

ESTIMATES OF BEST APPROXIMATIONS OF FUNCTIONS WITH LOGARITHMIC SMOOTHNESS IN THE LORENTZ SPACE WITH ANISOTROPIC NORM¹

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Abstract: In this paper, we consider the anisotropic Lorentz space $L_{\bar{p},\bar{\theta}}^*(\mathbb{I}^m)$ of periodic functions of m variables. The Besov space $B_{\bar{p},\bar{\theta}}^{(0,\alpha,\tau)}$ of functions with logarithmic smoothness is defined. The aim of the paper is to find an exact order of the best approximation of functions from the class $B_{\bar{p},\bar{\theta}}^{(0,\alpha,\tau)}$ by trigonometric polynomials under different relations between the parameters $\bar{p}, \bar{\theta}$, and τ .

The paper consists of an introduction and two sections. In the first section, we establish a sufficient condition for a function $f \in L_{\bar{p},\bar{\theta}(1)}^*(\mathbb{I}^m)$ to belong to the space $L_{\bar{p},\bar{\theta}(2)}^*(\mathbb{I}^m)$ in the case $1 < \theta^2 < \theta_j^{(1)}$, $j = 1, \dots, m$, in terms of the best approximation and prove its unimprovability on the class $E_{\bar{p},\bar{\theta}}^\lambda = \{f \in L_{\bar{p},\bar{\theta}}^*(\mathbb{I}^m) : E_n(f)_{\bar{p},\bar{\theta}} \leq \lambda_n, n = 0, 1, \dots\}$, where $E_n(f)_{\bar{p},\bar{\theta}}$ is the best approximation of the function $f \in L_{\bar{p},\bar{\theta}}^*(\mathbb{I}^m)$ by trigonometric polynomials of order n in each variable x_j , $j = 1, \dots, m$, and $\lambda = \{\lambda_n\}$ is a sequence of positive numbers $\lambda_n \downarrow 0$ as $n \rightarrow +\infty$. In the second section, we establish order-exact estimates for the best approximation of functions from the class $B_{\bar{p},\bar{\theta}(1)}^{(0,\alpha,\tau)}$ in the space $L_{\bar{p},\bar{\theta}(2)}^*(\mathbb{I}^m)$.

Key words: Lorentz space, Nikol’skii–Besov class, Best approximation.

1. Introduction

Let $\bar{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$, $\mathbb{I}^m = [0, 2\pi]^m$, $\bar{p} = (p_1, \dots, p_m)$, and $\bar{\theta} = (\theta_1, \dots, \theta_m)$, where $p_j \in (1, \infty)$ and $\theta_j \in [1, \infty)$ for $j = 1, 2, \dots, m$. Denote by $L_{\bar{p},\bar{\theta}}^*(\mathbb{I}^m)$ the Lorentz space of real-valued functions $f(\bar{x})$ that are 2π -periodic in each variable and

$$\|f\|_{\bar{p},\bar{\theta}}^* = \left\{ \int_0^{2\pi} t_m^{\frac{\theta_m}{p_m}-1} \left[\dots \left[\int_0^{2\pi} (f^{*1,\dots,*m}(t_1, \dots, t_m))^{\theta_1} t_1^{\frac{\theta_1}{p_1}-1} dt_1 \right]^{\frac{\theta_2}{p_2}} \dots \right]^{\frac{\theta_m}{p_m}} dt_m \right\}^{1/\theta_m} < +\infty,$$

where $f^{*1,\dots,*m}$ is a nonincreasing rearrangement of the function $|f(x_1, \dots, x_m)|$ in each of the variables x_j whereas the other variables are fixed (see [8, 18]).

In the case $p_1 = \dots = p_m = \theta_1 = \dots = \theta_m = p$, the Lorentz space $L_{\bar{p},\bar{\theta}}^*(\mathbb{I}^m)$ coincides with the Lebesgue space $L_p(\mathbb{I}^m)$ with the norm

$$\|f\|_p = \left[\int_0^{2\pi} \dots \int_0^{2\pi} |f(x_1, \dots, x_m)|^p dx_1 \dots dx_m \right]^{1/p},$$

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where $p \in [1, +\infty)$.

Instead of $L_{\bar{p}, \bar{\theta}}^*(\mathbb{I}^m)$, we will write $L_{p, \theta}^*(\mathbb{I}^m)$ in the case $p_1 = \dots = p_m = p$ and $\theta_1 = \dots = \theta_m = \theta$ and $L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$ if $\bar{p} = (p_1, \dots, p_m)$ and $\theta_1 = \dots = \theta_m = \theta^{(2)}$.

Given a natural number M , consider the set

$$\square_M = \{\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : |k_j| < M, j = 1, \dots, m\}.$$

Consider the multiple Dirichlet kernel

$$D_{\square_M}(\bar{x}) = \sum_{\bar{k} \in \square_M} e^{i\langle \bar{k}, \bar{x} \rangle}, \quad \bar{x} \in \mathbb{I}^m,$$

and its convolution with a function $f \in L_{\bar{p}, \bar{\theta}}^*(\mathbb{I}^m)$:

$$\sigma_s(f, \bar{x}) = \int_{\mathbb{I}^m} f(\bar{y})(D_{\square_{2^s}}(\bar{x} - \bar{y}) - D_{\square_{2^{s-1}}}(\bar{x} - \bar{y}))d\bar{y},$$

where $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and \mathbb{N} is the set of positive integers.

Let $M \in \mathbb{N}_0$, and let $T_M(\bar{x}) = \sum_{\bar{k} \in \square_M} a_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle}$ be a trigonometric polynomial of order M in each variable x_j , $j = 1, \dots, m$. Denote by \mathfrak{F}_{\square_M} the set of all such polynomials.

Let $E_{M, \dots, M}(f)_{\bar{p}, \bar{\theta}} = \inf_{T \in \mathfrak{F}_{\square_M}} \|f - T\|_{\bar{p}, \bar{\theta}}^*$ be the best approximation of a function $f \in L_{\bar{p}, \bar{\theta}}^*(\mathbb{I}^m)$ by the set \mathfrak{F}_{\square_M} . Sometimes, we will use the notation $E_M(f)_{\bar{p}, \bar{\theta}}$ instead of $E_{M, \dots, M}(f)_{\bar{p}, \bar{\theta}}$. For a given class $F \subset L_{\bar{p}, \bar{\theta}}^*(\mathbb{I}^m)$, let $E_M(F)_{\bar{p}, \bar{\theta}} = \sup_{f \in F} E_M(f)_{\bar{p}, \bar{\theta}}$.

Let $\alpha \geq 0$, $\gamma \in (-\infty, +\infty)$, and $0 < \tau < \infty$. Denote by $\mathbb{A}_{\bar{p}, \bar{\theta}}^{(\alpha, \gamma, \tau)}$ the space of all functions $f \in L_{\bar{p}, \bar{\theta}}^*(\mathbb{I}^m)$ such that the quasi-norm (see [9, 20])

$$\|f\|_{\mathbb{A}_{\bar{p}, \bar{\theta}}^{(\alpha, \gamma, \tau)}} = \left[\sum_{n=1}^{\infty} n^{-1} (n^\alpha (1 + \log n)^\gamma E_n(f)_{\bar{p}, \bar{\theta}})^\tau \right]^{1/\tau}$$

is finite, where $\log a$ is the logarithm of the number a to the base 2.

If $\tau = \infty$, then

$$\|f\|_{\mathbb{A}_{\bar{p}, \bar{\theta}}^{\alpha, \gamma, \tau}} = \sup_{n \geq 1} n^\alpha (1 + \log n)^\gamma E_n(f)_{\bar{p}, \bar{\theta}} < \infty.$$

It is known that $\mathbb{A}_{\bar{p}, \bar{\theta}}^{(\alpha, \gamma, \tau)}$ is a quasi-Banach space (see [9, 10, 20]). It is called an approximate space (see [11]).

In the anisotropic Lorentz space, we consider the space $B_{\bar{p}, \bar{\theta}}^{(0, \alpha, \tau)}$, $1 \leq \tau \leq \infty$, of all functions $f \in L_{\bar{p}, \bar{\theta}}^*(\mathbb{I}^m)$ representable in the form of series

$$\sum_{n=0}^{\infty} Q_{2^{2^n}}(f, \bar{x}), \quad Q_{2^{2^n}}(f) \in \mathfrak{F}_{\square_{2^{2^n}}} \tag{1.1}$$

and such that

$$\left[\sum_{n=0}^{\infty} (2^{n\alpha} \|Q_{2^{2^n}}(f)\|_{\bar{p}, \bar{\theta}}^*)^\tau \right]^{1/\tau} < +\infty \tag{1.2}$$

for $1 \leq \tau < \infty$ and

$$\sup_{n \in \mathbb{N}_0} 2^{n\alpha} \|Q_{2^{2^n}}(f)\|_{\bar{p}, \bar{\theta}}^* < \infty$$

for $\tau = \infty$. The infimum of expression (1.2) over all representations (1.1) defines a quasi-norm in this space:

$$\|f\|_{B_{\bar{p},\bar{\theta}}^{(0,\alpha,\tau)}} = \inf \left[\sum_{n=0}^{\infty} (2^{n\alpha} \|Q_{2^{2n}}(f)\|_{\bar{p},\bar{\theta}}^*)^\tau \right]^{1/\tau}.$$

The space $B_{\bar{p},\bar{\theta}}^{(0,\alpha,\tau)}$ is called the Besov space with logarithmic smoothness. In $B_{\bar{p},\bar{\theta}}^{(0,\alpha,\tau)}$, we consider the unit ball

$$\mathbb{B}_{\bar{p},\bar{\theta}}^{(0,\alpha,\tau)} = \left\{ f \in L_{\bar{p},\bar{\theta}}^*(\mathbb{I}^m) : \|f\|_{B_{\bar{p},\bar{\theta}}^{(0,\alpha,\tau)}} \leq 1 \right\}.$$

It is known that $f \in \mathbb{B}_{\bar{p},\bar{\theta}}^{(0,\gamma+1/\tau,\tau)}$ if and only if $f \in \mathbb{A}_{\bar{p},\bar{\theta}}^{(0,\gamma,\tau)}$ (see [10]).

The main aim of the present paper is to obtain an exact order of the best approximation of the function classes $\mathbb{A}_{\bar{p},\bar{\theta}^{(1)}}^{(0,\gamma,\tau)}$ and $\mathbb{B}_{\bar{p},\bar{\theta}^{(1)}}^{(0,\gamma,\tau)}$ in anisotropic Lorentz spaces.

In the one-dimensional case, sufficient conditions for a function $f \in L_p(I^1)$ to belong to the space $L_q(\mathbb{I}^1)$ for $1 \leq p < q < \infty$ in terms of the best approximation and the modulus of continuity were established by P.L. Ul'ynov [30]. This study was continued by V.I. Kolyada [15], V.A. Andrienko [5], N. Temirgaliev [27, 28], E.A. Storozhenko [26], M.F. Timan, P. Oswald, L. Leindler, S.V. Lapin, B.V. Simonov, and others (see the references in [16]).

N. Temirgaliev established [28] a necessary and sufficient condition for a univariate function $f \in L_p(\mathbb{I}^1)$ to belong to the Lorentz space $L_{q,\theta}(\mathbb{I}^1)$ in terms of the modulus of continuity for $1 \leq \theta < p < \infty$. L.A. Sherstneva studied [22] this problem in terms of the best approximation of a function. Such problems in the Lorentz space were investigated in [1, 4, 23].

Problems of estimating various approximative characteristics of function classes are well known and a survey of the results on this topic is given in [12, 29]. In particular, in the Lebesgue space $L_p(\mathbb{I}^m)$, exact estimates of the best approximation of functions of the Besov class $B_{p,\bar{\theta}^{(1)}}^r$ were established by A.S. Romanyuk [21]. In the case $\theta_j^{(1)} = p_j = p$, $j = 1, \dots, m$, estimates of approximative characteristics of the class $\mathbb{B}_{\bar{p},\bar{\theta}^{(1)}}^{0,\alpha}$ were obtained by S.A. Stasyuk [24, 25]. In [13], the embedding and characterization problems of the Besov space with logarithmic smoothness in the Lebesgue space $L_p(\mathbb{I}^m)$ were investigated.

Exact estimates of best approximations of functions from the Besov class in the Lorentz space with a mixed norm were obtained in [2, 6, 7].

The present paper consists of the introduction and two sections. In Section 1, we establish a sufficient condition for a function $f \in L_{\bar{p},\bar{\theta}}^*(\mathbb{I}^m)$ to belong to the space $L_{\bar{p},\bar{\theta}^{(2)}}^*(\mathbb{I}^m)$, $\theta^{(2)} < \theta_j^{(1)}$, $j = 1, \dots, m$, and prove its accuracy on the class

$$E_{\bar{p},\bar{\theta}}^\lambda = \left\{ f \in L_{\bar{p},\bar{\theta}}^*(\mathbb{I}^m) : E_n(f)_{\bar{p},\bar{\theta}} \leq \lambda_n, \quad n = 0, 1, \dots \right\},$$

where $\lambda = \{\lambda_n\}$ is a sequence of positive numbers $\lambda_n \downarrow 0$ as $n \rightarrow +\infty$.

In the case $p_j = \theta_j = p$, $j = 1, \dots, m$, V.I. Kolyada proved [15] a necessary and sufficient condition for the embedding of classes E_p^λ in the space $L_q(\mathbb{I}^1)$, $1 \leq p < q$.

In Section 2, we establish order-exact estimates of the value $E_n(\mathbb{B}_{\bar{p},\bar{\theta}^{(1)}}^{(0,\gamma,\tau)})_{\bar{q},\bar{\theta}^{(2)}}$ under various relations between coordinates of the parameters $\bar{p}, \bar{\theta}^{(1)}, \bar{q}, \bar{\theta}^{(2)}, \tau$ (see Theorems 5 and 6).

The notation $A(y) \asymp B(y)$ means that there exists positive constants C_1 and C_2 such that $C_1 A(y) \leq B(y) \leq C_2 A(y)$. If $B(y) \leq C_2 A(y)$ or $A(y) \geq C_1 B(y)$, then we write $B(y) \ll A(y)$ and $A(y) \gg B(y)$, respectively.

2. Conditions for embedding classes in the Lorentz space

Theorem 1 [19, Theorem 10]. *Let $1 \leq p_j < +\infty$ and $1 \leq \theta_j < q_j < +\infty$ for $j = 1, \dots, m$, let $\bar{p} = (p_1, \dots, p_m)$ and $\bar{q} = (q_1, \dots, q_m)$, and let $\theta = (\theta_1, \dots, \theta_m)$. Then a trigonometric polynomial*

$$T_{\bar{n}}(\bar{x}) = \sum_{k_1=-n_1}^{n_1} \dots \sum_{k_m=-n_m}^{n_m} b_{\bar{k}} e^{i\langle \bar{x}, \bar{k} \rangle}$$

satisfies the following inequality:

$$\|T_{\bar{n}}\|_{\bar{p}, \bar{\theta}}^* \leq C(p, q, \theta) \prod_{j=1}^m (\ln(1 + n_j))^{1/\theta_j - 1/q_j} \|T_{\bar{n}}\|_{\bar{p}, \bar{q}}^*$$

Lemma 1. *Let $1 < p_j < \infty$ and $1 < q_2 < q_j^{(1)} < +\infty$ for $j = 1, \dots, m$. Let $\{u_n\}$ be a sequence of non-negative measurable functions on the cube $\mathbb{I}^m = [0, 2\pi]^m$ such that*

(1)

$$\|u_n\|_{\bar{p}, \bar{q}^{(1)}}^* \leq \varepsilon_n, \quad \varepsilon_{n+1} \leq \beta \varepsilon_n, \quad \beta \in (0, 1);$$

(2) *there exists a sequence of positive numbers $\{\Delta_n\}$ such that*

$$\|u_n\|_{p, \theta}^* \leq C \Delta_n^{\sum_{j=1}^m (1/\theta_j - 1/q_j^{(1)})} \varepsilon_n, \quad n = 1, 2, 3, \dots,$$

for any $\theta_j \in (0, q_j^{(1)})$, $j = 1, \dots, m$.

Then the inequality

$$\|f\|_{p, q_2}^* \leq C \left\{ \sum_{n=1}^{\infty} \Delta_n^{\sum_{j=1}^m (1/q_2 - 1/q_j^{(1)})} \varepsilon_n^{q_2} \right\}^{1/q_2}$$

holds for every function of the form $f(\bar{x}) = \sum_{n=1}^{\infty} u_n(\bar{x})$.

This lemma is proved by V.I. Kolyada's method (see [15, Proof of Lemma 4]) as in [3].

Remark 1. Lemma 1 was proved by L.A. Sherstneva [22, Lemma 13] in the one-dimensional case and by the author [3] in the multi-dimensional case for $q_1^{(1)} = \dots = q_m^{(1)}$.

Now, let us consider a condition for a function $f \in L_{\bar{p}, \bar{\theta}^{(1)}}^*(\mathbb{I}^m)$ to belong to the space $L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$, $1 < \theta^{(2)} < \theta_j^{(1)} < +\infty$, $j = 1, \dots, m$.

Theorem 2. *Let $1 < \theta^{(2)} < \theta_j^{(1)} < +\infty$ and $1 < p_j < \infty$ for $j = 1, \dots, m$, and let $\bar{\theta}^{(1)} = (\theta_1^{(1)}, \dots, \theta_m^{(1)})$. Assume that $f \in L_{\bar{p}, \bar{\theta}^{(1)}}^*(\mathbb{I}^m)$ and*

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} E_{n, \dots, n}^{\theta^{(2)}}(f)_{\bar{p}, \bar{\theta}^{(1)}} < +\infty. \quad (2.1)$$

Then $f \in L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$ and

$$\|f\|_{\bar{p}, \theta^{(2)}}^* \ll \left\{ \|f\|_{\bar{p}, \theta^{(1)}}^* + \left[\sum_{k=2}^{\infty} \frac{(\ln(k+1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{k} E_{k, \dots, k}^{\theta^{(2)}}(f)_{\bar{p}, \bar{\theta}^{(1)}} \right]^{1/\theta^{(2)}} \right\}, \quad (2.2)$$

$$\begin{aligned}
E_{n,\dots,n}(f)_{\bar{p},\theta^{(2)}} &<< \left\{ (\ln(n+1))^{\sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})} E_{n,\dots,n}(f)_{\bar{p},\bar{\theta}^{(1)}} + \right. \\
&+ \left. \left[\sum_{k=n+1}^{\infty} \frac{(\ln(k+1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{k} E_{k,\dots,k}^{\theta^{(2)}}(f)_{\bar{p},\bar{\theta}^{(1)}} \right]^{1/\theta^{(2)}} \right\}.
\end{aligned} \tag{2.3}$$

P r o o f. Since $E_{n,\dots,n}(f)_{\bar{p},\bar{\theta}^{(1)}} \equiv \varepsilon_n \downarrow 0$ as $n \rightarrow +\infty$ for every function $f \in L_{\bar{p},\bar{\theta}^{(1)}}^*(\mathbb{I}^m)$, $1 < p_j, \theta_j^{(1)} < +\infty$, $j = 1, \dots, m$, there exists a numerical sequence $\{n_\nu\}$ such that (see [15, Sect. 2])

$$\varepsilon_{n_{\nu+1}} < \frac{1}{2}\varepsilon_{n_\nu}, \quad \varepsilon_{n_{\nu+1}-1} \geq \frac{1}{2}\varepsilon_{n_\nu}, \quad \nu = 1, 2, \dots$$

Let $T_n(f, \bar{x})$ be a trigonometric polynomial of the best approximation of a function $f \in L_{\bar{p},\bar{\theta}^{(1)}}^*(\mathbb{I}^m)$, $1 < p_j, \theta_j^{(1)} < +\infty$, $j = 1, \dots, m$. Consider the series

$$T_{n_1}(f, \bar{x}) + \sum_{\nu=1}^{\infty} (T_{n_{\nu+1}}(f, \bar{x}) - T_{n_\nu}(f, \bar{x})). \tag{2.4}$$

Let us prove the convergence of this series in the norm of the space $L_{\bar{p},\theta^{(2)}}^*(\mathbb{I}^m)$. Suppose that

$$u_\nu(\bar{x}) = |T_{n_{\nu+1}}(f, \bar{x}) - T_{n_\nu}(f, \bar{x})|, \quad \nu = 0, 1, \dots$$

Then

$$\|u_\nu\|_{\bar{p},\bar{\theta}^{(1)}}^* \leq 2\varepsilon_\nu, \quad \nu = 0, 1, \dots,$$

and, by Theorem 1,

$$\|u_\nu\|_{\bar{p},\bar{\tau}}^* << (\ln n_{\nu+1})^{\sum_{j=1}^m (1/\tau_j - 1/\theta_j^{(1)})} \varepsilon_\nu$$

for any $\tau_j \in (0, \theta_j^{(1)})$, $j = 1, \dots, m$. Hence, by Lemma 1, we obtain

$$\begin{aligned}
\left\| \sum_{\nu=k+1}^l (T_{n_{\nu+1}}(f) - T_{n_\nu}(f)) \right\|_{\bar{p},\theta^{(2)}}^* &\leq \left\| \sum_{\nu=k+1}^l u_\nu \right\|_{\bar{p},\theta^{(2)}}^* << \\
&<< \left\{ \sum_{\nu=k+1}^l (\ln n_{\nu+1})^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})} \varepsilon_\nu^{\theta^{(2)}} \right\}^{1/\theta^{(2)}}.
\end{aligned} \tag{2.5}$$

Condition (2.1) implies that

$$\sum_{\nu=1}^{\infty} (\ln n_{\nu+1})^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})} \varepsilon_{n_\nu}^{\theta^{(2)}} < +\infty. \tag{2.6}$$

It follows from (2.5) and (2.6) that series (2.4) converges to a function $g \in L_{\bar{p},\theta^{(2)}}^*(\mathbb{I}^m)$ in the norm. It is easy to see that $g(\bar{x}) = f(\bar{x})$ almost everywhere on \mathbb{I}^m . Hence, $f \in L_{\bar{p},\theta^{(2)}}^*(\mathbb{I}^m)$. Setting $k = 0$ in (2.5), we get

$$\|T_{n_{l+1}}(f)\|_{\bar{p},\theta^{(2)}}^* << \left[\|f\|_{\bar{p},\bar{\theta}^{(1)}}^* + \sum_{\nu=1}^l (\ln n_{\nu+1})^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})} \varepsilon_\nu^{\theta^{(2)}} \right]^{1/\theta^{(2)}} <<$$

$$\ll \left\{ \|f\|_{\bar{p}, \bar{\theta}^{(1)}}^* + \left[\sum_{n=2}^{\infty} \frac{(\ln(n+1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} E_{n, \dots, n}^{\theta^{(2)}}(f)_{\bar{p}, \bar{\theta}^{(1)}} \right]^{1/\theta^{(2)}} \right\}.$$

By tending l to $+\infty$ in this inequality, we obtain

$$\|f\|_{\bar{p}, \theta^{(2)}}^* \ll \left\{ \|f\|_{\bar{p}, \bar{\theta}^{(1)}}^* + \left[\sum_{n=2}^{\infty} \frac{(\ln(n+1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} E_{n, \dots, n}^{\theta^{(2)}}(f)_{\bar{p}, \bar{\theta}^{(1)}} \right]^{1/\theta^{(2)}} \right\}.$$

Thus, inequality (2.2) is proved.

Applying inequality (2.2) to the function $f - T_n(f) \in L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$, it is easy to prove inequality (2.3). The proof of Theorem 2 is complete. \square

Let us prove that condition (2.1) is exact on the classes $E_{\bar{p}, \bar{\theta}^{(1)}}^\lambda$.

Theorem 3. *Let $1 < p_j < \infty$ and $1 < \theta_j^{(2)} < \theta_j^{(1)} < +\infty$ for $j = 1, \dots, m$. The following condition is necessary and sufficient for the inclusion $E_{\bar{p}, \bar{\theta}^{(1)}}^\lambda \subset L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$:*

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} \lambda_n^{\theta^{(2)}} < +\infty. \quad (2.7)$$

P r o o f. The sufficiency of condition (2.7) follows from Theorem 2. Let us prove the necessity. Let $E_{\bar{p}, \bar{\theta}^{(1)}}^\lambda \subset L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$. Assume that condition (2.7) is violated, i.e.,

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} \lambda_n^{\theta^{(2)}} = +\infty. \quad (2.8)$$

We choose a sequence of numbers $\{\nu_k\}$ with the following properties (see [15]):

$$\lambda_{\nu_{k+1}} < \frac{1}{2} \lambda_{\nu_k}, \quad \lambda_{\nu_{k+1}-1} \geq \frac{1}{2} \lambda_{\nu_k}. \quad (2.9)$$

Since the function $(\ln x)^\beta/x$ with $\beta \in \mathbb{R}$ decreases to 0 as $x \rightarrow +\infty$, we have

$$\begin{aligned} \sum_{n=\nu_k+1}^{\nu_{k+1}} \frac{(\ln n)^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} &\leq \sum_{n=\nu_k+1}^{\nu_{k+1}} \frac{(\ln(n - \nu_k + 1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n - \nu_k} \ll \\ &\ll (\ln(\nu_{k+1} - \nu_k + 1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})}. \end{aligned}$$

Thus, (2.8) implies that

$$\sum_{k=1}^{\infty} (\ln(\nu_{k+1} - \nu_k + 1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})} \lambda_{\nu_k}^{\theta^{(2)}} = +\infty. \quad (2.10)$$

Let us consider the function

$$f_0(\bar{x}) = \sum_{k=0}^{\infty} \lambda_{\nu_k} (\ln(\nu_{k+1} - \nu_k + 1))^{-\sum_{j=1}^m 1/\theta_j^{(1)}} \tau_k(\bar{x}),$$

where

$$\tau_k(\bar{x}) = \prod_{j=1}^m \sum_{n_j=\nu_k+1}^{\nu_{k+1}} (n_j - \nu_k)^{\frac{1}{p_j}-1} \sin n_j x_j.$$

It is known that (see [22])

$$\|\tau_k\|_{\bar{p}, \bar{\theta}^{(1)}}^* \asymp (\ln(\nu_{k+1} - \nu_k + 1))^{\sum_{j=1}^m 1/\theta_j^{(1)}}, \quad 1 < p_j, \theta_j^{(1)} < +\infty, \quad j = 1, \dots, m. \quad (2.11)$$

Using this relation and (2.9), we can verify that

$$\|f_0\|_{\bar{p}, \bar{\theta}^{(1)}}^* \leq \sum_{k=0}^{\infty} \lambda_{\nu_k} (\ln(\nu_{k+1} - \nu_k + 1))^{-\sum_{j=1}^m 1/\theta_j^{(1)}} \|\tau_k\|_{\bar{p}, \bar{\theta}^{(1)}}^* \leq C \sum_{k=0}^{\infty} \lambda_{\nu_k} < \infty.$$

Hence, $f_0 \in L_{\bar{p}, \bar{\theta}^{(1)}}^*(\mathbb{I}^m)$, $1 < p_j, \theta_j^{(1)} < \infty$, $j = 1, \dots, m$.

Let a positive integer n satisfy the inequalities $\nu_l \leq n < \nu_{l+1}$. Then, by the best approximation property and according to relation (2.11) and inequality (2.9), we have

$$\begin{aligned} E_n(f_0)_{\bar{p}, \bar{\theta}^{(1)}} &\leq E_{\nu_l}(f_0)_{\bar{p}, \bar{\theta}^{(1)}} \leq \sum_{k=l}^{\infty} \lambda_{\nu_k} (\ln(\nu_{k+1} - \nu_k + 1))^{-\sum_{j=1}^m 1/\theta_j^{(1)}} \|\tau_k\|_{\bar{p}, \bar{\theta}^{(1)}}^* \ll \\ &\ll \sum_{k=l}^{\infty} \lambda_{\nu_k} \ll \lambda_{\nu_l} \ll 2\lambda_{\nu_{l+1}-1} \leq C_0 \lambda_n. \end{aligned}$$

Hence, $f_1 = C_0^{-1} f_0 \in E_{\bar{p}, \bar{\theta}^{(1)}}^\lambda$.

Let us show that $f_1 \notin L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$, $1 < \theta^{(2)} < \infty$. To this end, we consider the function

$$g_0(\bar{x}) = \sum_{k=0}^{\infty} (\ln(\nu_{k+1} - \nu_k + 1))^{\sum_{j=1}^m \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \lambda_{\nu_k}^{\theta^{(2)}-1} \xi_k(\bar{x}),$$

where

$$\xi_k(\bar{x}) = \prod_{j=1}^s \sum_{n_j=\nu_k+1}^{\nu_{k+1}} (n_j - \nu_k)^{\frac{1}{p'_j}-1} \sin n_j x_j, \quad p'_j = \frac{p_j}{p_j - 1}, \quad j = 1, \dots, m.$$

It is clear that (see (2.11))

$$\|\xi_k\|_{\bar{p}', \bar{\theta}}^* \asymp (\ln(\nu_{k+1} - \nu_k + 1))^{\sum_{j=1}^m 1/\theta_j}, \quad 1 < p_j < +\infty, \quad 1 < \theta_j < \infty, \quad j = 1, \dots, m.$$

Further, in view of the orthogonality of the trigonometric system, for any number N , we have

$$\begin{aligned} B_N &\equiv \int_{\mathbb{I}^m} f_1(\bar{x}) \sum_{k=0}^N \lambda_{\nu_k}^{\theta^{(2)}-1} (\ln(\nu_{k+1} - \nu_k + 1))^{\sum_{j=1}^m \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \xi_k(\bar{x}) d\bar{x} = \\ &= C \sum_{k=0}^N [\ln(\nu_{k+1} - \nu_k + 1)]^{-\theta^{(2)} \sum_{j=1}^m 1/\theta_j^{(1)}} \lambda_{\nu_k}^{\theta^{(2)}} \prod_{j=1}^m \sum_{n_j=\nu_k+1}^{\nu_{k+1}} \frac{1}{n_j - \nu_k} \gg \\ &\gg \sum_{k=0}^N [\ln(\nu_{k+1} - \nu_k + 1)]^{\theta^{(2)} \sum_{j=1}^m (1/\theta_j^{(2)} - 1/\theta_j^{(1)})} \lambda_{\nu_k}^{\theta^{(2)}}. \end{aligned} \quad (2.12)$$

Using the integral Hölder inequality, we obtain

$$B_N \ll \|f_1\|_{\bar{p},\theta^{(2)}}^* \left\| \sum_{k=0}^N (\ln(\nu_{k+1} - \nu_k + 1))^{\sum_{j=1}^m \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \lambda_{\nu_k}^{\theta^{(2)}-1} \xi_k \right\|_{\bar{p}',\theta^{(2)'}}^*, \quad (2.13)$$

where

$$\theta^{(2)'} = \frac{\theta^{(2)}}{\theta^{(2)} - 1}.$$

We set $u_k(\bar{x}) = (\ln(\nu_{k+1} - \nu_k + 1))^{\sum_{j=1}^m \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \lambda_{\nu_k}^{\theta^{(2)}-1} |\xi_k(\bar{x})|$. Then (see (2.11))

$$\|u_k\|_{\bar{p}',\frac{\theta^{(1)}}{\theta^{(2)}-1}}^* \ll \lambda_{\nu_k}^{\theta^{(2)}-1} \equiv \beta_k,$$

$$\|u_k\|_{\bar{p}',\bar{\tau}}^* \ll [\ln(\nu_{k+1} - \nu_k + 1)]^{\sum_{j=1}^m (\frac{1}{\tau_j} - \frac{\theta^{(2)}-1}{\theta_j^{(1)}})} \beta_k, \quad k = 0, 1, \dots$$

Thus, all the conditions of Lemma 1 hold for the sequence of functions $\{u_k(\bar{x})\}$. Therefore,

$$\begin{aligned} & \left\| \sum_{k=0}^N (\ln(\nu_{k+1} - \nu_k + 1))^{\sum_{j=1}^m \frac{1-\theta^{(2)}}{\theta_j^{(1)}}} \lambda_{\nu_k}^{\theta^{(2)}-1} \xi_k \right\|_{\bar{p}',\theta^{(2)'}}^* \ll \\ & \ll \left\{ \sum_{k=0}^N (\ln(\nu_{k+1} - \nu_k + 1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta_j^{(2)} - 1/\theta_j^{(1)})} \lambda_{\nu_k}^{\theta^{(2)}} \right\}^{1-1/\theta^{(2)}}. \end{aligned} \quad (2.14)$$

Now, it follows from inequalities (2.12), (2.13), and (2.14) that

$$\left\{ \sum_{k=0}^N (\ln(\nu_{k+1} - \nu_k + 1))^{\theta^{(2)} \sum_{j=1}^m (1/\theta_j^{(2)} - 1/\theta_j^{(1)})} \lambda_{\nu_k}^{\theta^{(2)}} \right\}^{1/\theta^{(2)}} \ll \|f_1\|_{\bar{p},\theta^{(2)}}^*.$$

By (2.10), we find that $f_1 \notin L_{\bar{p},\theta^{(2)}}^*(\mathbb{I}^m)$, $1 < \theta^{(2)} < \theta_j^{(1)} < +\infty$, $j = 1, \dots, m$. This contradicts the inclusion $E_{\bar{p},\bar{\theta}^{(1)}}^\lambda \subset L_{\bar{p},\theta^{(2)}}^*(\mathbb{I}^m)$. The proof of Theorem 3 is complete. \square

Remark 2. The results of L.A. Sherstneva [22] follow from Theorems 2 and 3 in the case $m = 1$.

3. Estimates of best approximations of functions with logarithmic smoothness

Now, let us prove estimates of the value $E_M(F)_{\bar{p},\bar{\theta}^{(2)}}$ for the classes $F = \mathbb{B}_{\bar{p},\bar{\theta}^{(1)}}^{(0,\alpha,\tau)}$ and $F = \mathbb{A}_{\bar{p},\bar{\theta}^{(1)}}^{(0,\gamma,\tau)}$.

Theorem 4. *Let $1 < p_j < \infty$ and $1 \leq \theta_j^{(2)} < \theta_j^{(1)} < \infty$ for $j = 1, \dots, m$, and let $1 \leq \tau \leq \infty$. If $\alpha > \sum_{j=1}^m (1/\theta_j^{(2)} - 1/\theta_j^{(1)})$, then $B_{\bar{p},\bar{\theta}^{(1)}}^{(0,\alpha,\tau)} \subset L_{\bar{p},\theta^{(2)}}^*(\mathbb{I}^m)$ and*

$$\|f\|_{\bar{p},\theta^{(2)}}^* \ll \|f\|_{B_{\bar{p},\bar{\theta}^{(1)}}^{(0,\alpha,\tau)}}.$$

P r o o f. Let $f \in B_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)}$. Then, by the definition of the class, this function can be represented in the form of the series

$$\sum_{\nu=0}^{\infty} Q_{2^{2\nu}}(f, \bar{x}), \quad Q_{2^{2\nu}}(f, \bar{x}) \in \mathfrak{F}_{\square_{2^{2n}}},$$

in the sense of convergence in the quasi-norm of the space $L_{\bar{p}, \bar{\theta}^{(1)}}^*(\mathbb{I}^m)$ and

$$\left[\sum_{\nu=0}^{\infty} (2^{\nu\alpha} \|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}}^*)^\tau \right]^{1/\tau} < +\infty.$$

If $\theta^{(2)} < \tau < \infty$, then, using the Hölder inequality and taking into account that $\alpha > \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})$, we obtain

$$\begin{aligned} & \left\{ \sum_{\nu=0}^{\infty} 2^{\nu \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\theta^{(2)}} (\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^*)^{\theta^{(2)}} \right\}^{1/\theta^{(2)}} \leq \\ & \leq \left\{ \sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha} (\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^*)^\tau \right\}^{1/\tau} \left\{ \sum_{\nu=0}^{\infty} 2^{\nu\theta^{(2)}\beta' (\sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - \alpha)} \right\}^{\frac{1}{\theta^{(2)}\beta'}} \leq \\ & \leq C \left\{ \sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha} (\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^*)^\tau \right\}^{1/\tau}, \end{aligned} \quad (3.1)$$

where

$$\beta = \frac{\tau}{\theta^{(2)}}, \quad \beta' = \frac{\beta}{\beta - 1}.$$

If $\tau = \infty$, then

$$\begin{aligned} & \left\{ \sum_{\nu=0}^{\infty} 2^{\nu \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\theta^{(2)}} (\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^*)^{\theta^{(2)}} \right\}^{1/\theta^{(2)}} \leq \\ & \leq \sup_{\nu \in \mathbb{N}_0} 2^{\nu\alpha} \|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \left\{ \sum_{\nu=0}^{\infty} 2^{\nu\theta^{(2)} (\sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - \alpha)} \right\}^{1/\theta^{(2)}}. \end{aligned} \quad (3.2)$$

If $\tau \leq \theta^{(2)}$, then, using the Jensen inequality (see [17, p. 125]), we obtain

$$\left\{ \sum_{\nu=0}^{\infty} 2^{\nu \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\theta^{(2)}} (\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^*)^{\theta^{(2)}} \right\}^{1/\theta^{(2)}} \leq \left\{ \sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha} (\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^*)^\tau \right\}^{1/\tau}. \quad (3.3)$$

Thus, (3.1)–(3.3) imply that the series

$$\sum_{\nu=0}^{\infty} 2^{\nu \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\theta^{(2)}} (\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^*)^{\theta^{(2)}} \quad (3.4)$$

is convergent for every function $f \in B_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)}$.

Taking into account the monotonicity of the best approximation and the properties of the norm, it is easy to verify that

$$\begin{aligned}
 \sum_{n=2}^{\infty} \frac{(\ln n)^{\sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1}}{n} E_{n, \dots, n}^{\theta^{(2)}}(f)_{\bar{p}, \bar{\theta}^{(1)}} &<< \sum_{\nu=0}^{\infty} 2^{\nu} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})^{\theta^{(2)}} E_{2^{2\nu}, \dots, 2^{2\nu}}^{\theta^{(2)}}(f)_{\bar{p}, \bar{\theta}^{(1)}} << \\
 &<< \sum_{\nu=0}^{\infty} 2^{\nu} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})^{\theta^{(2)}} \left(\left\| \sum_{l=\nu}^{\infty} Q_{2^{2l}}(f) \right\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\theta^{(2)}} << \\
 &<< \sum_{\nu=0}^{\infty} 2^{\nu} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})^{\theta^{(2)}} \left(\sum_{l=\nu}^{\infty} \|Q_{2^{2l}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\theta^{(2)}}.
 \end{aligned} \tag{3.5}$$

Since $\theta^{(2)} < \theta_j^{(1)}$, $j = 1, \dots, m$, we have

$$\sum_{\nu=0}^n 2^{\nu} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})^{\theta^{(2)}} << 2^n \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})^{\theta^{(2)}}, \quad n \in \mathbb{N}_0.$$

Therefore, according to [14, Lemma 2.2], we find from (3.5) that

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\theta^{(2)} - 1}}{n} E_{n, \dots, n}^{\theta^{(2)}}(f)_{\bar{p}, \bar{\theta}^{(1)}} << \sum_{\nu=0}^{\infty} 2^{\nu} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})^{\theta^{(2)}} \left(\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\theta^{(2)}}. \tag{3.6}$$

Since the series (3.4) converges, it follows from (3.6) that

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\theta^{(2)} - 1}}{n} E_{n, \dots, n}^{\theta^{(2)}}(f)_{\bar{p}, \bar{\theta}^{(1)}} < \infty.$$

Hence, by Theorem 3, we have $f \in L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$.

Let us estimate the quasi-norm $\|f\|_{\bar{p}, \bar{\theta}^{(1)}}^*$. By the quasi-norm property and the Hölder inequality, we obtain

$$\begin{aligned}
 \|f\|_{\bar{p}, \bar{\theta}^{(1)}}^* &<< \sum_{\nu=0}^{\infty} \|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* << \\
 &<< \left(\sum_{\nu=0}^{\infty} 2^{\nu} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})^{\theta^{(2)}} \left(\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\theta^{(2)}} \right)^{1/\theta^{(2)}}.
 \end{aligned} \tag{3.7}$$

Therefore, according to relations (2.2), (3.6), and (3.7), we have

$$\|f\|_{\bar{p}, \bar{\theta}^{(1)}}^* << \left\{ \sum_{\nu=0}^{\infty} 2^{\nu} \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})^{\theta^{(2)}} \left(\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\theta^{(2)}} \right\}^{1/\theta^{(2)}}. \tag{3.8}$$

Taking into account (3.1)–(3.3) and (3.8), we obtain

$$\|f\|_{\bar{p}, \theta^{(2)}}^* << \left\{ \sum_{\nu=0}^{\infty} 2^{\nu\tau(\gamma+1/\tau)} \left(\|Q_{2^{2\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\tau} \right\}^{1/\tau} \tag{3.9}$$

for every function $f \in B_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)}$. The proof of Theorem 4 is complete. \square

Theorem 5. *Let $1 < p_j < \infty$ and $1 \leq \theta^{(2)} < \theta_j^{(1)} < \infty$ for $j = 1, \dots, m$, and let $1 \leq \tau \leq \infty$. If $\alpha > \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})$, then*

$$E_M(\mathbb{B}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)})_{\bar{p}, \bar{\theta}^{(2)}} \asymp (\log(M+1))^{-\left(\alpha - \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\right)}, \quad M \in \mathbb{N}.$$

Proof. Let $f \in \mathbb{B}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)}$. We have $\alpha > \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})$; therefore, $f \in L_{\bar{p}, \bar{\theta}^{(2)}}^*(\mathbb{I}^m)$ by Theorem 4. Take a positive integer l such that $2^{2^l} \leq M < 2^{2^{l+1}}$. Then, using the best approximation property and inequality (3.9), we have

$$E_M(f)_{\bar{p}, \bar{\theta}^{(2)}} \leq E_{2^{2^l}}(f)_{\bar{p}, \bar{\theta}^{(2)}} << \left\{ \sum_{\nu=l}^{\infty} 2^{\nu \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) \theta^{(2)}} \left(\|Q_{2^{2^\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\theta^{(2)}} \right\}^{1/\tau}. \quad (3.10)$$

If $\theta^{(2)} < \tau$, then by the Hölder inequality and in view of the fact that $\alpha > \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})$, (3.10) implies that (see formula (3.1))

$$E_M(f)_{\bar{p}, \bar{\theta}^{(2)}} \leq \left\{ \sum_{\nu=0}^{\infty} 2^{\nu \tau \alpha} \left(\|Q_{2^{2^\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^\tau \right\}^{1/\tau} \times \left\{ \sum_{\nu=l}^{\infty} 2^{\nu \theta^{(2)} \beta' \left(\sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - \alpha \right)} \right\}^{\frac{1}{\theta^{(2)} \beta'}} << 2^{-l(\alpha - \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}))} \quad (3.11)$$

for every function $f \in \mathbb{B}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)}$ in the case $\theta^{(2)} < \tau$.

If $\tau \leq \theta^{(2)}$, then, arguing as in the proof of formula (3.3), by means of the Jensen inequality, we find from (3.10) that

$$E_M(f)_{\bar{p}, \bar{\theta}^{(2)}} \leq \left\{ \sum_{\nu=0}^{\infty} 2^{\nu \tau \alpha} \left(\|Q_{2^{2^\nu}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^\tau \right\}^{1/\tau} 2^{-l(\alpha - \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}))}. \quad (3.12)$$

Now, taking into account that $2^{2^l} \leq M < 2^{2^{l+1}}$, by formulas (3.11) and (3.12), we obtain

$$E_M(f)_{\bar{p}, \bar{\theta}^{(2)}} << (\log(M+1))^{-\left(\alpha - \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\right)}$$

for every function $f \in \mathbb{B}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)}$. Thus, the upper estimates are proved.

Let us prove the lower estimates. Consider the function

$$f_2(\bar{x}) = 2^{-(n+1)(\alpha + \sum_{j=1}^m 1/\theta_j^{(1)})} \sum_{s=2^{n+1}+1}^{2^{n+2}} \sum_{\bar{k} \in \square_{2^s} \setminus \square_{2^{s-1}}} \prod_{j=1}^m (k_j - 2^{s-1} + 1)^{\frac{1}{p_j} - 1} e^{i\langle \bar{k}, \bar{x} \rangle},$$

where $\bar{x} \in \mathbb{I}^m$ and $n \in \mathbb{N}_0$. It is well known that

$$\left\| \sum_{s=2^{n+1}}^{2^{n+2}} \sigma_s(f_2) \right\|_{\bar{p}, \bar{\theta}^{(1)}}^* = 2^{-(n+1)(\alpha + \sum_{j=1}^m 1/\theta_j^{(1)})} \left\| \sum_{s=2^{n+1}+1}^{2^{n+2}} \sum_{\bar{k} \in \square_{2^s} \setminus \square_{2^{s-1}}} \prod_{j=1}^m (k_j - 2^{s-1} + 1)^{\frac{1}{p_j} - 1} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}^{(1)}}^* <<$$

$$\ll 2^{-(n+1)(\alpha + \sum_{j=1}^m 1/\theta_j^{(1)})} (\log(2^{2^{n+2}} - 2^{2^{n+1}}))_{j=1}^{\sum_{j=1}^m 1/\theta_j^{(1)}} \ll 2^{-(n+1)\alpha}.$$

Thus,

$$\left\{ \sum_{\nu=0}^{\infty} 2^{\nu\tau\alpha} \left(\left\| \sum_{s=2^\nu}^{2^{\nu+1}} \sigma_s(f_2) \right\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^\tau \right\}^{1/\tau} = 2^{(n+1)\alpha} \left\| \sum_{s=2^{n+1}}^{2^{n+2}} \sigma_s(f_2) \right\|_{\bar{p}, \bar{\theta}^{(1)}}^* \leq C_1.$$

Hence, $C_1^{-1} f_2 \in \mathbb{B}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)}$ for $1 < \theta^{(2)} < \infty$ and $1 \leq \tau < \infty$. Next, by the definition of the best approximation and the estimate

$$\left\| \sum_{s=2^{n+1}+1}^{2^{n+2}} \sum_{\bar{k} \in \square_{2^s} \setminus \square_{2^{s-1}}} \prod_{j=1}^m (k_j - 2^{s-1} + 1)^{\frac{1}{p_j} - 1} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \theta^{(2)}}^* \gg 2^{n \frac{m}{\theta^{(2)}}},$$

we have

$$\begin{aligned} E_{2^{2n}}(f_2)_{\bar{p}, \theta^{(2)}} &= C_1^{-1} \|f_2\|_{\bar{p}, \theta^{(2)}}^* = \\ &= C_1^{-1} 2^{-(n+1)(\alpha + \sum_{j=1}^m 1/\theta_j^{(1)})} \left\| \sum_{s=2^{n+1}+1}^{2^{n+2}} \sum_{\bar{k} \in \square_{2^s} \setminus \square_{2^{s-1}}} \prod_{j=1}^m (k_j - 2^{s-1} + 1)^{\frac{1}{p_j} - 1} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \theta^{(2)}}^* \gg \\ &\gg 2^{-(n+1)(\alpha - \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}))}. \end{aligned}$$

Taking into account that $2^{2^n} \leq M < 2^{2^{n+1}}$, we obtain

$$E_M(f_2)_{\bar{p}, \theta^{(2)}} \gg (\log(M+1))^{\left(\alpha - \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)})\right)}$$

for $1 \leq \theta^{(2)} < \infty$ and $1 \leq \tau \leq \infty$. Thus, the proof of Theorem 5 is complete. \square

Theorem 6. *Let $1 < p_j < \infty$ and $1 \leq \theta^{(2)} < \theta_j^{(1)} < \infty$ for $j = 1, \dots, m$, and let $1 \leq \tau \leq \infty$. If $\gamma > \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}) - 1/\tau$, then*

$$E_M(\mathbb{A}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \gamma, \tau)})_{\bar{p}, \bar{\theta}^{(2)}} \asymp (\log(M+1))^{-(\gamma + 1/\tau - \sum_{j=1}^m (1/\theta^{(2)} - 1/\theta_j^{(1)}))}.$$

P r o o f. Since $\mathbb{A}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \gamma, \tau)}$ and $\mathbb{B}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \gamma + 1/\tau, \tau)}$ coincide, the statement of Theorem 6 follows from Theorem 5. \square

4. Conclusion

The best approximations of functions of the classes $\mathbb{B}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \alpha, \tau)}$ and $\mathbb{A}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \gamma, \tau)}$ in the space $L_{\bar{p}, \theta^{(2)}}^*(\mathbb{I}^m)$ have logarithmic order.

Note that estimates of the quantities $E_M(\mathbb{B}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \gamma, \tau)})_{\bar{p}, \bar{\theta}^{(2)}}$ and $E_M(\mathbb{A}_{\bar{p}, \bar{\theta}^{(1)}}^{(0, \gamma, \tau)})_{\bar{p}, \bar{\theta}^{(2)}}$ are unknown in the case $\theta_j^{(1)} = \theta^{(2)}$, $j = 1, \dots, m$.

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