'|_{F_i} = f(x_i)$. Since X is Y - z -Tychonoff, f' has an extension $\bar{f} \in C(X, Y)$. It is clear that \bar{f} is a desired extension of f . \square

Example 2.14. *There is a first countable \mathbb{R} - z -Tychonoff space which is not \mathbb{R} - z -normal.*

PROOF. Let L be the Niemytzki plane, see Example 1.2.4 of [10]. Since L is Tychonoff it is \mathbb{R} - z -Tychonoff by (iv) of Proposition 2.13. Denote by L_1 the line $y = 0$. Then L_1 is a discrete subspace of L . Example 1.5.9 of [10] implies that there are two disjoint subsets A and B of L_1 such that for every open subsets U, V of L such that $A \subseteq U$ and $B \subseteq V$ it follows that $U \cap V \neq \emptyset$. It is clear that L_1 is the zero-set of the function $(x, y) \mapsto y$. Define $f : L_1 \rightarrow \mathbb{R}$ by $f(A) = \{1\}$, $f(B) = \{2\}$, and $f(L_1 \setminus (A \cup B)) \subseteq \{0\}$. Then, by the choice of A and B , the function f cannot be extended to L . Thus the Tychonoff space L is not \mathbb{R} - z -normal. \square

If X is perfectly normal we can prove more. Recall that a topological space X is called *perfectly normal* if it is a normal space and every closed subset of X is a G_δ -set. It is clear that a perfectly normal space has countable pseudocharacter, also we recall that every metrizable space is perfectly normal.

Proposition 2.15. *For any perfectly normal space X the following conditions are equivalent:*

- (i) X is Y - z -Tychonoff;
- (ii) X is Y -normal;
- (iii) X is Y - z -normal.

PROOF. By Proposition 2.13, (ii) \Rightarrow (iii) \Rightarrow (i). Let us prove (i) \Rightarrow (ii).

Assume that X is a Y - z -Tychonoff space. Let F be a closed subset of X and let $f : F \rightarrow Y$ be a continuous function with finite image $f(F) = \{y_1, \dots, y_n\}$ where all y_i are distinct. Since X is a perfectly normal space and F is closed, the set $F_i = f^{-1}(y_i)$ is a zero-set of X for each $i = 1, \dots, n$. It is clearly that $F_i \cap F_j = \emptyset$ for all distinct $i, j \in \{1, \dots, n\}$. By (i), there exists a continuous function $\bar{f} : X \rightarrow Y$ such that $\bar{f}|_F = f$. Thus X is Y -normal. \square

3. Fréchet–Urysohnness and sequentiality in some spaces of continuous functions

Since we shall consider subspaces H of $C(X, Y)$, some kind of richness of H will be necessary. Below we generalize the notion of being a Y -Tychonoff space.

Definition 3.1. Let X, Y be topological spaces, \mathcal{I} be an ideal of compact sets of X , \mathcal{D} be a subspace of Y , and let $H \subseteq S$ be two subspaces of the power space Y^X . Then H is called a *relatively $\mathcal{D}_{\mathcal{I}}$ -Tychonoff subspace of S* if for every closed subset A of X , compact subset $F \in \mathcal{I}$ contained in $X \setminus A$, point $y_0 \in \mathcal{D}$ and each $f \in S$ such that $f(F)$ is a finite subset of \mathcal{D} there is a function $\bar{f} \in H$ such that $\bar{f}|_F = f|_F$ and $\bar{f}(A) \subseteq \{y_0\}$. \square

In other words, H is a relatively $\mathcal{D}_{\mathcal{I}}$ -Tychonoff subspace of S if for every closed subset A of X , every function $f \in S$ and each $F \in \mathcal{I}$ such that $f(F)$ is a finite subset of \mathcal{D} and $F \subseteq X \setminus A$, the restriction $f|_F$ can be extended to a function $\bar{f} \in H$ with the additional condition $\bar{f}(A) \subseteq \{y_0\}$ for some point $y_0 \in Y$. If $\mathcal{I} = \mathcal{F}(X)$ or $\mathcal{I} = \mathcal{K}(X)$ we shall say that H is a *relatively \mathcal{D}_p -* or *relatively \mathcal{D}_k -Tychonoff subspace of S* , respectively. Observe that X is Y -Tychonoff if and only if $H = C(X, Y)$ is a relatively \mathcal{D}_p -Tychonoff subspace of $S = Y^X$.

Let us recall that a topological space X is called *Fréchet–Urysohn* if for any cluster point $a \in X$ of a subset $A \subseteq X$ there is a sequence $\{a_n\}_{n \in \omega} \subseteq A$ which converges to a .

Proposition 3.2. *Let Y be a T_1 topological space containing a two element subset $\mathcal{D} = \{g_0, g_1\}$, and let X be a $Y_{\mathcal{I}}$ -Tychonoff space for some ideal \mathcal{I} of compact sets of X . Assume that H is a relatively $\mathcal{D}_{\mathcal{I}}$ -Tychonoff subspace of $C_{\mathcal{I}}(X, Y)$ containing the constant function \mathbf{g}_0 . If H is a Fréchet–Urysohn space, then X has the property $\gamma_{\mathcal{I}}$.*

PROOF. Our proof is based on the idea of the proof of Theorem 4.7.4 of [28]. Let γ be an open \mathcal{I} -cover of X . If $X \in \gamma$, then $\xi = \{X\}$ is a desired \mathcal{I} -sequence. Assume that $X \notin \gamma$. Set

$$P := \{f \in H : f^{-1}(Y \setminus \{g_1\}) \subseteq U \text{ for some } U \in \gamma\}.$$

We claim that $\mathbf{g}_0 \in \overline{P} \setminus P$. Indeed, $\mathbf{g}_0 \notin P$ since $\mathbf{g}_0^{-1}(Y \setminus \{g_1\}) = X \notin \gamma$. Let $W = W[\mathbf{g}_0; F, V] \cap H$ be a standard neighborhood of \mathbf{g}_0 , where $F \in \mathcal{I}$ and $V \subseteq Y$ is a neighborhood of g_0 . As γ is an \mathcal{I} -cover, there is $U \in \gamma$ such that $F \subseteq U$. Since X is $Y_{\mathcal{I}}$ -Tychonoff and H is relatively $\mathcal{D}_{\mathcal{I}}$ -Tychonoff, there is a function $f \in H$ such that $f(x) = g_0$ for every $x \in F$ and $f(X \setminus U) \subseteq \{g_1\}$. Then $f^{-1}(Y \setminus \{g_1\}) \subseteq X \setminus (X \setminus U) = U$. Therefore $f \in P \cap W$. Thus $\mathbf{g}_0 \in \overline{P} \setminus P$.

Since H is a Fréchet–Urysohn space, there is a sequence $\{f_n\}_{n \in \omega} \subseteq P$ such that $f_n \rightarrow \mathbf{g}_0$. For every $n \in \omega$, the choice of f_n implies that there is a $U_n \in \gamma$ such that $f_n^{-1}(Y \setminus \{g_1\}) \subseteq U_n$. Set $\xi := \{U_n\}_{n \in \omega}$. We show that ξ is an \mathcal{I} -sequence. Indeed, fix an arbitrary $F \in \mathcal{I}$ and choose an open neighborhood V of $g_0 \in Y$ such that $g_1 \notin V$ (recall that Y is T_1). Then there exists an $m \in \omega$ such that $f_n \in W[\mathbf{g}_0; F, V]$ for every $n \geq m$. In particular, this implies that $F \subseteq f_n^{-1}(V) \subseteq f_n^{-1}(Y \setminus \{g_1\}) \subseteq U_n$ for all $n \geq m$. Thus ξ is an \mathcal{I} -sequence as desired. \square

Below we prove the first main result of this section.

Theorem 3.3. *Let Y be a metrizable space containing a two element subset $\mathcal{D} = \{g_0, g_1\}$, and let X be a $Y_{\mathcal{I}}$ -Tychonoff space for some ideal \mathcal{I} of compact sets of X . Assume that H is a relatively $\mathcal{D}_{\mathcal{I}}$ -Tychonoff subspace of $C_{\mathcal{I}}(X, Y)$ containing the constant function \mathbf{g}_0 . Then H is a Fréchet–Urysohn space if and only if X has the property $\gamma_{\mathcal{I}}$.*

PROOF. If H is Fréchet–Urysohn, then X has the property $\gamma_{\mathcal{I}}$ by Proposition 3.2.

Conversely, assume that X has the property $\gamma_{\mathcal{I}}$. It is well known that any metrizable space Y can be isometrically embedded into some Banach space E . Therefore H is a subspace of $C_{\mathcal{I}}(X, E)$. Hence it suffices to prove that the space $C_{\mathcal{I}}(X, E)$ is Fréchet–Urysohn. Let $A \subseteq C_{\mathcal{I}}(X, E)$ and $f \in \overline{A} \setminus A$. Since $C_{\mathcal{I}}(X, E)$ being a locally convex space is homogenous, we can assume that $f = \mathbf{0}$ is the zero-function. Let $\{V_n\}_{n \in \omega}$ be a strictly decreasing open base at 0 in E .

Below we use the idea from the proof of Theorem 4.7.4 of [28]. For every $n \in \omega$ and each $F \in \mathcal{I}$, choose a continuous function $g_{n,F} \in [F; V_n] \cap A$, and set

$$W(n, F) := \{x \in X : g_{n,F}(x) \in V_n\} \text{ and } \mathcal{W}_n := \{W(n, F) : F \in \mathcal{I}\}.$$

Since $\mathbf{0} \in \overline{A} \setminus A$, it follows that \mathcal{W}_n is an open \mathcal{I} -cover of X for every $n \in \omega$. Now we define a sequence $\{\mathcal{U}_n\}_{n \in \omega}$ of open \mathcal{I} -covers of X as follows: $\mathcal{U}_0 := \mathcal{W}_0$ and $\mathcal{U}_n := \mathcal{U}_{n-1} \wedge \mathcal{W}_n$ for $n \geq 1$.

It can be assumed that $X \notin \mathcal{I}$ (otherwise, the space $C_{\mathcal{I}}(X, E)$ is metrizable). Then the family $\{X \setminus \{x\} : x \in X\}$ is an open \mathcal{I} -cover of X . Therefore, by hypothesis, there is a sequence $\{x_n\}_{n \in \omega}$ in X such that $\{X \setminus \{x_n\} : n \in \omega\}$ is an \mathcal{I} -sequence. For each $n \in \omega$, define $\mathcal{U}'_n := \{U \setminus \{x_n\} : U \in \mathcal{U}_n\}$, and let $\mathcal{V} := \bigcup_{n \in \omega} \mathcal{U}'_n$. Then \mathcal{V} is an open \mathcal{I} -cover of X . Now choose an \mathcal{I} -sequence $\{O_k\}_{k \in \omega}$ from \mathcal{V} .

For every $k \in \omega$, choose an $n_k \in \omega$ such that $O_k \subseteq U_{n_k} \setminus \{x_{n_k}\}$ for some $U_{n_k} \in \mathcal{U}_{n_k}$. So there is an $F_{n_k} \in \mathcal{I}$ such that $O_k \subseteq W(n_k, F_{n_k}) \setminus \{x_{n_k}\}$. Observe that the sequence $\{n_k\}_{k \in \omega}$ cannot be bounded (indeed, if $n_k \leq M$ for all $k \in \omega$, then the finite set $\{x_0, \dots, x_M\} \in \mathcal{I}$ is not contained in O_k for every $k \in \omega$, and hence $\{O_k\}_{k \in \omega}$ is not an \mathcal{I} -sequence, a contradiction). Take an increasing subsequence $\{n_{k_i}\}_{i \in \omega}$, and let $f_i := g_{n_{k_i}, F_{n_{k_i}}}$.

We claim that $f_i \rightarrow \mathbf{0}$. Indeed, take an $F \in \mathcal{I}$ and an open neighborhood V of 0 in E . Choose an $m \in \omega$ such that $V_{n_{k_i}} \subseteq V$ and $F \subseteq O_{k_i}$ for every $i \geq m$. Then, for every $i \geq m$, $f_i(F) \subseteq V_{n_{k_i}} \subseteq V$ and hence $f_i \in [F; V]$. Thus $f_i \rightarrow \mathbf{0}$. \square

The metrizability condition on Y in Theorem 3.3 is essential. Indeed, if X is a singleton and $Y = \{0, 1\}^{\omega_1}$, then $C_p(X, Y)$ is topologically isomorphic to the compact non-sequential space Y .

The next theorem gives a complete answer to Problem 1.6 for the property of being a Fréchet–Urysohn space.

Theorem 3.4. *Let Y be a metrizable space containing at least two points, and let X be a $Y_{\mathcal{I}}$ -Tychonoff space for some ideal \mathcal{I} of compact sets of X . Then the following statements are equivalent:*

- (i) $C_{\mathcal{I}}(X, Y)$ is a Fréchet–Urysohn space;
- (ii) $C_{\mathcal{I}}(X, Y)^{\omega}$ is a Fréchet–Urysohn space;
- (iii) $C_{\mathcal{I}}(X, Y^{\omega})$ is a Fréchet–Urysohn space;
- (iv) X has the property $\gamma_{\mathcal{I}}$.

PROOF. The equivalence (i) \Leftrightarrow (iv) follows from Theorem 3.3. The implication (ii) \Rightarrow (i) is clear since $C_{\mathcal{I}}(X, Y)$ is homeomorphic to a subspace of $C_{\mathcal{I}}(X, Y)^{\omega}$, and the equivalence (ii) \Leftrightarrow (iii) follows from Lemma 2.8. To prove (iv) \Rightarrow (ii) we note first that, by Lemma 2.8, $C_{\mathcal{I}}(X, Y)^{\omega}$ is homeomorphic to the space $C_{\mathcal{I}(Z)}(Z, Y)$ where $Z = \bigoplus \{X_i : X_i = X \text{ for each } i \in \omega\}$. Now Proposition 2.4 implies that the space Z has the property $\gamma_{\mathcal{I}(Z)}$. Finally, applying Theorem 3.3 we obtain that the space $C_{\mathcal{I}}(X, Y)^{\omega}$ is Fréchet–Urysohn. \square

Let $Y = \mathbb{R}$ or $Y = \mathbb{I} := [0, 1]$. Then, by Theorem 3.1.7 of [10], X is $Y_{\mathcal{I}}$ -Tychonoff if and only if it is a Tychonoff space. Now Theorem 3.4 implies the following extension of Theorem 4.7.4 of [28]:

Corollary 3.5. *Let X be a Tychonoff space, and let \mathcal{I} be an ideal of compact sets of X . Then the following statements are equivalent:*

- (i) $C_{\mathcal{I}}(X)$ is a Fréchet–Urysohn space;
- (ii) $C_{\mathcal{I}}(X)^{\omega}$ is a Fréchet–Urysohn space;
- (iii) $C_{\mathcal{I}}(X, \mathbb{I})$ is a Fréchet–Urysohn space;
- (iv) $C_{\mathcal{I}}(X, \mathbb{I})^{\omega}$ is a Fréchet–Urysohn space;
- (v) $C_{\mathcal{I}}(X, \mathbb{I}^{\omega})$ is a Fréchet–Urysohn space;
- (vi) X has the property $\gamma_{\mathcal{I}}$.

Proposition 2.10 and Theorem 3.4 imply

Corollary 3.6. *Let X be a zero-dimensional T_1 -space and let \mathcal{I} be an ideal of compact sets of X . Then the following statements are equivalent:*

- (i) $C_{\mathcal{I}}(X, \mathbf{2})$ is a Fréchet–Urysohn space;
- (ii) $C_{\mathcal{I}}(X, \mathbf{2})^{\omega}$ is a Fréchet–Urysohn space;
- (iii) $C_{\mathcal{I}}(X, \mathbf{2}^{\omega})$ is a Fréchet–Urysohn space;
- (iv) X has the property $\gamma_{\mathcal{I}}$.

Remark 3.7. Combining Theorem 1.5 with Corollary 3.6 we obtain that a zero-dimensional metric space X has the property γ_k if and only if X is a Polish locally compact space. It follows from [30, Theorem 9.3] that if X is an arbitrary separable metric space, then the space X has the property γ_k if and only if X is hemicompact. \square

Recall that a topological space X is called *sequential* if for each non-closed subset $A \subseteq X$ there is a sequence $\{a_n\}_{n \in \omega} \subseteq A$ converging to some point $a \in \bar{A} \setminus A$. Below we characterize sequentiality of $C_{\mathcal{I}}(X, \mathbf{2})$. We say that a cover γ of a topological space X is *clopen* if every $U \in \gamma$ is a clopen (=closed and open) subset of X . Recall (see Proposition 2.10) that a T_1 -space X is zero-dimensional if and only if it is $\mathbf{2}$ -Tychonoff.

Theorem 3.8. *Let X be a zero-dimensional T_1 -space, and let \mathcal{I} be an ideal of compact subsets of X . Then the space $C_{\mathcal{I}}(X, \mathbf{2})$ is sequential if and only if for every clopen \mathcal{I} -cover γ of X either $X \in \gamma$ or there is a sequence $\{U_n\}_{n \in \omega}$ in γ such that*

- (i) *the set $W_0 := \lim_n U_n$ is a clopen subset of X and $W_0 \notin \gamma$;*
- (ii) *the set W_0 is \mathcal{I} -covered by $\{U_n\}_{n \in \omega}$;*
- (iii) *the set $X \setminus W_0$ is \mathcal{I} -covered by $\{X \setminus U_n\}_{n \in \omega}$.*

PROOF. Assume that the space $C_{\mathcal{I}}(X, \mathbf{2})$ is sequential and let γ be a clopen \mathcal{I} -cover of X . If $X \in \gamma$, we are done. Assume now that $X \notin \gamma$. Since γ is clopen, for every $U \in \gamma$ we can define a function $f_U \in C_{\mathcal{I}}(X, \mathbf{2})$ setting

$$f_U^{-1}(0) := U \text{ and } f_U^{-1}(1) := X \setminus U.$$

Set $P := \{f_U : U \in \gamma\}$ and denote by $\mathbf{0} \in C_{\mathcal{I}}(X, \mathbf{2})$ the zero function. We show that $\mathbf{0}$ belongs to $\bar{P} \setminus P$. Indeed, $\mathbf{0} \notin P$ since $\mathbf{0}^{-1}(0) = X \notin \gamma$. Let $W = W[\mathbf{0}; F, \{0\}]$ be a standard neighborhood of $\mathbf{0}$, where $F \in \mathcal{I}$. As γ is an \mathcal{I} -cover, there is $U \in \gamma$ such that $F \subseteq U$. Then clearly $f_U \in W$. Thus $\mathbf{0} \in \bar{P} \setminus P$, and hence P is a non-closed subset of $C_{\mathcal{I}}(X, \mathbf{2})$.

Since $C_{\mathcal{I}}(X, \mathbf{2})$ is a sequential space, there is a sequence $\{f_{U_n}\}_{n \in \omega} \subseteq P$ converging to some function $h \in \bar{P} \setminus P$. Set $W_0 := h^{-1}(0)$ and observe that W_0 is a clopen subset of X . The

equality $h = \lim_n f_{U_n}$ and the discreteness of the space $\mathbf{2}$ easily imply that $W_0 = \lim_n U_n$ and $h^{-1}(1) = X \setminus W_0 = \lim_n X \setminus U_n$. To prove (i), we have to show that $W_0 \notin \gamma$. But if $W_0 \in \gamma$ we would have $h = f_{W_0} \in P$ that contradicts the choice of h .

To show (ii) and (iii), fix a compact set $F \in \mathcal{I}$ such that $F \subseteq W_0$ (respectively, $F \subseteq X \setminus W_0$). Since $f_{U_n} \rightarrow h$, there is an $m \in \omega$ such that $f_{U_n} \in W[h; F, \{0\}]$ (respectively, $f_{U_n} \in W[h; F, \{1\}]$) for every $n \geq m$. But this means that $F \subseteq f_{U_n}^{-1}(0) = U_n$ (respectively, $F \subseteq f_{U_n}^{-1}(1) = X \setminus U_n$) for every $n \geq m$. Thus (ii) and (iii) hold true.

Conversely, assume that for every clopen \mathcal{I} -cover γ of X either $X \in \gamma$ or there is a sequence $\{U_n\}_{n \in \omega}$ in γ satisfying (i)-(iii). To show that the space $C_{\mathcal{I}}(X, \mathbf{2})$ is sequential, for every non-closed subset A of $C_{\mathcal{I}}(X, \mathbf{2})$, we have to find a function $f \in \overline{A} \setminus A$ and a sequence $\{f_n\}_{n \in \omega}$ in A converging to f . Since $C_{\mathcal{I}}(X, \mathbf{2})$ being an abelian topological group is homogenous, we can assume that the zero-function $\mathbf{0}$ belongs to $\overline{A} \setminus A$.

Set

$$\mathcal{W} := \{g^{-1}(0) : g \in A\}.$$

Since $\mathbf{0} \in \overline{A} \setminus A$ and all $g^{-1}(0)$ are clopen, it follows that \mathcal{W} is a clopen \mathcal{I} -cover of X . Observe that $X \notin \mathcal{W}$ since, otherwise, we would have $g_0^{-1}(0) = X$ for some $g_0 \in A$, and hence $\mathbf{0} = g_0 \in A$, a contradiction. By assumption, there is a sequence $\{f_n\}_{n \in \omega}$ in A such that the set

$$W_0 = \lim_n f_n^{-1}(0) \notin \gamma$$

is a clopen subset of X and is \mathcal{I} -covered by $\{f_n^{-1}(0)\}_{n \in \omega}$, and the set $X \setminus W_0$ is \mathcal{I} -covered by the sequence $\{X \setminus f_n^{-1}(0)\}_{n \in \omega}$. Since W_0 is clopen, we can define $f \in C_{\mathcal{I}}(X, \mathbf{2})$ by $f^{-1}(0) = W_0$ and $f^{-1}(1) = X \setminus W_0$. Let us show that $f \notin A$. Indeed, otherwise, the set $W_0 = f^{-1}(0)$ belongs to \mathcal{W} that contradicts (i).

We claim that $f_n \rightarrow f$. Indeed, since W_0 is clopen and the range is the doubleton $\mathbf{2}$, for every neighborhood \mathcal{O} of f there are $F_0, F_1 \in \mathcal{I}$ such that $F_0 \subseteq W_0$, $F_1 \subseteq X \setminus W_0$ and $\tilde{\mathcal{O}} := W[f; F_0, \{0\}] \cap W[f; F_1, \{1\}] \subseteq \mathcal{O}$. By (ii) and (iii), choose an $m \in \omega$ such that $F_0 \subseteq f_n^{-1}(0)$ and $F_1 \subseteq X \setminus f_n^{-1}(0) = f_n^{-1}(1)$ for every $n \geq m$. Then, for every $x \in F_0 \cup F_1$ and each $n \geq m$, we have $f_n(x) = f(x)$ and hence $f_n \in \tilde{\mathcal{O}} \subseteq \mathcal{O}$. Thus $f_n \rightarrow f$. \square

Remark 3.9. Let X be a zero-dimensional T_1 -space. If X has the property $\gamma_{\mathcal{I}}$, then X satisfies the conditions of Theorem 3.8. Indeed, let μ be a clopen \mathcal{I} -cover of X . If $X \in \mu$, we are done. If $X \notin \mu$, then, by the property $\gamma_{\mathcal{I}}$, there is an \mathcal{I} -sequence $\{U_n\}_{n \in \omega}$ in μ . Thus the conditions (i)-(iii) of Theorem 3.8 are satisfied if we put $W_0 = X$. \square

Remark 3.10. By Corollary 3.6, if $C_{\mathcal{I}}(X, \mathbf{2})$ is Fréchet–Urysohn, then also $C_{\mathcal{I}}(X, \mathbf{2})^{\omega}$ is a Fréchet–Urysohn space. One can ask whether the same is true for sequentiality. In general the answer is “no”. Indeed, let X be a countable non-locally compact metric space with one non-isolated point. Then, by Theorem 1.5, the space $C_k(X, \mathbf{2})$ is sequential (and non-Fréchet–Urysohn). However, the space $C_k(X, \mathbf{2})^{\omega} = C_k(Y, \mathbf{2})$, where $Y = \bigoplus_{\omega} X$, is not sequential by the same Theorem 1.5. \square

We finish this section with the following problem.

Problem 3.11. Let X be a zero-dimensional (metric) T_1 -space such that the space $C_p(X, \mathbf{2})$ is sequential. (α) Is it true that $C_p(X, \mathbf{2})$ is Fréchet–Urysohn? (β) Is it true that $C_p(X, \mathbf{2})^{\omega}$ is sequential?

Of course, by Corollary 3.6, a positive answer to (α) of Problem 3.11 implies a positive answer to (β) of this problem. However, we conjecture that answers to both questions in Problem 3.11 are negative.

4. The k -space property, normality and some cardinal numbers for subspaces of $C_{\mathcal{I}}(X, Y)$

Recall that a topological space X is called a k -space if for each non-closed subset $A \subseteq X$ there is a compact subset $K \subseteq X$ such that $A \cap K$ is not closed in K . To consider the k -space property for subspaces of $C_{\mathcal{I}}(X, Y)$ we need two lemmas.

Lemma 4.1. *Let Y be a topological space containing a discrete and closed sequence $\mathcal{D} = \{g_n\}_{n \in \omega} \subseteq Y$, and let X be a $Y_{\mathcal{I}}$ -Tychonoff space for some ideal \mathcal{I} of compact sets of X . Assume that H is a relatively $\mathcal{D}_{\mathcal{I}}$ -Tychonoff subspace of $C_{\mathcal{I}}(X, Y)$ containing the constant function \mathbf{g}_0 . If H is a k -space, then X has the property $\varphi_{\mathcal{I}}$.*

PROOF. We use the idea of the proof of Theorem 3 in [15]. Suppose for a contradiction that X does not have the property $\varphi_{\mathcal{I}}$. Then there exists an open \mathcal{I} -cover $\eta = \{\eta_n : n \in \omega\}$ of X with $\eta_n \leq \eta_{n+1}$ such that for every \mathcal{I} -sequence $\xi = \{X_n : n \in \omega\}$, one can find an $m = m_{\xi} \in \omega$ such that X_m is not \mathcal{I} -covered by η_m . For every $n \in \omega$, set

$$A_n := \{f \in H : f^{-1}(Y \setminus \{g_k : k > n\}) \text{ is } \mathcal{I}\text{-covered by } \eta_n\},$$

and put $A := \bigcup_{n \in \omega} A_n$.

We claim that the constant function \mathbf{g}_0 belongs to $\overline{A} \setminus A$ and hence A is not closed in H . Indeed, first we observe that $\mathbf{g}_0 \notin A$ since $\mathbf{g}_0^{-1}(Y \setminus \{g_k : k > n\}) = X$ is not \mathcal{I} -covered by η_n for every $n \in \omega$ (if X is \mathcal{I} -covered by η_m for some $m \in \omega$, then X is \mathcal{I} -covered by η_n for all $n \geq m$ since $\eta_m \leq \eta_n$; but this contradicts the choice of the cover η , see also Remark 2.2). Let now $W = W[\mathbf{g}_0; F, \mathcal{O}] \cap H$ be a standard neighborhood of \mathbf{g}_0 , where $F \in \mathcal{I}$ and \mathcal{O} is an open neighborhood of $g_0 \in Y$. Since $\eta = \{\eta_n : n \in \omega\}$ is an open \mathcal{I} -cover, there are an $l \in \omega$ and an open set $U \in \eta_l$ such that $F \subseteq U$. As X is $Y_{\mathcal{I}}$ -Tychonoff and H is a relatively $\mathcal{D}_{\mathcal{I}}$ -Tychonoff subspace of $C_{\mathcal{I}}(X, Y)$, there exists a function $f \in H$ such that $f(F) = \{g_0\}$ and $f(X \setminus U) = \{g_{l+1}\}$. It is clear that $f \in A_l$ and $f \in W$. Thus $\mathbf{g}_0 \in \overline{A} \setminus A$.

Let us show that for every $n \in \omega$, A_n is closed in H . Indeed, fix an arbitrary $f \in H \setminus A_n$. Then the set $B_n := f^{-1}(Y \setminus \{g_k : k > n\})$ is not \mathcal{I} -covered by η_n . Hence there exists an $F \in \mathcal{I}$ with $F \subseteq B_n$ such that no member of η_n contains F . Set $U_F := Y \setminus \{g_k : k > n\}$ (so U_F is open because the sequence $\{g_n\}_{n \in \omega}$ is closed in Y) and put

$$W := W[f; F, U_F] \cap H.$$

Then W is an open neighborhood of f . Now, if $g \in W \cap A_n$, then $F \subseteq g^{-1}(U_F) = g^{-1}(Y \setminus \{g_k : k > n\})$ is \mathcal{I} -covered by η_n that contradicts the choice of the set F . Thus $W \cap A_n = \emptyset$ and hence A_n is closed in H .

Since H is a k -space, there is a compact subset K of H such that $K \cap A$ is not closed in K . As K is compact, for every $x \in X$, the set $\{f(x) : f \in K\}$ is compact in Y . Taking into account that the sequence $\{g_n\}_{n \in \omega}$ is closed and discrete in Y , it follows that for every $x \in X$, there is an $n(x) \in \omega$ such that

$$f(x) \in Y \setminus \{g_k : k > n(x)\} \quad \text{for every } f \in K.$$

For every $n \in \omega$, set $X_n := \{x \in X : n(x) \leq n\}$. Then $X_n \subseteq X_{n+1}$ and $X = \bigcup_{n \in \omega} X_n$. We show that the sequence $\xi = \{X_n : n \in \omega\}$ is also an \mathcal{I} -sequence. Indeed, let $C \in \mathcal{I}$. Then C is a compact subset of X and $C_{\mathcal{I}}(C, Y) = C_k(C, Y)$ because \mathcal{I} is an ideal. Now Theorems 2.6.11 and 3.4.3 of [10] imply that the evaluation map

$$\Omega : C_k(C, Y) \times C \rightarrow Y, \quad \Omega(f, x) := f(x),$$

is continuous. Therefore the image $\Omega(K, C)$ is a compact subset of Y . Since \mathcal{D} is closed and discrete, there is an $r \in \omega$ such that $n(x) \leq r$ for every $x \in C$, i.e. $C \subseteq X_r$. Thus ξ is an \mathcal{I} -sequence as stated.

The choice of the open \mathcal{I} -cover $\eta = \{\eta_n : n \in \omega\}$ and Remark 2.2 imply that there exists an $m \in \omega$ such that X_m is not \mathcal{I} -covered by η_n for every $n \in \omega$.

We show that $K \cap A_k = \emptyset$ for every natural number $k > m$. Indeed, fix $k > m$ and let $f \in A_k$. By the definition of A_k , the set $f^{-1}(Y \setminus \{g_i : i > k\})$ is \mathcal{I} -covered by η_k . Since X_m is not \mathcal{I} -covered by η_k , we obtain that

$$X_m \setminus f^{-1}(Y \setminus \{g_i : i > k\}) \neq \emptyset,$$

i.e. there is a point $x \in X_m$ such that $f(x) \in \{g_i : i > k > m\}$. Then the definition of X_m implies that $f \notin K$. Thus $K \cap A_k = \emptyset$.

Now $K \cap A = K \cap \bigcup_{k \in \omega} A_k = \bigcup_{k=0}^m (K \cap A_k)$ is closed in K since all A_n s are closed in H . But this contradicts the choice of K . Thus the space X must have the property $\varphi_{\mathcal{I}}$. \square

Lemma 4.2. *Let Y be a topological space containing a discrete two-element subset $\mathcal{D} = \{g_0, g_1\}$, X be a Tychonoff space satisfying the property φ , and let H be a subspace of $C_p(X, Y)$ such that $H \cap C_p(X, \mathcal{D})$ is a relatively \mathcal{D}_p -Tychonoff subspace of $C_p(X, \mathcal{D})$ and $\mathbf{g}_0 \in H$. If H is a k -space, then X satisfies the property γ .*

PROOF. Since X has the property φ , Corollary II.3.6 of [1] implies that X is zero-dimensional. Observe that the space $C_p(X, \mathcal{D})$ is a closed subspace of $C_p(X, Y)$, and hence $H \cap C_p(X, \mathcal{D})$ is closed in H . Therefore $H \cap C_p(X, \mathcal{D})$ is a k -space as well. Since X has the property φ , Proposition 2.1 implies that to prove the lemma it suffices to show that X has the property ε .

Suppose for a contradiction that the property ε is not fulfilled. Then there is an open ω -cover η of X which does not have a countable ω -subcover. Put

$$A := \{f \in H \cap C(X, \mathcal{D}) : f^{-1}(g_0) \text{ can be } \omega\text{-covered by a countable subfamily of } \eta\}.$$

Now repeating word for word the proof of Theorem 4 of [15], we obtain that $H \cap C_p(X, \mathcal{D})$ is not a k -space, a contradiction. Thus X has also the property ε . \square

Now we prove the first main result of this section, which applying to $H = C_p(X)$ immediately yields Gerlits–Nagy–Pytkeev Theorem 1.3.

Theorem 4.3. *Let Y be a non-compact metrizable space, and let X be a Y -Tychonoff space. Fix a closed and discrete sequence $\mathcal{D} = \{g_n\}_{n \in \omega}$ in Y , and let H be a relatively \mathcal{D}_p -Tychonoff subspace of $C_p(X, Y)$ containing the constant function \mathbf{g}_0 . Then the following assertions are equivalent:*

- (i) H is a Fréchet–Urysohn space;
- (ii) H is a sequential space;
- (iii) H is a k -space;
- (iv) X satisfies the property γ .

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear. Observe that $H \cap C_p(X, \{g_0, g_1\})$ is a relatively $\{g_0, g_1\}_p$ -Tychonoff subspace of $C_p(X, \{g_0, g_1\})$. Now the implication (iii) \Rightarrow (iv) follows from Lemmas 4.1 and 4.2. Finally, the implication (iv) \Rightarrow (i) follows from Theorem 3.3. \square

Let X be a Tychonoff space and let E be a locally convex space. Recall that we denote by $C^b(X, E)$ and $C^{rc}(X, E)$ the spaces of all functions $f \in C(X, E)$ such that $f(X)$ is a bounded or a relatively compact subset of E , respectively. We need also the next assertion.

Proposition 4.4. *Let E be a non-trivial locally convex space, X be a topological space, and let \mathcal{I} be an ideal of compact sets of X . Then $C_{\mathcal{I}}^b(X, E)$ and $C_{\mathcal{I}}^{rc}(X, E)$ are relatively $E_{\mathcal{I}}$ -Tychonoff subspaces of $C_{\mathcal{I}}(X, E)$.*

PROOF. The proof follows from the proof of (ii) of Proposition 2.9. Indeed, the extension $h \circ g$ constructed there belongs to $C_{\mathcal{I}}^{rc}(X, E)$ because $h \circ g(X)$ is contained in the compact subset $h([0, n])$ of E . \square

Theorem 4.5. *Let X be a Tychonoff space, \mathcal{I} be an ideal of compact subsets of X , Y be a metrizable locally convex space, and let $E = C_{\mathcal{I}}(X, Y)$, $E = C_{\mathcal{I}}^b(X, Y)$ or $E = C_{\mathcal{I}}^{rc}(X, Y)$. Then the following assertions are equivalent:*

- (i) E is a Fréchet–Urysohn space;
- (ii) E is a sequential space;
- (iii) E is a k -space;
- (iv) X has the property $\gamma_{\mathcal{I}}$;
- (v) E^λ is Fréchet–Urysohn (sequential or a k -space) for some $\lambda \in (0, \omega]$.
- (vi) E^λ is Fréchet–Urysohn (sequential or a k -space) for every $\lambda \in (0, \omega]$.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear, and the equivalence (i) \Leftrightarrow (iv) follows from Proposition 4.4 and Theorem 3.3.

(iii) \Rightarrow (i) It is well known that there is a linear subspace Y' of Y such that $Y = \mathbb{F} \oplus Y'$, where \mathbb{F} is the field of Y . Therefore $E = C_{\mathcal{I}}(X, Y') \oplus C_{\mathcal{I}}(X, \mathbb{F})$ ($E = C_{\mathcal{I}}^b(X, Y') \oplus C_{\mathcal{I}}^b(X, \mathbb{F})$ or $E = C_{\mathcal{I}}^{rc}(X, Y') \oplus C_{\mathcal{I}}^{rc}(X, \mathbb{F})$). Then the space $C_{\mathcal{I}}(X, \mathbb{F})$ ($C_{\mathcal{I}}^b(X, \mathbb{F})$ or $C_{\mathcal{I}}^{rc}(X, \mathbb{F})$) being a closed subspace of E must be a k -space. If $\mathbb{F} = \mathbb{C}$, then $C_{\mathcal{I}}(X, \mathbb{F}) = C_{\mathcal{I}}(X) \oplus C_{\mathcal{I}}(X)$ and $C_{\mathcal{I}}^b(X, \mathbb{F}) = C_{\mathcal{I}}^b(X) \oplus C_{\mathcal{I}}^b(X)$, and hence the space $C_{\mathcal{I}}(X)$ (or $C_{\mathcal{I}}^b(X)$) is a k -space. Now Pytkeev’s Theorem 1.4 implies that the space $C_{\mathcal{I}}(X)$ (or $C_{\mathcal{I}}^b(X)$) is Fréchet–Urysohn. Hence, by Proposition 4.4 and Theorem 3.3, the space X has the property $\gamma_{\mathcal{I}}$. Once again applying Proposition 4.4 and Theorem 3.3, we obtain that the space E is Fréchet–Urysohn.

(iv) \Rightarrow (v) By Lemma 2.8, the space $C_{\mathcal{I}}(X, Y)^\lambda$ is homeomorphic to $C_{\mathcal{I}(Z)}(Z, Y)$, where $Z = \bigoplus \{X_i : X_i = X \text{ for each } i \in \lambda\}$. Proposition 2.4 implies that the space Z has the property $\gamma_{\mathcal{I}(Z)}$. Now the equivalence (i) \Leftrightarrow (iv) implies that $C_{\mathcal{I}}(X, Y)^\lambda$ is a Fréchet–Urysohn space. Therefore, the subspaces $C_{\mathcal{I}}^b(X, Y)^\lambda$ and $C_{\mathcal{I}}^{rc}(X, Y)^\lambda$ of $C_{\mathcal{I}}(X, Y)^\lambda$ are also Fréchet–Urysohn.

The implication (v) \Rightarrow (vi) is clear, and (vi) implies (i) (respectively, (ii) or (iii)) because E is a closed subspace of E^λ . \square

Now we consider the tightness $t(H)$ of a subspace H of $C_{\mathcal{I}}(X, Y)$. Recall that the *tightness* $t(x, X)$ at a point $x \in X$ of a topological space X is the least infinite cardinality κ such that if x is in the closure of a subset A of X , then A contains a subset B of cardinality $\leq \kappa$ with $x \in \overline{B}$; the *tightness* $t(X)$ of X is the least upper bound of $\{t(x, X) : x \in X\}$.

Let X be a topological space and let \mathcal{I} be an ideal of compact sets in X . Define the \mathcal{I} -Lindelöf degree $\mathcal{I}\text{-Lin}(X)$ of X as the least infinite cardinal κ such that every open \mathcal{I} -cover of X has an \mathcal{I} -subcover of cardinality $\leq \kappa$ (see [28]). Theorem 4.7.1 of [28] states that $t(C_{\mathcal{I}}(X)) = \mathcal{I}\text{-Lin}(X)$. In [37], Pytkeev proved that if Y is a metrizable space, then $t(C_p(X, Y)) = \aleph_0$ if and only if the Lindelöf number $l(X^n)$ of X^n is countable for every $n \geq 1$. The next theorem extends these results and has a similar proof.

Theorem 4.6. *Let Y be a metric space containing at least two points, and let X be a $Y_{\mathcal{I}}$ -Tychonoff space for some ideal \mathcal{I} of compact sets in X . If H is a relatively $Y_{\mathcal{I}}$ -Tychonoff subspace of $C_{\mathcal{I}}(X, Y)$, then $t(H) = \mathcal{I}\text{-Lin}(X)$. In particular, $t(C_{\mathcal{I}}(X, Y)) = \mathcal{I}\text{-Lin}(X)$.*

PROOF. First we show that $\mathcal{I}\text{-Lin}(X) \leq t(H)$. To this end, fix two distinct points $y, z \in Y$ and let \mathcal{U} be an open \mathcal{I} -cover of X (we assume that $X \notin \mathcal{U}$). Then for each $F \in \mathcal{I}$, there is a $U_F \in \mathcal{U}$ such that $F \subseteq U_F$. Since X is $Y_{\mathcal{I}}$ -Tychonoff and H is relatively $Y_{\mathcal{I}}$ -Tychonoff, for each $F \in \mathcal{I}$, choose an $f_F \in H$ such that $f_F(F) = \{y\}$ and $f_F(X \setminus U_F) = \{z\}$. In particular, the constant function \mathbf{y} belongs to the closure of the family $A = \{f_F : F \in \mathcal{I}\}$, and $\mathbf{y} \notin A$ because $U_F \neq X$. Then there exists a subfamily B of A with cardinality $\leq t(H)$, which contains \mathbf{y} in its closure. Define $\mathcal{V} := \{U_F : f_F \in B\}$. To see that \mathcal{V} is an \mathcal{I} -subcover of \mathcal{U} , let $F \in \mathcal{I}$ and fix an open neighborhood V of y such that $z \notin V$. Set $W := W[\mathbf{y}; F, V]$. Then W is a neighborhood of \mathbf{y} , and therefore there is an $f_G \in B \cap W$ for some $G \in \mathcal{V}$. Now for any $x \in F$, we have $f_G(x) \in V$; while $f_G(x) = z$ for any $x \in X \setminus U_G$. Therefore $F \subseteq U_G$, so that \mathcal{V} is an \mathcal{I} -subcover of \mathcal{U} and has cardinality $\leq t(H)$. Thus $\mathcal{I}\text{-Lin}(X) \leq t(H)$.

Let us prove the reverse inequality $t(H) \leq \mathcal{I}\text{-Lin}(X)$. Since Y embeds into a Banach space E , it follows that H is a subspace of $C_{\mathcal{I}}(X, E)$. Therefore it suffices to prove that $t(C_{\mathcal{I}}(X, Y)) \leq \mathcal{I}\text{-Lin}(X)$, where Y is a Banach space. As the space $C_{\mathcal{I}}(X, Y)$ is homogeneous, it is sufficient to show that the tightness of $C_{\mathcal{I}}(X, Y)$ at zero function $\mathbf{0}$ is less than or equal to $\mathcal{I}\text{-Lin}(X)$. To this end, let A be a subset of $C_{\mathcal{I}}(X, Y)$ such that $\mathbf{0} \in \overline{A} \setminus A$. For every $\varepsilon > 0$, denote by D_{ε} the open ball of radius ε centered at $\mathbf{0} \in Y$. For each $F \in \mathcal{I}$ and a natural number $n \geq 1$, choose a function $h_{n,F} \in A \cap [F; D_{1/n}]$, and set

$$W(n, F) := \{x \in X : h_{n,F}(x) \in D_{1/n}\}.$$

Then for each $n \geq 1$, the family $\mathcal{W}_n := \{W(n, F) : F \in \mathcal{I}\}$ is an open \mathcal{I} -cover of X . So each \mathcal{W}_n has an \mathcal{I} -subcover \mathcal{V}_n of cardinality $\leq \mathcal{I}\text{-Lin}(X)$. Define

$$A' := \{h_{n,F} : n \geq 1 \text{ and } W(n, F) \in \mathcal{V}_n\}$$

Clearly, $A' \subseteq A$ and $|A'| \leq \mathcal{I}\text{-Lin}(X)$, and $\mathbf{0}$ is in the closure of A' in $C_{\mathcal{I}}(X, Y)$. Thus $t(C_{\mathcal{I}}(X, Y)) \leq \mathcal{I}\text{-Lin}(X)$ as desired. \square

Corollary 4.7. *Let X be a Tychonoff space, \mathcal{I} be an ideal of compact subsets of X , Y be a metrizable locally convex space, and let $E = C_{\mathcal{I}}(X, Y)$, $C_{\mathcal{I}}^b(X, Y)$ or $C_{\mathcal{I}}^{rc}(X, Y)$. Then $t(E) = \mathcal{I}\text{-Lin}(X)$.*

PROOF. By Proposition 2.9, the space X is $Y_{\mathcal{I}}$ -Tychonoff. By Proposition 4.4, the space E is a relatively $Y_{\mathcal{I}}$ -Tychonoff subspaces of $C_{\mathcal{I}}(X, E)$. Now Theorem 4.6 applies. \square

Below we shall use the well-known fact that the Tychonoff product ω^{ω_1} of ω_1 discrete spaces ω is not normal (for a more general assertion see [12, Theorem 3.10]).

Proposition 4.8. *Let Y be a non-pseudocompact Tychonoff space, X be a Y -Tychonoff space containing a discrete family $\mathcal{U} = \{U_i\}_{i \in \omega_1}$ of open subsets, and let \mathcal{I} be an ideal of compact subsets of X . Then each subspace H of $(Y^X, \tau_{\mathcal{I}})$ consisting of $C_{\mathcal{I}}(X, Y)$ is not normal.*

PROOF. As Y is not pseudocompact, it contains an infinite, closed and discrete subspace $D = \{y_n\}_{n \in \omega}$. Since X is Tychonoff by Proposition 2.6, for every $i \in \omega_1$, choose a point $x_i \in U_i$ and an open neighborhood V_i of x_i such that $\overline{V_i} \subseteq U_i$. For every $i \in \omega_1$ and $n \in \omega$, by Y -Tychonoffness of X , there is a continuous function $f_{n,i} \in C(X, Y)$ such that

$$f_{n,i}(x_i) = y_n \text{ and } f_{n,i}(X \setminus V_i) = \{y_0\}.$$

For every function $\chi : \omega_1 \rightarrow \omega$, define a function $F_\chi : X \rightarrow Y$ by

$$F_\chi(x) := \begin{cases} f_{\chi(i),i}(x), & \text{if } x \in V_i \text{ for some } i \in \omega_1, \\ y_0, & \text{if } x \in X \setminus \bigcup_{i \in \omega_1} V_i. \end{cases}$$

Since the family \mathcal{U} is discrete, it is easy to see that the function F_χ is continuous.

We claim that the map $p : \omega^{\omega_1} \rightarrow H$ defined by $p(\chi) := F_\chi$ is a homeomorphism onto its image $p(\omega^{\omega_1}) \subseteq H$. Indeed, it is clear that p is a bijection. To prove that p is continuous, fix $\chi \in \omega^{\omega_1}$ and a standard neighborhood $W[p(\chi); \mathcal{F}, \mathcal{V}]$ of $p(\chi)$ in $(Y^X, \tau_{\mathcal{I}})$, where \mathcal{F} is a finite subfamily of \mathcal{I} and \mathcal{V} is a corresponding finite family of open sets in Y . Since the family \mathcal{U} is discrete, the set J of all $i \in \omega_1$ for which the set $K \cap U_i$ is not empty for some $K \in \mathcal{F}$ is finite. For every $j \in J$, set $W_j := \{\chi(j)\}$ and let $\mathcal{W} = \{W_j : j \in J\}$. Then $p(W[\chi; J, \mathcal{W}]) \subseteq W[p(\chi); \mathcal{F}, \mathcal{V}]$. Thus p is continuous.

To show that p is also open, let $\chi \in \omega^{\omega_1}$ and let

$$W := \{\eta \in \omega^{\omega_1} : \eta(i) = \chi(i) \text{ if } i \in F \text{ for some finite } F \subseteq \omega_1\}$$

be a standard neighborhood of χ . For every $i \in F$, choose an open neighborhood \mathcal{O}_i of $F_\chi(x_i)$ such that $\mathcal{O}_i \cap D = \{F_\chi(x_i)\}$. Then $p(W)$ contains the standard neighborhood

$$\bigcap_{i \in F} W[F_\chi; \{x_i\}, \{\mathcal{O}_i\}] \cap p(\omega^{\omega_1})$$

of F_χ . Thus p is open.

Since the space ω^{ω_1} is not normal, to show that also H is not normal it suffices to prove that $p(\omega^{\omega_1})$ is closed in H . Fix an arbitrary $h \in \overline{p(\omega^{\omega_1})}$. It is clear that $h(x) = y_0$ for every $x \in X \setminus \bigcup_{i \in \omega_1} V_i$. Fix an $i \in \omega_1$. Since D is closed and discrete in Y it follows that there is an $n(i) \in \omega$ such that $h(x_i) = f_{n(i),i}(x_i) = y_{n(i)} \in D$. Take an open neighborhood O_i of $y_{n(i)}$ such that $O_i \cap D = \{y_{n(i)}\}$. Then for every $x \in V_i$ and each neighborhood O_x of x , the standard neighborhood $W[h; \{x_i\}, \{O_i\}] \cap W[h; \{x\}, \{O_x\}] \cap H$ of h contains only those functions $f \in p(\omega^{\omega_1})$ for which $f(x) = f_{n(i),i}(x)$. Therefore $h|_{V_i} = f_{n(i),i}|_{V_i}$. Hence $h = F_\chi$, where $\chi \in \omega^{\omega_1}$ is defined by $\chi(i) = n(i)$ for all $i \in \omega_1$. Thus $p(\omega^{\omega_1})$ is closed in H . \square

Remark 4.9. In Proposition 4.8 the assumption on Y to be non-pseudocompact is essential. Indeed, let $Y = [0, 1]$ and $X = \bigcup_{i \in \omega_1} S_i$ be the topological sum of countable compact spaces S_i . It follows from Proposition 5.10 below that $B_1(X, Y) = \prod_{i \in \omega_1} Y^{S_i} = Y^X$, and hence $B_1(X, Y)$ is a compact space. However, if we consider the case $H = C_{\mathcal{I}}(X, Y)$, then the space H is still not normal because all spaces $C_{\mathcal{I}}(S_i, Y)$ contain infinite, closed and discrete subspaces, see Lemma 1 of [33]. \square

To prove the next proposition we need the following mild extension property.

Definition 4.10. Let X and Y be topological spaces, and let \mathcal{I} be an ideal of compact subsets in X . We say that X has the $Y_{\mathcal{I}}$ -extension property if for every $F \in \mathcal{I}$ and each $f \in C(F, Y)$ there is an $\bar{f} \in C(X, Y)$ such that $\bar{f}|_F = f$. \square

Let X be a Tychonoff space. As usual we denote by $w(X)$ and $d(X)$ the *weight* of X and the *density* of X , respectively.

Proposition 4.11. *Let Y be a Hausdorff topological space containing at least two points, and let X be a $Y_{\mathcal{I}_X}$ -Tychonoff space for some ideal \mathcal{I}_X of compact subsets in X . Let S be a $Y_{\mathcal{I}_S}$ -Tychonoff space for some ideal \mathcal{I}_S of compact subsets in S such that*

- (a) *there is a continuous bijection ϕ from X onto S such that $\phi(\mathcal{I}_X) \subseteq \mathcal{I}_S$;*
- (b) *S has the $Y_{\phi(\mathcal{I}_X)}$ -extension property.*

Then $d(C_{\mathcal{I}_X}(X, Y)) \leq d(C_{\mathcal{I}_S}(S, Y))$.

PROOF. It is sufficient to show that if D is a dense subset of $C_{\mathcal{I}_S}(S, Y)$, then $\phi^*(D)$ is dense in $C_{\mathcal{I}_X}(X, Y)$, where $\phi^* : C_{\mathcal{I}_S}(S, Y) \rightarrow C_{\mathcal{I}_X}(X, Y)$, $\phi^*(f) := f \circ \phi$, is the adjoint map of ϕ . Fix an $f \in C_{\mathcal{I}_X}(X, Y)$ and its standard neighborhood $W[f; \mathcal{F}, \mathcal{U}]$, where $\mathcal{F} \subseteq \mathcal{I}_X$ is finite. Then the bijectivity of ϕ and (b) imply that there is an $h \in C_{\mathcal{I}_S}(S, Y)$ such that $h|_{\bigcup \mathcal{F}} = f|_{\bigcup \mathcal{F}}$. Take $g \in D$ such that $g \in W[h; \mathcal{F}, \mathcal{U}] \subseteq C_{\mathcal{I}_S}(S, Y)$. It is clear that $\phi^*(g) = g \circ \phi \in W[f; \mathcal{F}, \mathcal{U}]$. Thus $\phi^*(D)$ is dense in $C_{\mathcal{I}_X}(X, Y)$ and hence $d(C_{\mathcal{I}_X}(X, Y)) \leq d(C_{\mathcal{I}_S}(S, Y))$. \square

The *weak weight* $ww(X)$ of a Tychonoff space X is the least infinite cardinality of $w(Z)$ of a Tychonoff space Z such that there is a continuous bijection from X onto Z , see [28]. Analogously we define

Definition 4.12. Let X and Y be topological spaces, and let \mathcal{I} be an ideal of compact subsets in X such that X has the $Y_{\mathcal{I}}$ -extension property. The $Y_{\mathcal{I}}$ -*weak weight* of X (for short $Y_{\mathcal{I}}\text{-}ww(X)$) is the least infinite cardinality of $w(Z)$ of a Y -Tychonoff space Z such that there is a continuous bijection ϕ from X onto Z and Z has the $Y_{\phi(\mathcal{I})}$ -extension property. \square

For the sake of simplicity we set $Y_p\text{-}ww(X) := Y_{\mathcal{F}(X)}\text{-}ww(X)$ and $Y_k\text{-}ww(X) := Y_{\mathcal{K}(X)}\text{-}ww(X)$.

Lemma 4.13. (i) *Every Y -Tychonoff space X has the $Y_{\mathcal{F}(X)}$ -extension property.*

- (ii) *Each Tychonoff space X has the $\mathbb{R}_{\mathcal{I}}$ -extension property for every ideal \mathcal{I} of compact subsets of X .*
- (iii) *Let \mathcal{I} be an ideal of compact subsets of a Tychonoff space X . Then $\mathbb{R}_{\mathcal{I}}\text{-}ww(X) = ww(X)$.*
- (iv) *$ww(X) \leq Y_{\mathcal{I}}\text{-}ww(X)$ for every Tychonoff space Y and each ideal \mathcal{I} of compact subsets of a Y -Tychonoff space X with the $Y_{\mathcal{I}}$ -extension property.*
- (v) *If X is a Y -Tychonoff (for example, Y - z -Tychonoff) compact space, then $Y_p\text{-}ww(X) = w(X)$.*

PROOF. (i) follows from definitions, (ii) follows from [16, 3.11(c)], and (iii) follows from (ii). Finally, (iv) follows from (ii) and the fact that every Y -Tychonoff space is Tychonoff, see Proposition 2.6.

(v) Recall that, by Proposition 2.13, every Y - z -Tychonoff space is Y -Tychonoff. By (i), X has the $Y_{\mathcal{F}(X)}$ -extension property. Since every continuous bijection of a compact space is a homeomorphism, by Definition 4.12, we have $Y_p\text{-}ww(X) = w(X)$. \square

Taking into account Lemma 4.13, for $Y = \mathbb{R}$, the next assertion is proved in Theorem 4.2.1 of [28].

Proposition 4.14. *Let Y be a Hausdorff topological space containing at least two points, and let X be a $Y_{\mathcal{I}_X}$ -Tychonoff space for some ideal \mathcal{I}_X of compact subsets in X . If X has the $Y_{\mathcal{I}_X}$ -extension property, then*

- (i) $d(Y) \leq d(C_{\mathcal{I}_X}(X, Y)) \leq Y_{\mathcal{I}}\text{-}ww(X) \cdot w(Y)$;
- (ii) $ww(X) \leq w(Y) \cdot d(C_{\mathcal{I}_X}(X, Y))$.

PROOF. (i) First we show that $d(C_{\mathcal{I}_X}(X, Y)) \leq Y_{\mathcal{I}}\text{-}ww(X) \cdot w(Y)$. To this end, let $\phi : X \rightarrow Z$ be a continuous bijection, where Z is a Y -Tychonoff space with the $Y_{\phi(\mathcal{I}_X)}$ -extension property and such that $w(Z) = Y_{\mathcal{I}}\text{-}ww(X)$. For simplicity, set $\mathcal{I}_Z := \phi(\mathcal{I}_X)$ (so \mathcal{I}_Z is an ideal of compact sets in Z). By Proposition 4.11, it suffices to show that $d(C_{\mathcal{I}_Z}(Z, Y)) \leq w(Z) \cdot w(Y)$. Choose an open base \mathcal{B}_Z for Z which is closed under taking finite unions and such that $|\mathcal{B}_Z| = w(Z)$, and let \mathcal{B}_Y be an open base of Y such that $|\mathcal{B}_Y| = w(Y)$. For every finite subfamilies $\mathcal{U} = \{U_1, \dots, U_n\} \subseteq \mathcal{B}_Z$ and $\mathcal{V} = \{V_1, \dots, V_n\} \subseteq \mathcal{B}_Y$, choose (if this is possible) a function $f_{\mathcal{U}, \mathcal{V}} \in C(Z, Y)$ such that $f_{\mathcal{U}, \mathcal{V}}(U_i) \subseteq V_i$ for every $i = 1, \dots, n$ (i.e., $f_{\mathcal{U}, \mathcal{V}} \in [\mathcal{U}; \mathcal{V}]$). Set

$$D := \{f_{\mathcal{U}, \mathcal{V}} : \mathcal{U} \subseteq \mathcal{B}_Z \text{ and } \mathcal{V} \subseteq \mathcal{B}_Y \text{ are finite}\}.$$

By construction, $|D| \leq w(Z) \cdot w(Y)$. We show that D is dense in $C_{\mathcal{I}_Z}(Z, Y)$. Indeed, let $f \in C_{\mathcal{I}_Z}(Z, Y)$ and let $W[f; \mathcal{F}, \mathcal{V}]$ be a basic open neighborhood of f in $C_{\mathcal{I}_Z}(Z, Y)$, where $\mathcal{F} = \{F_1, \dots, F_n\} \subseteq \mathcal{I}(Z)$ and $\mathcal{V} = \{V_1, \dots, V_n\}$ is a finite family of open sets in Y . Since f is continuous and \mathcal{B}_Z is closed under taking finite unions, for every $i = 1, \dots, n$, there is an open $U_i \in \mathcal{B}_Z$ such that $F_i \subseteq U_i$ and $f(U_i) \subseteq V_i$. Therefore, the set $[\mathcal{U}; \mathcal{V}]$ is not empty. It is clear that $f_{\mathcal{U}, \mathcal{V}} \in [\mathcal{U}; \mathcal{V}] \subseteq W[f; \mathcal{F}, \mathcal{V}]$, and hence D is dense in $C_{\mathcal{I}_Z}(Z, Y)$. Thus $d(C_{\mathcal{I}_Z}(Z, Y)) \leq w(Z) \cdot w(Y)$.

To prove the inequality $d(Y) \leq d(C_{\mathcal{I}_X}(X, Y))$, fix a dense subset \mathcal{R} of $C_{\mathcal{I}_X}(X, Y)$ such that $|\mathcal{R}| = d(C_{\mathcal{I}_X}(X, Y))$. Take an arbitrary $x_0 \in X$ and put $D_Y := \{f(x_0) : f \in \mathcal{R}\}$. Then D_Y is dense in Y since, otherwise, there would be an open subset U of Y such that $U \cap D_Y = \emptyset$. But then the open subset $\{x_0\}; U$ of $C_{\mathcal{I}_X}(X, Y)$ is nonempty (since it contains the constant function y for every $y \in U$) and does not contain elements of the dense family \mathcal{R} , a contradiction. Thus $d(Y) \leq |D_Y| \leq |\mathcal{R}| = d(C_{\mathcal{I}_X}(X, Y))$.

(ii) As above, let \mathcal{R} be a dense subset of $C_{\mathcal{I}_X}(X, Y)$ of cardinality $d(C_{\mathcal{I}_X}(X, Y))$. Define a map $T : X \rightarrow Y^{\mathcal{R}}$ by $T(x) := (f(x))_{f \in \mathcal{R}}$, where $Y^{\mathcal{R}}$ is endowed with the product topology. Clearly, T is continuous. Since X is $Y_{\mathcal{I}_X}$ -Tychonoff and Y is Hausdorff, the family \mathcal{R} separates the points of X , and hence T is also one-to-one. Therefore,

$$ww(X) \leq w(T(X)) \leq w(Y) \cdot |\mathcal{R}| = w(Y) \cdot d(C_{\mathcal{I}_X}(X, Y)). \square$$

Corollary 4.15. *Let Y be a separable metric space containing at least two points and let X be Y -Tychonoff space. Then $ww(X) \leq d(C_p(X, Y)) \leq Y_p\text{-}ww(X)$. In particular, if X is compact, then $d(C_p(X, Y)) = w(X)$.*

PROOF. The first inequality $ww(X) \leq d(C_p(X, Y))$ follows from (ii) of Proposition 4.14. The second inequality $d(C_p(X, Y)) \leq Y_p\text{-}ww(X)$ follows from Lemma 4.13(i), Proposition 4.14(i) and the fact $w(Y) = \aleph_0$. Finally, if additionally X is compact, the last assertion follows from (v) of Lemma 4.13 and the trivial fact that $ww(X) = w(X)$. \square

Lemma 4.13(iii) and Proposition 4.14 immediately imply the next result which is also proved in Theorem 4.2.1 of [28].

Corollary 4.16. *If X is a Tychonoff space and \mathcal{I} is an ideal of compact subsets of X , then $d(C_{\mathcal{I}}(X)) = ww(X)$.*

5. Topological properties of spaces of Baire functions

Recall that the *cellularity* $c(X)$ of a topological space X is the minimal infinite cardinal κ such that every disjoint family of open sets in X has cardinality less than or equal to κ . Denote by $\psi(x, X)$ and $\psi(X)$ the pseudocharacter of X at a point $x \in X$ and the pseudocharacter of X , respectively. We shall use the following results.

Proposition 5.1. *Let Z be a Tychonoff P -space. If $\psi(Z) > \aleph_0$, then $c(Z) > \aleph_0$. In particular, if X is a Tychonoff space such that $\psi(X_{\aleph_0}) > \aleph_0$, then $c(X_{\aleph_0}) > \aleph_0$.*

PROOF. Fix a point $z \in Z$ in which the pseudocharacter $\psi(z, Z)$ of Z is uncountable. We construct an uncountable family of pairwise disjoint open sets in Z by transfinite induction. For $i = 0$, set $V_0 := Z$. Choose a clopen neighborhood V_1 of z such that $V_0 \setminus V_1 \neq \emptyset$ and set $U_1 := V_0 \setminus V_1$. Assume that for every countable ordinal α and each $i < \alpha$ we have constructed a family $\{V_i\}_{i < \alpha}$ of clopen neighborhoods of z and a disjoint family $\{U_i\}_{i < \alpha}$ of clopen subsets of Z such that

$$U_i \subseteq \bigcap_{j < i} V_j \setminus V_i$$

for every $i < \alpha$. Since Z is a P -space, the set $W := \bigcap_{j < \alpha} V_j$ is an open neighborhood of z . As $\psi(z, Z) > \aleph_0$, there is a clopen neighborhood $V_\alpha \subseteq W$ of z such that $W \setminus V_\alpha \neq \emptyset$. Choose a clopen nonempty subset U_α of $W \setminus V_\alpha$. It is clear that the uncountable family $\{U_i : i < \omega_1\}$ is disjoint. Thus $c(Z) > \aleph_0$.

The last assertion follows from the fact that X_{\aleph_0} is a P -space. \square

Corollary 5.2. *Let X be a Tychonoff space. Then:*

- (i) *If $\psi(X_{\aleph_0}) = \aleph_0$ (for example $\psi(X) = \aleph_0$), then X_{\aleph_0} is discrete.*
- (ii) *$c(X_{\aleph_0}) = \aleph_0$ if and only if X is countable.*

PROOF. (i) immediately follows from the fact that X_{\aleph_0} is a P -space.

(ii) Assume that X_{\aleph_0} has countable cellularity. Then, by Proposition 5.1, $\psi(X_{\aleph_0}) = \aleph_0$ and hence, by (i), X_{\aleph_0} is discrete. But since $c(X_{\aleph_0}) = \aleph_0$ it follows that X is countable.

Conversely, if X is countable, then the equality $c(X_{\aleph_0}) = \aleph_0$ holds trivially. \square

Let X and Y be topological spaces. Define $B(X, Y) := \bigcup_{\alpha \in \omega_1} B_\alpha(X, Y)$. A function $f : X \rightarrow Y$ is called *Baire* if $f \in B(X, Y)$. If $Y = \mathbb{R}$, set $B(X) := B(X, \mathbb{R})$. If X is Tychonoff, it is easy to see that every function $f \in B(X)$ is continuous in the Baire topology. We shall use the following easy assertion which allows us to reduce the study of topological properties of the corresponding spaces of Baire functions to topological properties of spaces of continuous functions.

Proposition 5.3. *Let Y be a topological space, and let X be a Tychonoff space. Then the adjoint map i^* of the identity mapping $i : X_{\aleph_0} \rightarrow X$ is an embedding of $B(X, Y)$ into $C_p(X_{\aleph_0}, Y)$.*

PROOF. Clearly, the adjoint map $i^* : Y^X \rightarrow Y^{X_{\aleph_0}}$, $i^*(f) = f \circ i$, is a topological isomorphism and $i^*(C_p(X, Y)) \subseteq C_p(X_{\aleph_0}, Y)$. Since X is Tychonoff, the space X_{\aleph_0} is a P -space. Therefore the pointwise limit of a sequence of continuous functions in $C_p(X_{\aleph_0}, Y)$ is continuous. Thus, by the definition of $B(X, Y)$, the map i^* is an embedding. \square

Besides the classes $B_\alpha(X, Y)$ we shall consider also the *stable* Baire- α classes $B_\alpha^{st}(X, Y)$ of functions, which were introduced and studied by Császár and Laczkovich [9]. Set $B_0^{st}(X, Y) := C_p(X, Y)$. We say that a sequence $\{f_n\}_{n \in \omega} \subseteq Y^X$ *stably converges* to a function $f \in Y^X$ if for every $x \in X$ the set $\{n \in \omega : f_n(x) \neq f(x)\}$ is finite. Then the class $B_1^{st}(X, Y)$ is defined as the family of all functions from Y^X which are limits of stably convergent sequences of continuous functions. For every countable ordinal $\alpha > 1$, denote by $B_\alpha^{st}(X, Y) \subseteq Y^X$ the family of all functions

$f : X \rightarrow Y$ which are pointwise limits of function sequences from $\bigcup_{\beta < \alpha} B_\beta^{st}(X, Y)$. Functions in the family $B_\alpha^{st}(X, Y)$ are called the *functions of stable Baire- α class*. It is clear that

$$C_p(X, Y) \subseteq B_\alpha^{st}(X, Y) \subseteq B_\alpha(X, Y) \subseteq B(X, Y)$$

for every countable ordinal α . Note that the inclusion $B_1^{st}(X, Y) \subseteq B_1(X, Y)$ can be strict. If $Y = \mathbb{R}$, set $B_1^{st}(X) := B_1^{st}(X, \mathbb{R})$. For a function $f : X \rightarrow Y$, we set

$$\sigma(f) := \{h \in Y^X : \text{the set } \{x \in X : h(x) \neq f(x)\} \text{ is finite}\}.$$

It is very useful to know concrete constructions of Baire one functions. The following lemma extends Lemmas 3.3 and 3.4 of [12].

Lemma 5.4. *Let Y be a metric space containing at least two points, and let X be an infinite Y -Tychonoff space of countable pseudocharacter. Let $\{U_n\}_{n < N}$, $0 < N \leq \infty$, be a disjoint family of open subsets of X and let $x_n \in U_n$ for every $n < N$. Then, for every $g_0, g_1, \dots, g_N \in Y$, the function*

$$f(x) := \begin{cases} g_n, & \text{if } x = x_n \text{ for some } n < N, \\ g_N, & \text{if } x \in X \setminus \{x_n : n < N\} \end{cases}$$

belongs to $B_1^{st}(X, Y)$. In particular, $\sigma(\mathbf{g}) \subseteq B_1^{st}(X, Y)$ for every $g \in Y$.

PROOF. We prove the lemma only for the case $N = \omega$. For every $n \in \omega$, choose a decreasing sequence $\{V_{k,n} : k \in \omega\}$ of open neighborhoods of x_n such that $V_{0,n} \subseteq U_n$ and $\bigcap_{k \in \omega} V_{k,n} = \{x_n\}$. Since X is Y -Tychonoff, for every $k \in \omega$, there is a function $f_k \in C(X, Y)$ such that

$$f_k(x_i) = g_i \text{ for } i = 0, \dots, k, \text{ and } f_k\left(X \setminus \bigcup_{i=0}^k V_{k,i}\right) \subseteq \{g_\omega\}.$$

Now the choice of the open sets $V_{k,n}$ easily implies that the sequence $\{f_k\}_{k \in \omega}$ stably converges to f . \square

Remark 5.5. The condition on $\{U_n\}_{n \in \omega}$ to be disjoint in Lemma 5.4 is essential. Indeed, consider $X = Y = \mathbb{R}$, $g_i = 1$ for all finite i and $g_\omega = 0$, and let $\{x_n : n \in \omega\}$ be an enumeration of the rational numbers. Then the function f (= the Dirichlet function) does not belong to $B_1(X)$ because it does not have points of continuity. \square

Lemma 5.6. *Let Y be a normal space containing at least two points, and let X be a Y - z -Tychonoff space. Then for every disjoint zero-sets F_1, \dots, F_n in X and each points $y_0, \dots, y_n \in Y$ there exists a function $f \in B_1^{st}(X, Y)$ such that $f(X \setminus \bigcup_{i=1}^n F_i) \subseteq \{y_0\}$ and $f(F_i) = \{y_i\}$ for every $i = 1, \dots, n$.*

PROOF. Since the set $F := \bigcup_{i=1}^n F_i$ is a zero-set in X , there is a $t \in C(X)$ such that $F = t^{-1}(0)$. Therefore the set $X \setminus F = \bigcup_{n \in \omega} S_n$, where $S_n = \{x \in X : |t(x)| \geq \frac{1}{n+1}\}$, is the union of the increasing sequence $\{S_n\}_{n \in \omega}$ of zero-sets in X . Since $S_n \cap F = \emptyset$, the Y - z -Tychonoffness of X implies that there is an $f_n \in C(X, Y)$ such that $f_n(S_n) \subseteq \{y_0\}$ and $f_n(F_i) = \{y_i\}$ for every $i = 1, \dots, n$. It is clear that f_n stably converges to the function f . \square

The following lemma plays a crucial role in the proofs of the main results of this section.

Lemma 5.7. *Let Y be a Tychonoff space of countable pseudocharacter containing at least two points, and let X be a Y -Tychonoff space. Assume that $A = \{a_1, \dots, a_n\}$ is a subset of a zero-set F in X , $h : A \rightarrow Y$ is a function and $y_0 \in Y$. Let $h(A) := \{y_1, \dots, y_k\}$ with distinct $y_1, \dots, y_k \in Y$. Then there exist disjoint zero-sets Z_1, \dots, Z_k in X and a function $f \in B_1^{st}(X, Y)$ such that*

- (i) *for every $j = 1, \dots, n$, there is an $i_j \in \{1, \dots, k\}$ such that $a_j \in Z_{i_j}$;*
- (ii) $\bigcup_{i=1}^k Z_i \subseteq F$ and $f(Z_i) = \{y_i\}$ for every $i = 1, \dots, k$;
- (iii) $f(X \setminus \bigcup_{i=1}^k Z_i) \subseteq \{y_0\}$ and $f|_A = h$.

Moreover, if $A = \{a\}$, then there is a sequence $\{f_n\}_{n \in \omega} \subseteq C(X, Y)$ such that $f_n(a) = h(a)$ for all $n \in \omega$ and $\bigcap_{n \in \omega} f_n^{-1}(h(a)) \subseteq F$.

PROOF. We shall use the following fact: a Tychonoff space has countable pseudocharacter if and only if each its point is a zero-set. So $\{y\}$ is a zero-set in Y for each $y \in Y$.

Set $F_0 := F$. Then, as in the proof of Lemma 5.6, $X \setminus F_0 = \bigcup_{n \in \omega} S_{n,0}$, where $\{S_{n,0}\}_{n \in \omega}$ is an increasing sequence of zero-sets in X . Since X is Y -Tychonoff, there is a function $f_0 \in C(X, Y)$ such that $f_0|_A = h$ and $f_0(S_{0,0}) \subseteq \{y_0\}$.

For every $i = 1, \dots, k$, set $F_0^i := F_0 \cap f_0^{-1}(y_i)$. Since $\{y_i\}$ is a zero-set in Y and y_1, \dots, y_k are distinct, it follows that the sets F_0^1, \dots, F_0^k are disjoint zero-sets in X , and hence the set $F_1 := \bigcup_{i=1}^k F_0^i$ is also a zero-set in X . Clearly, $A \subseteq F_1 \subseteq F$ and $f_0(F_0^i) = \{y_i\}$ for every $i = 1, \dots, k$. Then $X \setminus F_1 = \bigcup_{n \in \omega} S_{n,1}$, where $\{S_{n,1}\}_{n \in \omega}$ is an increasing sequence of zero-sets in X . Since X is Y -Tychonoff, there is a function $f_1 \in C(X, Y)$ such that $f_1|_A = h$ and $f_1(S_{1,0} \cup S_{0,1}) \subseteq \{y_0\}$.

For every $i = 1, \dots, k$, set $F_1^i := F_0^i \cap f_1^{-1}(y_i)$. Since $\{y_i\}$ is a zero-set in Y and y_1, \dots, y_k are distinct, it follows that the sets F_1^1, \dots, F_1^k are disjoint zero-sets in X , and hence the set $F_2 := \bigcup_{i=1}^k F_1^i$ is also a zero-set in X . Clearly, $A \subseteq F_2 \subseteq F_1$ and $f_1(F_1^i) = \{y_i\}$ for every $i = 1, \dots, k$. Then $X \setminus F_2 = \bigcup_{n \in \omega} S_{n,2}$, where $\{S_{n,2}\}_{n \in \omega}$ is an increasing sequence of zero-sets in X . Since X is Y -Tychonoff, there is a function $f_2 \in C(X, Y)$ such that $f_2|_A = h$ and

$$f_2(S_{2,0} \cup S_{1,1} \cup S_{0,2}) \subseteq \{y_0\}.$$

Proceeding by induction, for every $t \in \omega$, we construct disjoint zero-sets F_t^1, \dots, F_t^k in X and a function $f_t \in C(X, Y)$ such that

- (a) $F_0^i \cap F_0^j = \emptyset$ for all distinct $i, j \in \{1, \dots, k\}$;
- (b) $X \setminus \bigcup_{i=1}^k F_t^i = \bigcup_{n \in \omega} S_{n,t}$, where $\{S_{n,t}\}_{n \in \omega}$ is an increasing sequence of zero-sets in X ;
- (c) $F_{t+1}^i = F_t^i \cap f_{t+1}^{-1}(y_i)$ for every $i = 1, \dots, k$;
- (d) $f_t|_A = h$ and $A \subseteq \bigcup_{i=1}^k F_t^i \subseteq F$;
- (e) $f_t(S_{t,0} \cup S_{t-1,1} \cup \dots \cup S_{0,t}) \subseteq \{y_0\}$.

For every $i = 1, \dots, k$, set $Z_i := \bigcap_{t \in \omega} F_t^i$. It is clear that Z_i is a zero-set in X and, by (a) and (c), the sets Z_1, \dots, Z_k are disjoint. By (d), for every $j = 1, \dots, n$, there is an $i_j \in \{1, \dots, k\}$ such that $a_j \in Z_{i_j}$. This proves (i). Also, by (d), $\bigcup_{i=1}^k Z_i \subseteq F$.

If $x \in Z_i$ for some $i = 1, \dots, k$, then (c) implies that $f_t(x) = y_i$ for all $t \geq 1$. If $x \in X \setminus \bigcup_{i=1}^k Z_i$, (c) implies that there is a $t_x \in \omega$ such that $x \in X \setminus \bigcup_{i=1}^k F_{t_x}^i$ and hence, by (b), there is an $n_x \in \omega$ such that $x \in S_{n_x, t_x}$. Therefore, for every $t > n_x + t_x$, (b) implies $S_{n_x, t_x} \subseteq S_{t-t_x, t_x}$ and hence, by (e), we have $f_t(x) = y_0$. Thus the sequence $\{f_t\}$ stably converges to a function $f \in B_1^{st}(X, Y)$ which satisfies (ii) and (iii). The equality $f|_A = h$ follows from (d).

To prove the last assertion, assume that $A = \{a\}$. Choose $y_0 \in Y$ such that $y_0 \neq h(a)$. By construction, $X \setminus F = \bigcup_{n \in \omega} S_{n,0}$, where $S_{n,0} \subseteq S_{n+1,0}$ for all $n \in \omega$. Now, (e) implies $f_n(S_{n,0}) \subseteq \{y_0\}$, and hence $f_n^{-1}(h(a)) \subseteq X \setminus f_n^{-1}(y_0) \subseteq X \setminus S_{n,0}$ for every $n \in \omega$. Therefore

$$\bigcap_{n \in \omega} f_n^{-1}(h(a)) \subseteq \bigcap_{n \in \omega} X \setminus S_{n,0} = X \setminus \bigcup_{n \in \omega} S_{n,0} = X \setminus (X \setminus F) = F.$$

Finally, by (d), $f_n(a) = h(a)$ holds for every $n \in \omega$. □

The next proposition gives examples of spaces X for which $\tau_Y = \tau_b$.

Proposition 5.8. *Let Y be a perfectly normal space containing at least two points, and let X be a Y -Tychonoff space. Then:*

- (i) $\tau_Y = \tau_b$, i.e. $X_Y = X_{\aleph_0}$.
- (ii) X_{\aleph_0} is a Y -Tychonoff space.

PROOF. (i) To prove that $\tau_Y \leq \tau_b$ it is sufficient to show that every set of the form $f^{-1}(F)$, where F is a closed set in Y and $f \in C(X, Y)$, is a zero-set in X . Since Y is perfectly normal, there is a function $h \in C(Y)$ such that $F = h^{-1}(0)$. Setting $g := h \circ f \in C(X)$ we obtain that $f^{-1}(F) = g^{-1}(0)$ is a zero-set in X .

To show the converse inclusion $\tau_Y \geq \tau_b$, fix two distinct points $a, b \in Y$. Let F be a zero-set of X and let x be any point of F . Applying Lemma 5.7 to $A = \{x\}$, $y_0 = b$ and $h : A \rightarrow Y$ defined by $h(x) = a$, we can find a sequence $\{f_n\}_{n \in \omega} \subseteq C(X, Y)$ such that

$$f_n(x) = a \ (n \in \omega) \text{ and } x \in \bigcap_{n \in \omega} f_n^{-1}(a) \subseteq F.$$

As the set $\{a\}$ is closed in Y and Y is perfectly normal, it follows that $f_n^{-1}(a) \in Z_Y$. Therefore $\bigcap_{n \in \omega} f_n^{-1}(a) \in \tau_Y$. Since x was arbitrary we obtain $F \in \tau_Y$. Thus $\tau_Y \geq \tau_b$.

(ii) Let $A = \{a_1, \dots, a_n\}$ be a finite subset of X_{\aleph_0} , D be a closed subset of X_{\aleph_0} such that $D \subseteq X \setminus A$, $h : A \rightarrow Y$ be a function, and let $y_0 \in Y$. Since X is Tychonoff by Proposition 2.6, there are disjoint zero-sets S_1, \dots, S_n in X such that $a_i \in S_i$ and $S_i \subseteq X \setminus D$ for every $i = 1, \dots, n$. Setting $F := \bigcup_{i=1}^n S_i$ and applying Lemma 5.7, one can find a zero-set S in X and a function $f \in B_1^{st}(X, Y)$ such that $A \subseteq S \subseteq \bigcup_{i=1}^n S_i$, $f|_A = h$ and $f(X \setminus S) \subseteq \{y_0\}$. Since $B_1^{st}(X, Y) \subseteq C_p(X_{\aleph_0}, Y)$ (see Proposition 5.3), we obtain that X_{\aleph_0} is a Y -Tychonoff space. □

Recall some of the most important types of local networks in a topological space. For historical remarks and numerous results related to generalized metric space theory we referee the reader to [18] or [13]. A family \mathcal{N} of subsets of a topological space X is

- a *network at a point* $x \in X$ if for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$; \mathcal{N} is a *network* in X if \mathcal{N} is a network at each point $x \in X$.
- a *cs*-network at a point* $x \in X$ if for each sequence $(x_n)_{n \in \mathbb{N}}$ in X converging to x and for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite; \mathcal{N} is a *cs*-network* in X if \mathcal{N} is a cs*-network at each point $x \in X$.
- a *cn-network at a point* $x \in X$ if for each neighborhood O_x of x the set $\bigcup\{N \in \mathcal{N} : x \in N \subseteq O_x\}$ is a neighborhood of x ; \mathcal{N} is a *cn-network* in X if \mathcal{N} is a cn-network at each point $x \in X$.

- a *ck-network* at a point $x \in X$ if for any neighborhood O_x of x there is a neighborhood U_x of x such that for each compact subset $K \subseteq U_x$ there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{N}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subseteq \bigcup \mathcal{F} \subseteq O_x$; \mathcal{N} is a *ck-network* in X if \mathcal{N} is a *ck-network* at each point $x \in X$.
- a *cp-network* at a point $x \in X$ if either x is an isolated point of X and $\{x\} \in \mathcal{N}$, or for each subset $A \subset X$ with $x \in \overline{A} \setminus A$ and each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$ and $N \cap A$ is infinite; \mathcal{N} is a *cp-network* in X if \mathcal{N} is a *cp-network* at each point $x \in X$.

To unify notations we call a network (at a point x of) X by 0-network.

Notation 5.9. If \mathcal{N} is either a *cp*-, *ck*-, *cn*-, *cs**-network or network (at a point x) in a topological space X , we will say that \mathcal{N} is an **n-network** (at x). Set $\mathfrak{N} = \{cp, ck, cn, cs^*, 0\}$.

Let X be a topological space. Recall that a collection \mathcal{N} of subsets of X is said to be *locally finite*, if each point in the space has a neighborhood that intersects only finitely many of the sets in \mathcal{N} ; and \mathcal{N} is called *σ -locally finite* if it is the union of a countable family of locally finite collections of subsets of X . For $\mathfrak{n} \in \mathfrak{N}$, the space X

- is an **n- σ -space** if X has a σ -locally finite **n-network**;
- has *countable n-character* if X has a countable **n-network** at each point $x \in X$.

So *cp- σ -spaces* are \mathfrak{P} -spaces, *cs*- σ -spaces* are \aleph -spaces, and *0- σ -spaces* are σ -spaces, see [13]. Recall also that X is called a *cosmic space* (an \aleph_0 -space or a \mathfrak{P}_0 -space) if it is regular and has a countable network (resp. a countable *k*- or *cp*-network).

Below we shall use repeatedly the following assertion.

Proposition 5.10 ([4]). *If X is a countable functionally Hausdorff space, then*

$$B_1^{st}(X, Y) = B_1(X, Y) = Y^X$$

for any topological space Y .

Theorem 5.11. *Let Y be a perfectly normal space containing at least two points, X be a Y -Tychonoff space, $\mathfrak{n} \in \mathfrak{N}$, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then H is an **n- σ -space** if and only if X is countable and Y is an **n- σ -space**. In this case $H = Y^X$.*

PROOF. Assume that H is an **n- σ -space**. First we show that X is countable. To this end, observe that every **n- σ -space** is a σ -space. Fix two distinct points $a, b \in Y$. Let us prove that X_{\aleph_0} has countable cellularity. Suppose for a contradiction that $c(X_{\aleph_0})$ is uncountable. Then there is an uncountable disjoint family $\mathcal{V} = \{V_i : i \in I\}$ of zero-sets in X . By Lemma 5.7, for every $i \in I$ there exist a zero-set $Z_i \subseteq V_i$ in X and a function $f_i \in B_1^{st}(X, Y) \subseteq H$ such that

$$f_i(Z_i) = \{b\} \text{ and } f_i(X \setminus Z_i) = \{a\}.$$

Set $K := \{f_i : i \in I\} \cup \{\mathbf{a}\}$. Since the family \mathcal{V} is disjoint, any standard neighborhood $W[\mathbf{a}; F, U]$ of the constant function \mathbf{a} contains all but finitely many functions from K . Therefore, K is a compact subset of H . Observe also that K does not have countable base at \mathbf{a} because the set I is uncountable. Hence K is not metrizable. Since every compact subset of a σ -space is metrizable ([18, Corollary 4.7]), we obtain that H is not a σ -space. This contradiction shows that $c(X_{\aleph_0}) = \aleph_0$.

Since $c(X_{\aleph_0}) = \aleph_0$, (ii) of Corollary 5.2 implies that X is countable. Now Proposition 5.10 implies that $B_1^{st}(X, Y) = Y^X$. Therefore $H = Y^X$. Thus Y is an \mathfrak{n} - σ -space, see Corollary 5.6 of [13].

Conversely, assume that X is countable and Y is an \mathfrak{n} - σ -space. Then, by Proposition 5.10, $H = Y^X$ and hence H is an \mathfrak{n} - σ -space by Corollary 5.6 of [13]. \square

Theorem 5.12. *Let Y be a perfectly normal space containing at least two points, X be a Y -Tychonoff space, $\mathfrak{n} \in \{cp, ck, cs^*\}$, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then H has countable \mathfrak{n} -character if and only if X is countable and Y has countable \mathfrak{n} -character. In this case $H = Y^X$.*

PROOF. Assume that H has countable \mathfrak{n} -character. First we show that X is countable. To this end, observe that in all cases X has countable cs^* -character. Now we prove that X_{\aleph_0} has countable cellularity. Suppose for a contradiction that $c(X_{\aleph_0})$ is uncountable. Consider the compact subset K of H defined in the proof of Theorem 5.11. Then also K being a subspace of H has countable cs^* -character. However, by Proposition 9 of [5], the cs^* -character of K is uncountable. This contradiction shows that $c(X_{\aleph_0}) = \aleph_0$.

Since $c(X_{\aleph_0}) = \aleph_0$, (ii) of Corollary 5.2 implies that X is countable. Now Proposition 5.10 implies that $B_1^{st}(X, Y) = Y^X$. Therefore $H = Y^X$. Thus Y has countable \mathfrak{n} -character, see Corollary 5.5 of [13].

Conversely, assume that X is countable and Y has countable \mathfrak{n} -character. Then, by Proposition 5.10, $H = Y^X$, and hence H has countable \mathfrak{n} -character by Corollary 5.5 of [13]. \square

Theorems 5.11 and 5.12 immediately imply

Corollary 5.13. *Let Y be a metric space containing at least two points, X be a Y -Tychonoff space, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then the following assertions are equivalent:*

- (i) H is metrizable and $H = Y^X$;
- (ii) H is a σ -space;
- (iii) H has countable cs^* -character;
- (iv) X is countable.

Corollary 5.14. *Let Y be a metric space containing at least two points, X be a Y -Tychonoff space, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then H is a cosmic space if and only if X is a countable Tychonoff space and Y is separable. Consequently, $H = Y^X$ is a separable metric space and therefore H is a \mathfrak{P}_0 -space.*

PROOF. Assume that H is cosmic. Then, by Corollary 5.13, X is countable and $H = Y^X$ is a metric space. Being cosmic the space H and hence also Y must be separable. Conversely, if X is countable and Y is separable, then Corollary 5.13 implies that $H = Y^X$. Therefore H is separable and metrizable. Thus H is a \mathfrak{P}_0 -space and hence cosmic. \square

To obtain the Fréchet–Urysohnness of spaces of Baire function we need the next proposition.

Proposition 5.15. *Let Y be a perfectly normal space containing at least two points, X be a Y -Tychonoff space, \mathcal{D} be a finite subset of Y , and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then H considered as a subspace of $C_p(X_{\aleph_0}, Y)$ has the following property:*

(i) $H \cap C_p(X_{\aleph_0}, \mathcal{D})$ is a relatively \mathcal{D}_p -Tychonoff subspace of $C_p(X_{\aleph_0}, \mathcal{D})$.

PROOF. Since X is Tychonoff (see Proposition 2.6), Proposition 5.3 implies that H is a subspace of $C_p(X_{\aleph_0}, Y)$. So it suffices to prove that $B_1^{st}(X, Y)$ satisfies (i).

Fix a closed subset A of X_{\aleph_0} , a point $y_0 \in \mathcal{D}$ and a function $f : F \rightarrow \mathcal{D}$ defined on a finite subset $F = \{x_1, \dots, x_n\}$ of $X \setminus A$. Since X is Tychonoff, there are disjoint zero-sets V_1, \dots, V_n in X such that $x_i \in V_i$ for every $i = 1, \dots, n$. Choose a zero-set V in X such that $F \subseteq V$ and $V \cap A = \emptyset$. For every $i = 1, \dots, n$, set $U_i := V_i \cap V$. Then U_1, \dots, U_n are disjoint zero-sets in X such that $A \cap U_i = \emptyset$ and $x_i \in U_i$ for every $i = 1, \dots, n$. By Lemma 5.7, there exist a zero-set Z in X and a function $\bar{f} \in B_1^{st}(X, Y) \subseteq C_p(X_{\aleph_0}, Y)$ such that $F \subseteq Z \subseteq \bigcup_{i=1}^n U_i$ and

$$\bar{f}|_F = f, \bar{f}(X \setminus Z) \subseteq \{y_0\} \text{ and } \bar{f}(X) \subseteq \{y_0\} \cup f(F) \subseteq \mathcal{D}.$$

Taking into account that $A \subseteq X \setminus Z$, we obtain that \bar{f} is a desired extension of f . \square

Theorem 5.16. *Let Y be a metrizable space containing at least two points, X be a Y -Tychonoff space, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then the following assertions are equivalent:*

- (i) H is a Fréchet–Urysohn space;
- (ii) H is a sequential space;
- (iii) H has countable tightness;
- (iv) X_{\aleph_0} is a Lindelöf space;
- (v) X_{\aleph_0} has the property γ .

Moreover, if X is scattered, then (i)–(v) are equivalent to

- (vi) X is a Lindelöf space.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear.

(iii) \Rightarrow (iv) Let $\xi = \{V_i : i \in I\}$ be an open cover of X_{\aleph_0} . By the definition of the Baire topology τ_b , we can assume that all V_i are zero-sets in X . Moreover we can assume that ξ is closed under taking finite unions. Fix two distinct points $a, b \in Y$. By Lemma 5.7, for every $i \in I$ and each finite subset $F \subseteq V_i$ there are a zero-set $Z_{i,F}$ and a function $f_{i,F} \in B_1^{st}(X, Y) \subseteq H$ such that

$$Z_{i,F} \subseteq V_i, f_{i,F}(Z_{i,F}) = \{a\} \text{ and } f_{i,F}(X \setminus Z_{i,F}) = \{b\}. \quad (5.1)$$

Since the cover ξ is closed under taking finite unions, the constant function \mathbf{a} belongs to the closure of the family $\mathcal{F} = \{f_{i,F} : i \in I, F \in [V_i]^{<\omega}\}$. As H has countable tightness, there is a sequence $S = \{f_{i_n, F_n} : n \in \omega\}$ in \mathcal{F} such that $\mathbf{a} \in \overline{S}$. Set $X_0 := \bigcup_{n \in \omega} V_{i_n}$. We claim that $X_0 = X$. Indeed, assuming the converse we can find a point $z \in X \setminus X_0$. Then, by (5.1), $f_{i_n, F_n}(z) = b$ for every $n \in \omega$. Choose an open neighborhood U of a such that $b \notin U$. Then $W[\mathbf{a}; \{z\}, U] \cap S = \emptyset$ that contradicts the inclusion $\mathbf{a} \in \overline{S}$. Thus $X_0 = X$ and hence X_{\aleph_0} is Lindelöf.

(iv) \Rightarrow (v) Since X_{\aleph_0} is a P -space and Lindelöf, the Galvin lemma [14] implies that X_{\aleph_0} has the property γ .

(v) \Rightarrow (i) First we note that the space X_{\aleph_0} is Y -Tychonoff by Proposition 5.8. Since X_{\aleph_0} has the property γ , Theorem 3.3 implies that the space $C_p(X_{\aleph_0}, Y)$ is Fréchet–Urysohn. As $H \subseteq B(X, Y) \subseteq C_p(X_{\aleph_0}, Y)$ it follows that also H is a Fréchet–Urysohn space.

Finally, if X is scattered, then (iv) and (vi) are equivalent by a result of Uspenskiĭ (see the proof of Lemma II.7.14 of [1]) which states that a scattered space X is Lindelöf if and only if X_{\aleph_0} is Lindelöf. \square

Corollary 5.17. *Let Y be a metrizable space containing at least two points, X be a Y -Tychonoff space, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then the following assertions are equivalent:*

- (i) H^k is Fréchet–Urysohn for each $k \in (0, \omega]$;
- (ii) H^k is a sequential space for each $k \in (0, \omega]$;
- (iii) H^k has countable tightness for each $k \in (0, \omega]$;
- (iv) X_{\aleph_0} has the property γ .
- (v) H^k is Fréchet–Urysohn (sequential or has countable tightness) for some $k \in (0, \omega]$.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) and (i) \Rightarrow (v) ((ii) \Rightarrow (v) and (iii) \Rightarrow (v), respectively) are clear.

(v) \Rightarrow (iv) Since H is a closed subspace of H^k , also the space H is Fréchet–Urysohn (sequential or has countable tightness). Therefore, by Theorem 5.16, X_{\aleph_0} has the property γ .

(iv) \Rightarrow (i) Since X_{\aleph_0} has the property γ , Theorem 3.4 implies that the space $C_p(X_{\aleph_0}, Y)^\omega$ is Fréchet–Urysohn. As $H \subseteq B(X, Y) \subseteq C_p(X_{\aleph_0}, Y)$ it follows that H^k embeds into $C_p(X_{\aleph_0}, Y)^\omega$ for every $k \in (0, \omega]$. Thus H^k is a Fréchet–Urysohn space. \square

Example 5.18. Let $X = [0, \omega_1]$ and $Z = [0, \omega_1)$. Note that X and Z are scattered spaces. Since X and Z are zero-dimensional T_1 -spaces ([10, 6.2.18]), by Proposition 2.13(iii), X and Z are Y - z -normal for any topological space Y . By Theorem 5.16, for each metrizable space Y containing at least two points (for example $Y = \mathbb{R}$ or $Y = \mathbf{2}$) and every ordinal $\alpha > 0$, we obtain

- (i) $B_\alpha^{st}(X, Y)$ and $B_\alpha(X, Y)$ are Fréchet–Urysohn spaces, but
- (ii) $B_\alpha^{st}(Z, Y)$ and $B_\alpha(Z, Y)$ have uncountable tightness. \square

In the general case the tightness of spaces of Baire functions is computed in the next proposition which generalizes (B) of Pestryakov’s Theorem 1.8 (see also [31]).

Proposition 5.19. *Let Y be a metrizable space containing at least two points, X be a Y -Tychonoff space, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then*

$$t(H) = p\text{-Lin}(X_{\aleph_0}) = \sup\{l(X_{\aleph_0}^n) : n \in \omega\}.$$

PROOF. By Proposition 5.15, H is a relatively Y_p -Tychonoff subspace of $C_p(X_{\aleph_0}, Y)$. Then the first equality $t(H) = p\text{-Lin}(X_{\aleph_0})$ follows from Theorem 4.6. Now the second equality immediately follows from Theorem 4.7.1 and Corollary 4.7.3 of [28] applied to $C_p(X_{\aleph_0})$. \square

For compact spaces we have the following result.

Theorem 5.20. *Let Y be a metrizable space containing at least two points, K be a Y -Tychonoff compact space, and let H be a subspace of $B(K, Y)$ containing $B_1^{st}(K, Y)$. Then the following assertions are equivalent:*

- (i) H is a Fréchet–Urysohn space;
- (ii) H is a sequential space;
- (iii) H has countable tightness;
- (iv) K is scattered.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear.

(iii) \Rightarrow (iv) Suppose for a contradiction that K is not scattered. Then, by Theorem 8.5.4 of [40], there is a continuous surjective map $T : K \rightarrow \mathbb{I} = [0, 1]$. Since \mathbb{I} is a normal space, $K = \bigcup_{t \in \mathbb{I}} T^{-1}(t)$ is a continual disjoint union of zero-sets. Therefore the space K_{\aleph_0} is not Lindelöf. Now Theorem 5.16 implies that the tightness of H is uncountable, a contradiction.

(iv) \Rightarrow (i) Since K is a compact scattered space, Theorem 5.7 of [27] states that K_{\aleph_0} is Lindelöf and Theorem 5.16 applies. \square

Theorem 5.21. *Let Y be a non-compact metrizable space, X be a Y -Tychonoff space, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then the following assertions are equivalent:*

- (i) H is a Fréchet–Urysohn space;
- (ii) H is a sequential space;
- (iii) H is a k -space;
- (iv) H has countable tightness;
- (v) X_{\aleph_0} satisfies the property γ .

PROOF. Observe that $H \subseteq B(X, Y) \subseteq C_p(X_{\aleph_0}, Y)$. Since Y is a metric space, Proposition 5.8 implies that the space X_{\aleph_0} is Y -Tychonoff. By Proposition 5.15, for every countable (finite or not) subset $\mathcal{D} \subseteq Y$, $H \cap C_p(X_{\aleph_0}, \mathcal{D})$ is a relatively \mathcal{D}_p -Tychonoff subspace of $C_p(X_{\aleph_0}, \mathcal{D})$. Now Theorems 4.3 and 5.16 apply. \square

Since the class of Tychonoff spaces coincides with the class of \mathbb{R} -Tychonoff spaces, Theorem 5.21 implies the following strengthening of Pestryakov’s Theorem 1.8.

Corollary 5.22. *Let X be a Tychonoff space, and let H be a subspace of $B(X)$ containing $B_1^{st}(X)$. Then the following assertions are equivalent:*

- (i) H is a Fréchet–Urysohn space;
- (ii) H is a sequential space;
- (iii) H is a k -space;
- (iv) H has countable tightness;
- (v) X_{\aleph_0} satisfies the property γ .

If X has countable pseudocharacter, then X_{\aleph_0} satisfies the property γ if and only if X is countable because, by Corollary 5.2, X_{\aleph_0} is discrete. Therefore, for subspaces of $B(X, Y)$, Theorem 5.21 and Corollary 5.13 imply the following extension of Theorem 1.9.

Corollary 5.23. *Let Y be a non-compact metrizable space, X be a Y -Tychonoff space of countable pseudocharacter, and let H be a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. Then the following assertions are equivalent:*

- (i) H is metrizable and $H = Y^X$;
- (ii) H is a k -space;
- (iii) H has countable tightness;
- (iv) H is a σ -space;
- (v) H has countable cs^* -character;
- (vi) X is countable.

Now we consider bounded Baire functions. Let X be a Tychonoff space and E be a locally convex space. A map $f : X \rightarrow E$ is called *bounded (relatively compact)* if the image $f(X)$ is a bounded (respectively, relatively compact) subset of E . For every countable ordinal α , we denote by $B_\alpha^b(X, E)$, $B_\alpha^{rc}(X, E)$, $B_\alpha^{st,b}(X, E)$ or $B_\alpha^{st,rc}(X, E)$ the family of all functions from $B_\alpha(X, E)$ or $B_\alpha^{st}(X, E)$ which are bounded or relatively compact, respectively. It is clear that

$$B_\alpha^{st,rc}(X, E) \subseteq B_\alpha^{st,b}(X, E) \subseteq B_\alpha^{st}(X, E) \quad \text{and} \quad B_\alpha^{rc}(X, E) \subseteq B_\alpha^b(X, E) \subseteq B_\alpha(X, E).$$

For the special case when Y is a metrizable locally convex space, the next result strengthens Corollary 5.13 and Theorem 5.21.

Theorem 5.24. *Let E be a metrizable locally convex space, X be a Tychonoff space, and let H be a subspace of $B(X, E)$ containing $B_1^{st,rc}(X, E)$. Then:*

- (A) *The following assertions are equivalent:*
 - (i) H is metrizable;
 - (ii) H is a σ -space;
 - (iii) H has countable cs^* -character;
 - (iv) X is countable.
- (B) *The following assertions are equivalent:*
 - (i) H is a Fréchet–Urysohn space;
 - (ii) H is a sequential space;
 - (iii) H is a k -space;
 - (iv) H has countable tightness;
 - (v) X_{\aleph_0} satisfies the property γ .

PROOF. The proof is actually given in the proofs of Theorems 5.11, 5.12, 5.16 and Proposition 5.15. Indeed, all functions used there have only finite image and hence belong to $B_1^{st,rc}(X, E)$. \square

We finish this section with the following analogue of Proposition 5.10.

Proposition 5.25. *Let E be a locally convex space and let X be a countable Tychonoff space. Then a map $f : X \rightarrow E$ is bounded (relatively compact) if and only if it belongs to $B_1^{st,b}(X, E)$ (respectively, $B_1^{st,rc}(X, E)$).*

PROOF. The assertion is trivial if X is finite. So, we assume that X is infinite. The sufficiency follows from the definition of $B_1^{st,b}(X, E)$ ($B_1^{st,rc}(X, E)$). To prove the necessity, let $f : X \rightarrow E$ be a bounded (relatively compact) map. Write X as the union $X = \bigcup_{n \in \omega} X_n$ of an increasing sequence $\{X_n\}_{n \in \omega}$ of nonempty finite subsets X_n of X . Since X is countable and Tychonoff, it is zero-dimensional by Corollary 6.2.8 of [10]. For every $n \in \omega$, by Proposition 2.10, the identity map $s_n : F_n \rightarrow F_n$ can be extended to a continuous function $r_n : X \rightarrow X_n$. It follows that the function sequence $\{f \circ r_n\}_{n \in \omega}$ stably converges to f and hence $f \in B_1^{st}(X, E)$. Since $f(X)$ is a bounded (respectively, relatively compact) subset of E , we obtain $f \in B_1^{st,b}(X, E)$ (respectively, $f \in B_1^{st,rc}(X, E)$). \square

6. Normality of spaces of Baire one functions

In [10, 3.1.H], Engelking showed that the space $\omega^{\mathfrak{c}}$ contains a closed discrete subspace of cardinality \mathfrak{c} , where ω is endowed with the discrete topology. The next proposition essentially generalizes this fact (just apply the proposition to $Y = \omega$, $X = 2^\omega$ and $H = Y^X$).

Proposition 6.1. *Let Y be a Tychonoff space containing a closed and discrete subspace $D = \{y_n\}_{n \in \omega}$, X be an uncountable Y - z -Tychonoff metrizable compact space, and let H be a subspace of Y^X containing $B_1^{st}(X, Y)$. Then H contains a discrete and closed subspace $\mathcal{F} \subseteq B_1^{st}(X, Y)$ of cardinality \mathfrak{c} such that $f(X) \subseteq D$ for every $f \in \mathcal{F}$ (in fact, \mathcal{F} is closed and discrete in Y^X).*

PROOF. First we note that the metrizable space X is Y -normal by Proposition 2.15. Fix a metric ρ on X such that $\rho(x, y) \leq 1$ for all points $x, y \in X$. For every $t \in X$, similar to [10, 3.1.H], define a function $f_t : X \rightarrow D$ as follows:

$$f_t(t) := y_0, \text{ and } f_t(x) := y_n \text{ if } \frac{1}{n+1} < \rho(x, t) \leq \frac{1}{n}.$$

Claim 1. $f_t \in B_1^{st}(X, Y)$ for every $t \in X$. Indeed, fix $t \in X$. For every natural numbers n, k such that $1 \leq n \leq k$, set

$$U_{n,k}^t := \left\{ x \in X : \frac{1}{n+1} + \frac{1}{4k^2} \leq \rho(x, t) \leq \frac{1}{n} \right\}$$

and

$$V_k^t := \left\{ x \in X : \rho(x, t) \leq \frac{1}{k+1} \right\}.$$

It is easy to see that the family $\{U_{n,k}^t\}_{n=1}^k \cup \{V_k^t\}$ of closed subsets of X is disjoint and $t \in V_k^t$. Since X is a Y -normal space, for every $k \geq 1$, there exists a continuous function $f_{t,k} : X \rightarrow Y$ such that

$$f_{t,k}(x) = y_n \text{ if } x \in U_{n,k}^t \text{ for } 1 \leq n \leq k, \text{ and } f_{t,k}(x) = y_0 \text{ if } x \in V_k^t.$$

To prove the claim we shall show that $f_{t,k}$ stably converges to f_t . Indeed, $f_{t,k}(t) = y_0$ for every $k \geq 1$. Now let $x \neq t$ and choose $n \geq 1$ such that $\frac{1}{n+1} < \rho(x, t) \leq \frac{1}{n}$. Take $k_0 \geq 1$ such that $\frac{1}{n+1} + \frac{1}{4k_0^2} < \rho(x, t)$. Then for every $k \geq \max\{n, k_0\}$, we have $x \in U_{n,k}^t$ and hence $f_{t,k}(x) = y_n = f_t(x)$. Thus $f_t \in B_1^{st}(X, Y)$. Claim 1 is proved.

Let $\mathcal{F} := \{f_t : t \in X\}$. We shall show that \mathcal{F} satisfies the conditions of the proposition. First we show that \mathcal{F} is discrete in Y^X . To this end, consider two distinct points $t, t' \in X$. Then $f_t(t) = y_0$ but $f_{t'}(t) \in D \setminus \{y_0\}$. If \mathcal{O}_0 is an open neighborhood of y_0 such that $\mathcal{O}_0 \cap D = \{y_0\}$, then $W[f_t; \{t\}, \mathcal{O}_0] \cap \mathcal{F} = \{f_t\}$. Thus \mathcal{F} is discrete.

To show that \mathcal{F} is closed, we shall prove that \mathcal{F} is closed in Y^X . Suppose for a contradiction that \mathcal{F} is not closed in Y^X . Fix a function $\chi \in \overline{\mathcal{F}} \setminus \mathcal{F}$. Observe that $\chi(X) \subseteq D$ because D is closed and, by construction, $f(X) \subseteq D$ for every $f \in \mathcal{F}$.

Claim 2. $\chi(X) \subseteq D \setminus \{y_0\}$. Indeed, assuming the converse we can find a point $t \in X$ such that $\chi(t) = y_0$. Since $\chi \neq f_t$, there is an $x \in X$ such that $\chi(x) \neq f_t(x)$. Choose an open neighborhood $U_{\chi(x)}$ of $\chi(x)$ in Y such that $U_{\chi(x)} \cap D = \{\chi(x)\}$. Then

$$(W[\chi; \{t\}, \mathcal{O}_0] \cap W[\chi; \{x\}, U_{\chi(x)}]) \cap \mathcal{F} = \emptyset$$

that contradicts the inclusion $\chi \in \overline{\mathcal{F}}$. Thus $\chi(X) \subseteq D \setminus \{y_0\}$.

Claim 3. There is an $m \geq 1$ such that $\chi(X) \subseteq \{y_1, \dots, y_m\}$. Indeed, suppose for a contradiction that there exist a sequence $\{x_k\}_{k \geq 1} \subseteq X$ and an increasing sequence $\{n_k\}_{k \geq 1} \subseteq \omega$ such that

$\chi(x_k) = y_{n_k}$. Since X is compact, without loss of generality we can assume that $x_k \rightarrow x_0$. Note that $\chi(x_0) = y_{n_0}$ for some $n_0 > 0$. Choose an open neighborhood \mathcal{O}_{n_0} of y_{n_0} such that $\mathcal{O}_{n_0} \cap D = \{y_{n_0}\}$. Since $\chi \in \overline{\mathcal{F}}$, the standard neighborhood $W[\chi; \{x_0\}, \mathcal{O}_{n_0}]$ of χ contains only those $f_t \in \mathcal{F}$ for which $f_t(x_0) = y_{n_0}$. Denote by T the set of all $t \in X$ such that

$$\frac{1}{n_0+1} < \rho(x_0, t) \leq \frac{1}{n_0},$$

so $t \in T$ if and only if $f_t(x_0) = y_{n_0}$. As the metric ρ is continuous and X is compact, there is an open neighborhood V of x_0 such that

$$\frac{1}{n_0+2} < \rho(x, t) \leq \min \left\{ \frac{1}{n_0-1}, 1 \right\} \quad (6.1)$$

for every $x \in V$ and each $t \in T$. But the definition of f_t and (6.1) imply that

$$f_t(x_k) \in \{y_{\max\{n_0-1, 1\}}, y_{n_0}, y_{n_0+1}\} \quad (6.2)$$

for every $t \in T$ and all $k \in \omega$ for which $x_k \in V$. Choose $k \in \omega$ such that $k > n_0 + 1$ and $x_k \in V$, and take an open neighborhood \mathcal{O}_{n_k} of y_{n_k} such that $\mathcal{O}_{n_k} \cap D = \{y_{n_k}\}$. Then (6.2) implies

$$(W[\chi; \{x_0\}, \mathcal{O}_{n_0}] \cap W[\chi; \{x_k\}, \mathcal{O}_{n_k}]) \cap \mathcal{F} = \emptyset$$

which contradicts the inclusion $\chi \in \overline{\mathcal{F}}$. Claim 3 is proved.

Since X is compact, choose a finite subset $\{z_1, \dots, z_s\}$ of X such that for every $x \in X$ there is an $i_x \in \{1, \dots, s\}$ for which $\rho(x, z_{i_x}) < \frac{1}{2m}$, where $m \in \omega$ is defined in Claim 3. Choose open neighborhoods O_1, \dots, O_s of $\chi(z_1), \dots, \chi(z_s)$, respectively, such that $O_i \cap D = \{\chi(z_i)\}$ for every $i = 1, \dots, s$. Consider the standard neighborhood

$$W = \bigcap_{i=1}^s W[\chi; \{z_i\}, O_i]$$

of the function χ . Now, for every $t \in X$, choose $i_t \in \{1, \dots, s\}$ such that $\rho(t, z_{i_t}) < \frac{1}{2m}$. Then either $t = z_{i_t}$ and hence $f_t(z_{i_t}) = y_0$, or $t \neq z_{i_t}$ and hence $f_t(z_{i_t}) = y_{n_t}$ for some $n_t > m$. So, by Claims 2 and 3 and the construction of W , it follows that $f_t \notin W$ for every $t \in X$. Therefore $\chi \notin \overline{\mathcal{F}}$ that contradicts the choice of the function χ . Thus the family \mathcal{F} is closed in Y^X . \square

To show that a given space is not normal we shall use the following Jones' Lemma, see [20, Lemma 3.5].

Lemma 6.2. *If X is a normal space, then $2^{|D|} \leq 2^{d(X)}$ for every closed discrete $D \subseteq X$. In particular, if X is normal and separable, then: (1) X cannot have a closed discrete set of cardinality $\geq \mathfrak{c}$; (2) $2^\omega < 2^{\omega_1}$ implies that X cannot have a closed discrete set of cardinality ω_1 .*

Proposition 6.3. *Let Y be a separable non-compact metric space, X be a Y - z -Tychonoff metrizable compact space, and let H be a subspace of Y^X containing $B_1^{st}(X, Y)$. Then H is a normal space if and only if X is countable. In this case $H = Y^X$.*

PROOF. Let H be a normal space. If X is uncountable, then, by Proposition 6.1, H contains a closed and discrete subspace \mathcal{F} of cardinality \mathfrak{c} . Since X is Y -Tychonoff (Proposition 2.13), Corollary 4.15 implies that the space $C_p(X, Y)$ is separable. Since $C_p(X, Y)$ is dense in H , the space H is also separable. Therefore, by Lemma 6.2, H is not normal. This contradiction shows that X must be countable. Conversely, if X is countable, then, by Proposition 5.10, $H = Y^X$, and therefore H is metrizable and hence normal. \square

Recall that a topological space X is called *k-scattered* if every compact Hausdorff subspace of X is scattered. It is proved in [8, p.34] that a Čech-complete space X is scattered if and only if it is *k-scattered* (for a short proof of this result see Theorem 8.7 in [4]).

Proposition 6.4. *Let Y be a path-connected non-compact separable metric space, and let X be a Tychonoff space. Assume that H is a subspace of $B(X, Y)$ containing $B_1^{st}(X, Y)$. If H is a normal space, then X is *k-scattered*.*

PROOF. We have to show that every compact subspace of X is scattered. Assuming the converse we can find a non-scattered compact subspace K of X . By Theorem 8.5.4 of [40], there is a continuous surjective map $h : K \rightarrow \mathbb{I} = [0, 1]$. The Tietze–Urysohn Theorem [10, 2.1.8] implies that h has a continuous (surjective) extension $g : X \rightarrow \mathbb{I}$. Observe that the adjoint map $g^* : Y^{\mathbb{I}} \rightarrow Y^X$, $g^*(f) := f \circ g$, is a homeomorphism of $Y^{\mathbb{I}}$ onto a *closed* subspace of Y^X . By Proposition 2.13, \mathbb{I} is Y - z -Tychonoff. Since Y is not compact, it contains a closed and discrete countably infinite subset. Now we can apply Proposition 6.1 to get that the space $B_1^{st}(\mathbb{I}, Y)$ contains a subset A of cardinality \mathfrak{c} which is closed and discrete in $Y^{\mathbb{I}}$. Since g is continuous we also have $g^*(B_1^{st}(\mathbb{I}, Y)) \subseteq B_1^{st}(X, Y)$. Therefore $g^*(A)$ is also a closed and discrete subset of H . Note that a closed subset of a normal space is a normal space. Therefore the space $Z := \overline{g^*(B_1^{st}(\mathbb{I}, Y))}^H$ is normal. Since $B_1^{st}(\mathbb{I}, Y)$ is separable (because it contains a dense subspace $C_p(\mathbb{I}, Y)$ which is separable by Corollary 4.15), we obtain that also Z is separable. Therefore $g^*(A)$ is a closed and discrete subset of the separable space Z and has cardinality \mathfrak{c} . Now Lemma 6.2 implies that Z is not normal, a contradiction. \square

Recall (see [27]) that a Tychonoff space X is called *functionally countable* if the set $f(X)$ is countable for each function $f \in C(X)$. In [4], topological spaces with this property are called \mathbb{R} -countable. In Theorem 4.1 of [7], Choban proved that a Tychonoff space X is functionally countable if and only if $f(X)$ is countable for each function $f \in B_1(X)$. Proposition 4.4 of [7] states that if X_{\aleph_0} is a Lindelöf space, then X and X_{\aleph_0} are functionally countable spaces.

Recall that a subset A of a topological space X is called a Z_σ -set in X if $A = \bigcup \{Z_i : i \in \omega\}$, where Z_i is a zero-set of X for each $i \in \omega$. For topological spaces X and Y , let $\mathcal{A}_1^0(X, Y)$ be the spaces of all functions $f : X \rightarrow Y$ such that $f^{-1}(U)$ is a Z_σ -set in X . We need the following assertion.

Proposition 6.5. *Let Y be a perfectly normal space containing at least two points, and let X be a Y -Tychonoff space such that $B_1(X, Y) = \mathcal{A}_1^0(X, Y)$. If X is functionally countable, then $B_1(X, Y) = B(X, Y)$. In particular, $B_1(X, Y) = B_2(X, Y)$.*

PROOF. Let $f \in B(X, Y)$. Since Y is perfectly normal, for every open subset $U \subseteq Y$ there is a continuous function $g : Y \rightarrow \mathbb{R}$ such that $U = g^{-1}(\mathbb{R} \setminus \{0\})$. By induction on $\alpha \in \omega_1$, one can check that the function $g \circ f : X \rightarrow \mathbb{R}$ is also Baire. Hence $f^{-1}(U) = (g \circ f)^{-1}(\mathbb{R} \setminus \{0\})$ is a Baire subset of X . Therefore, by the equality $B_1(X, Y) = \mathcal{A}_1^0(X, Y)$, to prove the proposition it is sufficient to show that every Baire subset A of X is a Z_σ -set in X . So let A be a Baire subset of X .

Claim 1. *There are $g = (g_i)_{i \in \omega} \in C(X, \mathbb{R}^\omega)$ and a Baire subset H of \mathbb{R}^ω such that $A = g^{-1}(H)$.* Indeed, by the definition of Baire sets of a topological space, the set A is generated by a family $\{Z_i : i \in \omega\}$ of zero-sets of X , i.e. $A = \varphi[Z_0, \dots, Z_i, \dots]$ where φ is an action in the hierarchy on the Baire sets. For every $i \in \omega$, choose a function $g_i \in C(X)$ such that Z_i is the zero-set of g_i and consider the diagonal mapping $g = (g_i)_{i \in \omega} : X \rightarrow \mathbb{R}^\omega$. For each $i \in \omega$, set $P_i := \{x = (x_i) \in \mathbb{R}^\omega : x_i = 0\}$. It is clear that for each $i \in \omega$, P_i is a zero-set of \mathbb{R}^ω and $g^{-1}(P_i) = Z_i$. Let $H = \varphi[P_1, \dots, P_i, \dots]$, so H

is a Baire subset of \mathbb{R}^ω . Since the operation of taking preimage preserves unions, intersections and differences we obtain that $A = g^{-1}(H)$. The claim is proved.

As X is functionally countable, all the sets $g_i(X) \subseteq \mathbb{R}$ are countable. Therefore the product $\prod_{i \in \omega} g_i(X)$ is a separable, zero-dimensional metrizable space, and hence so is its subspace $g(X)$. By Theorem 6.2.16 in [10], there is an embedding p of $g(X)$ into 2^ω . We can assume that $2^\omega \subseteq \mathbb{R}$. As X is functionally countable, the set $p \circ g(X)$ is countable. Since p is a bijection, we obtain that $g(X)$ and hence also H are countable. But every countable subset of \mathbb{R}^ω is a Z_σ -set in \mathbb{R}^ω . Hence $A = g^{-1}(H)$ is a Z_σ -set in X . \square

Let X and Y be sets. For a function $f : X \rightarrow Y$ and a subset $T \subseteq Y$, we set

$$\sigma(f, T) := \{h \in \sigma(f) : h(x) \in T \text{ for every } x \in X \text{ such that } h(x) \neq f(x)\}.$$

In the next theorem we shall use the following theorem (although this result was proved in [12] for the case when Y is an abelian metrizable group containing an infinite uniformly discrete subset, exactly the same proof works also for non-compact metrizable spaces).

Theorem 6.6 ([12]). *Let Y be a non-compact metrizable space. Choose a discrete and closed sequence $T = \{g_n\}_{n \in \omega}$ in Y . Assume that X is a set and let H be a subspace of Y^X containing pointwise limits of sequences from $\sigma(\mathbf{g}_1, T) \cup \sigma(\mathbf{g}_0, T)$. Then H is a normal space if and only if X is countable.*

Recall that a topological space X is called *K-analytic* if X is a continuous image of a Lindelöf Čech-complete space.

Theorem 6.7. *Let Y be a path-connected non-compact separable metric space, and let X be a Tychonoff K-analytic space of countable pseudocharacter. If $B_1(X, Y) = \mathcal{A}_1^0(X, Y)$, then the following assertions are equivalent:*

- (i) X is countable, so $B_1(X, Y) = Y^X$ is separable and metrizable;
- (ii) $B_1(X, Y)$ is Lindelöf;
- (iii) $B_1(X, Y)$ is normal.

PROOF. (i) \Rightarrow (ii) If X is countable, then, by Proposition 5.10, $B_1(X, Y) = Y^X$. Thus $B_1(X, Y)$ is separable and metrizable and hence Lindelöf. The implication (ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (i) Assume that $B_1(X, Y)$ is normal. Then, by Proposition 6.4, X is k -scattered. By Theorem 9.3 of [4], the space X is functionally countable. Now Proposition 6.5 implies that $B_1(X, Y) = B_2(X, Y)$. Therefore the space $B_2(X, Y)$ is also normal. By Proposition 2.9, the space X is Y -Tychonoff. Finally, applying Theorem 6.6 and Lemma 5.4, we obtain that X is countable. \square

To prove the next theorem we shall use the following result which is proved in [24, 25, 26].

Proposition 6.8. *If Y is a separable metrizable space and X is Y -dimensional, then $B_1(X, Y) = \mathcal{A}_1^0(X, Y)$.*

Recall that a metrizable space X is called an *absolute retract* if X is a retract of any metrizable space containing X as a closed subspace.

Theorem 6.9. *Let Y be a path-connected non-compact Polish absolute retract, and let X be a normal K -analytic space of countable pseudocharacter. Then $B_1(X, Y)$ is a normal space if and only if X is countable. In this case $B_1(X, Y) = Y^X$ is a separable metrizable space.*

PROOF. By Theorem 16.1(d) of Section II in [21], the space X is Y -dimensional. Therefore, by Proposition 6.8, $B_1(X, Y) = \mathcal{A}_1^0(X, Y)$. Now Theorem 6.7 applies. \square

Theorem 6.9 immediately implies

Corollary 6.10. *Let X be a Lindelöf Čech-complete space of countable pseudocharacter (for example, X is a Polish space). Then $B_1(X)$ is normal if and only if X is countable. In this case $B_1(X) = \mathbb{R}^X$ is a separable metric space.*

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