On generalization of theorems of Pestryakov

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Abstract

In 1987 A.V. Pestryakov proved a series of theorems for cardinal functions of the space $B_\alpha(X)$ of all real-valued functions of Baire class α ($\alpha > 0$), and he conjectured that most of these theorems are true for spaces containing all finite linear combinations of characteristic functions of zero-sets in X. In this paper we investigate for which theorems of Pestryakov generalizations are valid. Also we prove some additional propositions for function spaces applying the theory of selection principles.

Keywords: space of Baire functions, density, tightness, Lindelöf number, spread, G_δ-modification, selection principles, cardinal functions
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1. Introduction

In this paper by a space we shall always mean a Tychonoff space. Let $C_p(X)$ denote the space of continuous real-valued functions $C(X)$ on a space X with the topology of pointwise convergence. Let $B_0(X) = C(X)$ and inductively define $B_\alpha(X)$ for each ordinal $\alpha \leq \omega_1$ to be the space of pointwise limits of sequences of functions in $\bigcup\limits_{\beta<\alpha} B_\beta(X)$. So $B_\alpha(X)$ a set of all functions of Baire class α, defined on a Tychonoff space X, provided with the pointwise convergence topology.

The family of Baire sets of a space X is the smallest family of sets containing the zero sets of continuous real-valued functions (i.e. of the form $Z(f) = \{x \in X : f(x) = 0\}$), and closed under countable unions and countable intersections. The Baire sets of X of multiplicative class 0, denoted

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$Z(X)$, are the zero-sets of continuous real-valued functions. The sets of additive class 0, denoted $CZ(X)$, are the complements of the sets in $Z(X)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

The following prototype of many classical properties is called "$\mathcal{A}$ choose $\mathcal{B}$" in [22].

$\binom{\mathcal{A}}{\mathcal{B}}$: For each $A \in \mathcal{A}$ there exists $B \subset A$ such that $B \in \mathcal{B}$.

Clearly that $S_1(\mathcal{A}, \mathcal{B})$ implies $\binom{\mathcal{A}}{\mathcal{B}}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

A cover $\mathcal{U}$ of a space X is:

- an *ω-cover* if every finite subset of X is contained in a member of $\mathcal{U}$.
- a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Note that every γ-cover contains a countably γ-cover.

For a topological space X we denote:

- Ω — the family of all open ω-covers of X;
- Γ — the family of all open γ-covers of X;
- Z_Ω — the family of all countable ω-covers of X by zero-sets in X;
- Z_Γ — the family of all countable γ-covers of X by zero-sets in X.

Let (X, τ) be a topological space. The *Baire topology* τ_b on X is the topology on the underlying set X having for a basis the family of all zero-sets of X. Since the countable intersection of zero-sets is also a zero-set, it follows that the space X endowed with the Baire topology and denoted by $X_{\aleph_0}$ is a P-space. Recall that a topological space is called a *P-space* if the intersection of a countable family of open sets is open. Let us recall also that the family of G_δ-sets in X forms a base of the topology τ_δ on X, and the space X with the topology τ_δ is called the *P-modification of* X and is denoted by PX or X_δ (see [4, 6, 9]). Clearly, PX is a P-space and τ_δ is finer than the Baire topology τ_b. If X is a Tychonoff space, then $X_{\aleph_0} = PX$ and $X_{\aleph_0}$ is a Tychonoff space. Note that the topology $X_{\aleph_0}$ coincides with the weak topology generated by $B_\alpha(X)$ for each $\alpha > 0$ ([16]).

Further, we consider the spaces $C(X_{\aleph_0})$ and $B_\alpha(X)$ with the topology of pointwise convergence.

We will use the standard notation for usual cardinal invariants, so c, χ,

$\pi\chi$, ψ, w, πw, ψw, nw, d, t, l, s, denote cellularity, character, π-character, pseudocharacter, weight, π-weight, pseudoweight, network weight, density, tightness, the Lindelöf number, spread, respectively, see [7, 8]. For a cardinal function ϵ denoted by $h\epsilon(Y) = \{\epsilon(Z) : Z \subseteq Y\}$, $i\epsilon(Y) = \min\{\epsilon(Z) : Y$ admits a one-to-one continuous mapping onto a space $Z\}$ and $\epsilon^*(Y) = \sup\{\epsilon(Y^n) : n \in \mathbb{N}\}$.

Since $B_\alpha(X)$ is dense in $\mathbb{R}^X$ $(0 \leq \alpha \leq \omega_1)$, $c(B_\alpha(X)) = \omega_0$, $\pi\chi(B_\alpha(X)) = \chi(B_\alpha(X)) = \pi w(B_\alpha(X)) = w(B_\alpha(X)) = |X|$.

In 1987 A.V. Pestryakov proved the following theorems (P1-P9) for a space $B_\alpha(X)$ $(0 < \alpha \leq \omega_1)$([17, 18]).

Theorem 1.1. (**P1**) *$t(B_\alpha(X)) = l^*(X_{\aleph_0})$.*

Theorem 1.2. (**P2**) *$hd(B_\alpha(X)) = hl^*(X_{\aleph_0})$.*

Theorem 1.3. (**P3**) *$hl(B_\alpha(X)) = hd^*(X_{\aleph_0})$.*

Theorem 1.4. (**P4**) *$s(B_\alpha(X)) = s^*(X_{\aleph_0})$.*

Theorem 1.5. (**P5**) *The followings statements are equivalent.*

1. *$B_\alpha(X)$ is Fréchet-Urysohn;*
2. *$B_\alpha(X)$ is sequential;*
3. *$B_\alpha(X)$ is a k-space;*
4. *$B_\alpha(X)$ is a ω_1-k-space;*
5. *$B_\alpha(X)$ has countable tightness;*
6. *$X_{\aleph_0}$ satisfies $\binom{\Omega}{\Gamma}$;*
7. *$X_{\aleph_0}$ is Lindelöf.*

Theorem 1.6. (**P6**) *$d(B_\alpha(X)) = iw(X)$.*

Theorem 1.7. (**P7**) *For $0 < \alpha \leq \omega_1$, $\psi(B_\alpha(X)) = \psi w(B_\alpha(X)) = i\chi(B_\alpha(X)) = iw(B_\alpha(X)) = d(X_{\aleph_0})$.*

Theorem 1.8. (**P8**) *$nw(B_\alpha(X)) = nw(X_{\aleph_0})$.*

Theorem 1.9. (**P9**) *$l(B_\alpha(X)) \geq t^*(X_{\aleph_0})$.*

Put $L(X) = \{n_1 \cdot f_{Z_1} + ... + n_k \cdot f_{Z_k} : f_{Z_i}$ is the characteristic function of Z_i, $Z_i \in Z(X)$, $k, i, n_k \in \mathbb{N}\}$. For a space X, define $\mathbb{B} = \{Y : L(X) \subseteq$

$Y \subseteq C(X_{\aleph_0})\}$. For example, $B_\alpha(X) \in \mathbb{B}$ for each $\alpha > 0$, $C(X_{\aleph_0}) \in \mathbb{B}$ and $[C_p(X)]'_{\omega_0} = \bigcup\{cl_{\mathbb{R}^X} B : B \subset C_p(X), |B| \leq \omega_0\} \in \mathbb{B}$.

Pestryakov conjectured that most of theorems (P1-P9) are true for any $\mathbb{B}(X) \in \mathbb{B}$. In this paper we check for which Pestryakov's theorems generalizations are valid. Also we prove some additional propositions for spaces in the class $\mathbb{B}$.

2. Tightness

The following result is well known [2].

Theorem 2.1. *(Arhangel'skii-Pytkeev)* $t(C_p(X)) = l^*(X)$.

We prove an analogue of Theorem P1 and the Arhangel'skii-Pytkeev Theorem for a space $\mathbb{B}(X) \in \mathbb{B}$.

Theorem 2.2. $t(\mathbb{B}(X)) = l^*(X_{\aleph_0})$.

Proof. Since $t(C_p(Y)) = l^*(Y)$ for a space Y (the Arhangel'skii-Pytkeev Theorem) and $\mathbb{B}(X) \subseteq C(X_{\aleph_0})$, then $t(\mathbb{B}(X)) \leq l^*(X_{\aleph_0})$.

Fix $n \in \mathbb{N}$. Assume that η is an open cover of $X^n_{\aleph_0}$. Clearly that, whenever $V \in \eta$ and $x = (x_1, ..., x_n) \in V$ there exists $W_x = \prod\limits_{i=1}^{n}\{V_{x_i} : V_{x_i}$ is an open in $X_{\aleph_0}$ and $x_i \in V_{x_i}\}$ such that $x \in W_x \subset V$. Then we can consider the cover $\mu = \{W_x : x \in X^n\}$ of $X^n_{\aleph_0}$ such that μ is a refinement of η.

For each $x = (x_1, ..., x_n) \in X^n$ denote $\widetilde{x} = \{x_1, ..., x_n\} \subset X$.

Let $m \in \mathbb{N}$, $z = (z_1, ..., z_m) \in X^m$.

Fix $F(z_i) \in Z(X)$ such that $z_i \in F(z_i)$ $(1 \leq i \leq m)$ and if $\widetilde{x} \subset \widetilde{z}$ (i.e. $x = (z_{i_1}, ..., z_{i_n})$) then $F(z_{i_k}) \subset V_{z_{i_k}}$ $(k = 1, ..., n)$.

Let f_z be the characteristic function of $\bigcup\{F(z_i) : 1 \leq i \leq m\}$. The symbol $\mathbf{1}$ stands for the constant function to 1. Note that $F = \{f_z : z \in \bigcup\{X^m : 1 \leq m < \omega\}\} \subset \mathbb{B}(X)$ and $\mathbf{1} \in \overline{F}$. Then there exists $F' \subset F$ such that $\mathbf{1} \in \overline{F'}$, $|F'| \leq \tau = t(\mathbb{B}(X))$. Then there is $A \subset X^m$ such that $|A| \leq \tau$ and $F' = \{f_z \in F : z \in A\}$. Let $Y = \{y \in X^n : \widetilde{y} \subset \widetilde{z}, z \in A\}$. Clearly that $|Y| \leq \tau$.

We claim that $\{W_y : y \in Y\} \subset \mu$ is a cover of X^n. Let $x \in X^n$. Then $W = \{f \in \mathbb{B}(X) : f(t) > 0$ for all $t \in \widetilde{x}\}$ is a neighborhood of $\mathbf{1}$. There is an $f_z \in F' \bigcap W$. We have $\widetilde{x} \subset \bigcup\limits_{i=1}^{m} F(z_i)$. Let $x_k \in F(z_{i_k})$ for $1 \leq k \leq n$,

$y = (z_{i_1}, ..., z_{i_n})$. Then $y \in Y$ and $x \in \prod\{F(z_{i_k}) : 1 \leq k \leq n\} \subset \prod\{V_{y_k} : 1 \leq k \leq n\} = W_y$.

□

Corollary 2.3. $t(B_\alpha(X)) = t([C(X)]'_{\omega_0}) = t(C_p(X_{\aleph_0})) = l^*(X_{\aleph_0})$ $(\alpha > 0)$.

Recall that a space is said to be scattered if every nonempty subspace of it has an isolated point.

Note that if X is scattered, then $l(X) = l(X_{\aleph_0})$ [9]. Then we have the following result.

Corollary 2.4. If X is scattered, then $t(\mathbb{B}(X)) = l^*(X)$.

Note that $l(X_{\aleph_0}) = \omega_0$ implies that $l(X^n_{\aleph_0}) = \omega_0$ [10].

Corollary 2.5. $t(\mathbb{B}(X)) = t(C_p(X_{\aleph_0})) = \omega_0$ if and only if $X_{\aleph_0}$ is Lindelöf.

3. Hereditary density

The following result is well known in C_p-theory [2] (for $hd(C_p(X)) = \omega_0$ see [23]).

Theorem 3.1. $hd(C_p(X)) = hl^*(X)$.

We prove an analogue of this theorem and Theorem P2 for a space in class $\mathbb{B}$.

Theorem 3.2. $hd(\mathbb{B}(X)) = hl^*(X_{\aleph_0})$.

Proof. Since $hd(C_p(Y)) = hl^*(Y)$ for a space Y and $\mathbb{B}(X) \subseteq C_p(X_{\aleph_0})$, then $hd(\mathbb{B}(X)) \leq hl^*(X_{\aleph_0})$.

Let Y be a subspace of $X^n_{\aleph_0}$, we consider the family $\mu = \{W_y : y \in Y\}$ of sets in Y where $W_y = \prod\limits_{i=1}^{n}\{Z_{y_i} : Z_{y_i}$ is a zero-set in X, $y_i \in Z_{y_i}\}$ such that $Y \subseteq \bigcup \mu$ and $|\mu| = l(Y)$. Then $\gamma = \{W_y \cap Y : W_y \in \mu\}$ is an open cover of Y.

Suppose $A = \{f_{Z_y} : Z_y = \bigcup\limits_{i=1}^{n} Z_{y_i}, W_y \cap Y \in \gamma$ and f_{Z_y} is the characteristic function of $Z_y\}$. Note that $A \subset \mathbb{B}(X)$. Let $D \subset A$ such that $|D| < |A| \leq |\gamma|$. Then the family $\eta = \{W_y \cap Y : f_{Z_y} \in D\}$ is not a cover of Y. Hence, there are $W_y \in \mu$ and $y \in Y$ such that $W_y \cap Y \notin \eta$ and $y = (y_1, ..., y_n) \in W_y \cap (Y \setminus \bigcup \eta)$. Then $\{h : |h(y_i) - f_{Z_y}(y_i)| < 1, 1 \leq i \leq n\} \cap D = \emptyset$. It follows that D is not dense in A, and $d(A) \geq l(Y)$.

□

Corollary 3.3. $hd(B_\alpha(X)) = hd([C(X)]'_{\omega_0}) = hd(C_p(X_{\aleph_0})) = hl^*(X_{\aleph_0})$ $(\alpha > 0)$.

Note that if X is scattered, then $hl(X) = hl(X_{\aleph_0})$ [9]. Then we have the following result.

Corollary 3.4. If X is scattered, then $hd(\mathbb{B}(X)) = hd(C_p(X_{\aleph_0})) = hl^*(X)$.

4. Hereditary Lindelöf degree

The following result is well known in C_p-theory [2, 23]

Theorem 4.1. *If $ind(X) = 0$, then $hl(C_p(X)) = hd^*(X)$.*

Note that $ind(X_{\aleph_0}) = 0$ for any space X. Then we have the following theorem.

Theorem 4.2. *$hl(\mathbb{B}(X)) = hd^*(X_{\aleph_0})$.*

Proof. Since $ind(X_{\aleph_0}) = 0$ and $\mathbb{B}(X) \subseteq C_p(X_{\aleph_0})$, then, by Theorem 4.1, $hl(\mathbb{B}(X)) \leq hd^*(X_{\aleph_0})$.

First we prove an auxiliary proposition.

Proposition 4.3. *If $Y \subset X^n$ is such that whenever $y = (y_1, ..., y_n) \in Y$ and $y_i \neq y_j$ for $i \neq j$, then $hl(\mathbb{B}(X)) \geq d(Y_{\aleph_0})$.*

Proof. For each $y = (y_1, ..., y_n) \in Y$ we fix a local base $\beta(y)$ at y in $X^n_{\aleph_0}$ the following of the form $\beta(y) = \{V = \prod\limits_{i=1}^{n} V_i : y_i \in V_i \in Z(X), V_i \cap V_j = \emptyset$ for $i \neq j\}$. Let

$$f_V(x) = \begin{cases} i, & if \;\; x \in V_i, \\ 0, & if \; x \in X \setminus \bigcup\limits_{i=1}^{n} V_i, \end{cases}$$

and $A = \{f_V : V \in \beta(y), y \in Y\}$. Clearly that $A \subset \mathbb{B}(X)$. Let $U(y) = \{f \in \mathbb{B}(X) : |f(y_i) - i| < 1\}$ and $\gamma = \{U(y) : y \in Y\}$. Then γ is a cover of A. There is a subcover $\gamma' \subseteq \gamma$ such that $|\gamma'| = l(A) \leq hl(\mathbb{B}(X))$. Consider $S = \{y \in Y : U(y) \in \gamma'\}$. Note that $|S| \leq |\gamma'|$. We claim that S is dense in $Y_{\aleph_0}$. Fix $z = (z_1, ..., z_n) \in Y$, $V \in \beta(z)$. Then $f_V \in A$ and there is $U(y) \in \gamma'$ such that $f_V \in U(y)$. It follows that $y \in V \cap S$ and S is dense in $Y_{\aleph_0}$. ◻

Now, by indiction on n, we claim that $hl(\mathbb{B}(X)) \geq hd(X^n_{\aleph_0})$.

For $n = 1$ by Proposition 4.3.

Suppose that $hl(\mathbb{B}(X)) \geq hd(X^k_{\aleph_0})$ for $k < n$. Note that $X^n_{ij} = \{(x_1, ..., x_n) : x_i = x_j\}$ for $i \neq j$ is homeomorphic to the space X^{n-1}. Set $D = \bigcup\{X^n_{ij} : 1 \leq i \neq j \leq n\}$. Let $Z \subseteq X^n$. Then, by Proposition 4.3, $d(Z_{\aleph_0} \setminus D) \leq hl(\mathbb{B}(X))$ and, by the inductive hypothesis, $d(Z_{\aleph_0}) \leq d(Z_{\aleph_0} \setminus D) + \sum\limits_{1 \leq i \neq j \leq n} d(Z_{\aleph_0} \cap X^n_{ij}) \leq hl(\mathbb{B}(X))$.

□

Corollary 4.4. $hl(B_\alpha(X)) = hl([C(X)]'_{\omega_0}) = hl(C_p(X_{\aleph_0})) = hd^*(X_{\aleph_0})$ $(\alpha > 0)$.

5. Spread

The well-known the following result in C_p-theory [2].

Theorem 5.1. *If $ind(X) = 0$, then $s(C_p(X)) = s^*(X)$.*

Since $ind(X_{\aleph_0}) = 0$ for any space X, we have the following theorem for the spread $s(\mathbb{B}(X))$ of a space $\mathbb{B}(X)$ from the class $\mathbb{B}$.

Theorem 5.2. $s(\mathbb{B}(X)) = s^*(X_{\aleph_0})$.

Proof. Since $ind(X_{\aleph_0}) = 0$ and $\mathbb{B}(X) \subseteq C_p(X_{\aleph_0})$, then, by Theorem 5.1, $s(\mathbb{B}(X)) \leq s^*(X_{\aleph_0})$.

First we prove an auxiliary proposition.

Proposition 5.3. *Assume that $Y \subset X^n$ is such that whenever $y = (y_1, ..., y_n) \in Y$ and $y_i \neq y_j$ for $i \neq j$. If Y is discrete in $X^n_{\aleph_0}$, then $|Y| \leq s(\mathbb{B}(X))$.*

Proof. For each $y = (y_1, ..., y_n) \in Y$ we fix $V = \prod\limits_{i=1}^{n} V_i$ such that $V \cap Y = \{y\}$, $y_i \in V_i \in Z(X), V_i \cap V_j = \emptyset$ for $i \neq j$. Let

$$f_V(x) = \begin{cases} i, & if \ \ x \in V_i, \\ 0, & if \ x \in X \setminus \bigcup\limits_{i=1}^{n} V_i, \end{cases}$$

and $A = \{f_V : y \in Y\}$. Clearly that $A \subset \mathbb{B}(X)$ and $|A| = |Y|$. We claim that A is discrete. If $f_U \in (f_V, y, 1) \cap A$, then $|f_U(y_i) - i| < 1$ for $1 \leq i \leq n$. Hence, $y = (y_1, ..., y_n) \in U$. It follows that $U = V$ and $f_U = f_V$. □

Now, by indiction on n, we claim that $s(\mathbb{B}(X)) \geq s(X^n_{\aleph_0})$.

For $n = 1$ by Proposition 5.3.

Suppose that $s(\mathbb{B}(X)) \geq s(X^k_{\aleph_0})$ for $k < n$. Note that $X^n = \widetilde{X}^n \cup D$, where $D = \bigcup\{X^n_{ij} : 1 \leq i \neq j \leq n\}$, $X^n_{ij} = \{(x_1, ..., x_n) : x_i = x_j\}$, $\widetilde{X}^n = X^n \setminus D$. Let Y be discrete in $X^n_{\aleph_0}$. Put $Y_1 = Y \cap \widetilde{X}^n$, $Y_2 = Y \cap D$, then $Y = Y_1 \cap Y_2$. By Proposition 5.3, $|Y_1| \leq s(\mathbb{B}(X))$. If $i \neq j$, then X^n_{ij} is homeomorphic to the space X^{n-1} and, hence, $|Y_2| \leq s(X^{n-1}_{\aleph_0})$. By the inductive hypothesis, $|Y_2| \leq s(\mathbb{B}(X))$. Note that $|Y| = |Y_1| + |Y_2|$. It follows that $s(\mathbb{B}(X)) \geq s^*(X_{\aleph_0})$.

□

Corollary 5.4. $s(B_\alpha(X)) = s([C(X)]'_{\omega_0}) = s(C_p(X_{\aleph_0})) = s^*(X_{\aleph_0})$ $(\alpha > 0)$.

6. Modification of Theorem P5.

Let κ be an infinite cardinal. A space C is said to be *κ-initially compact* (see [12]) if every open cover $\mathcal{V}$ of C with $|\mathcal{V}| \leq \kappa$ has a finite subcover. A space E is a *κ-k* (*κ-k-space*), if whenever the subspace A is non-closed in E, there is a κ-initially compact subspace C of E with $C \cap A$ non-closed in C ([15]).

In 1982, Pytkeev [19] and Gerlits [15] independently proved the following result.

Theorem 6.1. *(Pytkeev-Gerlits) For a space X, the following are equivalent:*

1. *$C_p(X)$ is Fréchet-Urysohn;*
2. *$C_p(X)$ is sequential;*
3. *$C_p(X)$ is a k-space.*

Gerlits and Nagy defined three properties [14, 15]:

• the property (γ): for every open ω-cover $\mathcal{V}$ of X there exists a sequence $G_n \in \mathcal{V}$ such that $\{G_n : n \in \omega\}$ is a γ-cover of X ($\binom{\Omega}{\Gamma}$ in terminology of selection principles).

• the property (ϵ) is one of the following equivalent properties:

(a) X^n is Lindelöf for all $n \in \omega$ ($l^*(X) = \omega_0$);

(b) Every open ω-cover of X contains a countable ω-subcover;

(c) $t(C_p(X)) = \omega_0$.

• the property (φ): whenever $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ is an open ω-cover of $X, \mathcal{U}_n \subset \mathcal{U}_{n+1}$ $(n \in \mathbb{N})$, there exists a sequence $X_n \subset X$ such that $\underline{\lim} X_n = X$ and X_n is ω-covered by $\mathcal{U}_n$.

Gerlits proved that a space X has the property (γ) if and only if it has both (φ) and (ϵ) (Theorem 1 in [15]).

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collection:
- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

In 1982 Gerlits and Nagy [14] proved

Theorem 6.2. *(Gerlits-Nagy) For a space X, the following are equivalent:*

1. *$C_p(X)$ satisfies $S_1(\Omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;*
2. *$C_p(X)$ is Fréchet-Urysohn;*
3. *X satisfies $S_1(\Omega, \Gamma)$;*
4. *X has the property (γ), i.e. X satisfies $\binom{\Omega}{\Gamma}$.*

In 1984, A.V. Arhangel'skii [1] proved the following theorem in the class of P-spaces.

Theorem 6.3. *(Arhangel'skii) For a P-space X, the following are equivalent:*

1. *$C_p(X)$ has countable tightness;*
2. *$C_p(X)$ is Fréchet-Urysohn;*
3. *X is Lindelöf.*

Similar to Theorem 3 in [15], we get the next result.

Theorem 6.4. *If $\mathbb{B}(X)$ is a ω-k-space, then $X_{\aleph_0}$ has the property (φ).*

Proof. Otherwise $\mathbb{B}(X)$ is a ω-k-space, yet X has not the property (φ), and let $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ witness this. Put for $n \in \mathbb{N}$, $n \geq 1$ and $A_n = \{f \in \mathbb{B}(X) : f^{-1}(-\infty, n)$ is ω-covered by $\mathcal{U}_n\}$, $A = \bigcup\{A_n : n \in \mathbb{N}\}$. Then A_n is closed in $\mathbb{B}(X)$ for any n. On the other hand, A is not closed in $\mathbb{B}(X)$, because $\mathbf{0} \in \overline{A} \setminus A$. As $\mathbb{B}(X)$ is a ω-k-space, there is a countably compact subset C of $\mathbb{B}(X)$ such that $C \cap A$ is non-closed in C. As C is countably compact, so also is each of its projections on the real line: for each $x \in X$ there is an $n(x) \in \mathbb{N}$ such that for each $f \in C$, $f(x) \leq n(x)$. Put $X_n = \{x \in X : n(x) \leq n\}$. As the sets X_n monotonically increase and their union is X, we have $\underline{\lim} X_n = X$. Using now that $\{\mathcal{U}_n\}$ witnesses that X has not (φ), we get an $m \in \omega$ such that no $\mathcal{U}_k$ ω-covers X_m.

Note that $C \cap A_k = \emptyset$ if $m < k < \omega$. Indeed, let $f \in A_k$, $m < k < \omega$. $f^{-1}(-\infty, k)$ is ω-covered by $\mathcal{U}_k$, but X_m is not, so $X_m \setminus f^{-1}(-\infty, k) \neq \emptyset$, hence, there is a point $x \in X_m$ such that $f(x) \geq k > m$ and $n(x) \leq m$. The definition of X_m implies now that $f \notin C$.

However, this is impossible because then $C \cap A = \bigcup\{C \cap A_k : k \leq m\}$ would be closed in C, contrary to the choice of C.

□

The following theorem is proved similarly to Theorem 4 in [15]; therefore, we omit the proof of this theorem.

Theorem 6.5. *If $\mathbb{B}(X)$ is a ω_1-k-space, then $X_{\aleph_0}$ has the property $S_1(\Omega, \Gamma)$.*

Now we can consider a modification of Theorem P5.

Theorem 6.6. *Let X be a Tychonoff space and $\mathbb{B}(X) \in \mathbb{B}$. Then the following are equivalent.*

1. $\mathbb{B}(X)$ *is Fréchet-Urysohn;*
2. $\mathbb{B}(X)$ *is sequential;*
3. $\mathbb{B}(X)$ *is a k-space;*
4. $\mathbb{B}(X)$ *is a ω_1-k-space;*
5. $\mathbb{B}(X)$ *has countable tightness;*
6. $X_{\aleph_0}$ *satisfies $S_1(\Omega, \Gamma)$;*
7. $X_{\aleph_0}$ *is Lindelöf.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4), (1) ⇒ (5) are immediate. Since $\mathbb{B}(X) \subseteq C(X_{\aleph_0})$, then, by Theorem 6.2, we have that (6) ⇒ (1) holds.

By Lemma 2.2, we have that (5) ⇒ (7).

By Theorem 6.5, if $\mathbb{B}(X)$ is a ω_1-k-space, then $X_{\aleph_0}$ satisfies $S_1(\Omega, \Gamma)$, i.e. (4) ⇒ (6) holds.

(7) ⇒ (1). Since $X_{\aleph_0}$ is a P-space, then, by Theorem 6.3, we have that $C_p(X_{\aleph_0})$ is Fréchet-Urysohn. But $\mathbb{B}(X) \subseteq C_p(X_{\aleph_0})$, hence $\mathbb{B}(X)$ is Fréchet-Urysohn, too.

□

Corollary 6.7. Let X be a Tychonoff space. Then the following are equivalent.

1. $C_p(X_{\aleph_0})$ is Fréchet-Urysohn;

2. $C_p(X_{\aleph_0})$ is sequential;
3. $C_p(X_{\aleph_0})$ is a k-space;
4. $C_p(X_{\aleph_0})$ is a ω_1-k-space;
5. $C_p(X_{\aleph_0})$ has countable tightness;
6. $X_{\aleph_0}$ satisfies $S_1(\Omega, \Gamma)$;
7. $X_{\aleph_0}$ is Lindelöf.

Corollary 6.8. Let $\mathbb{B}(X)$ be a ω_1-k-space and $B_1(X) \subseteq \mathbb{B}(X)$. Then $B_1(X) = \mathbb{B}(X) = C(X_{\aleph_0})$.

Corollary 6.9. Assume that $X_{\aleph_0}$ satisfies $S_1(\Omega, \Gamma)$. Then $B_1(X) = C(X_{\aleph_0})$.

Corollary 6.10. Assume that X is a perfectly normal space and $B_\alpha(X)$ is k-space for some $1 \leq \alpha \leq \omega_1$. Then X is countable.

Proposition 6.11. *There exists a space X such that $B_\alpha(X)$ is a ω-k-space, but not a ω_1-k-space.*

Proof. Let X be the space $\omega_2 \setminus L$, where L denotes the set of ω-limits in ω_2; then $X = X_{\aleph_0}$, $C_p(X) = B_\alpha(X)$ $(0 \leq \alpha \leq \omega_1)$ and $B_\alpha(X)$ is ω-k but not ω_1-k (see Example in [15]). □

Proposition 6.12. *There exists a space X such that $\omega_0 = t(C_p(X)) < t(B_\alpha(X))$ for $\alpha > 0$.*

Proof. The space $C_p([0,1])$ is not sequential (Fréchet-Urysohn, k-space), but $\omega_0 = t(C_p([0,1])) < t(B_\alpha([0,1])) = \mathfrak{c}$ for any $\alpha > 0$. □

Proposition 6.13. *(MA+¬ CH) There exists a set of reals X such that $C_p(X)$ is sequential, but $t(\mathbb{B}(X)) > \omega_0$ for any $\mathbb{B}(X) \in \mathbb{B}$.*

Proof. By Theorem 1 in [13], assuming Martin's axiom, there exists a set of reals X of cardinality the continuum such that X has the property $S_1(\Omega, \Gamma)$. Then $X_{\aleph_0}$ is not Lindelöf and, hence, by Theorem 6.6, $t(\mathbb{B}(X)) > \omega_0$ for any $\mathbb{B}(X) \in \mathbb{B}$. □

Theorem 6.14. *If $\mathbb{B}(X)$ is a ω-k-space, then X satisfies $S_1(Z_\Omega, Z_\Gamma)$.*

Proof. Let $\alpha = \{F_i : i \in \mathbb{N}\}$ be a ω-cover of X by zero-sets of X. Consider $A = \{h_n : h_n = n \cdot f_n,\ f_n$ is the characteristic function of $X \setminus F_n$, $F_n \in \alpha$, $n \in \mathbb{N}\}$. Note that $\mathbf{0} \in \overline{A} \setminus A$. Hence, there exists a countably compact set C such that $A\bigcap C$ is not a closed subset of C. Since C is a countably compact set, whenever $x \in X$ there is $n(x) \in \mathbb{N}$ such that $f(x) < n(x)$ for each $f \in C$. Let $X_n = \{x \in X : f(x) < n \text{ for each } f \in C\}$. Then $X_{n+1} \supseteq X_n$ and $X = \bigcup_n X_n$.

If for every n there exists $i(n)$ such that $X_n \subseteq F_{i(n)}$, then $\{F_{i(n)} : n \in \mathbb{N}\}$ is a γ-cover of X. Otherwise, there is an n' such that $X_{n'} \setminus F_i \neq \emptyset$ for each $i \in \mathbb{N}$. Fix an $n \in \mathbb{N}$ such that $n > n'$. There is an $x \in X_{n'} \setminus F_n$ such that $h_n(x) = n > n'$. It follows that $h_n \notin C$. Thus, we have that $A\bigcap C = \{h_i : i < n' + 1\}\bigcap C$ is not a closed subset of C, a contradiction.

□

Recall that a space X is called *proper analytic* if it admits a perfect map onto an analytic subset of a complete separable metric space. A space X is *disjoint analytic* if and only if it is a one-to-one continuous image of a proper analytic space [16]. Note that any K-Lusin space is a disjoint analytic space.

Theorem 6.15. *Let X be a disjoint analytic space and $B_1(X) \subseteq \mathbb{B}(X)$. Then the following are equivalent:*

1. *X is scattered;*
2. *$\mathbb{B}(X)$ is Fréchet-Urysohn.*

Proof. If X is scattered, then $l(X) = l(X_{\aleph_0})$ [9]. By Theorem 6.6, $\mathbb{B}(X)$ is Fréchet-Urysohn.

If $\mathbb{B}(X)$ is Fréchet-Urysohn, then, by Theorem 6.6 and Corollary 6.8, $B_1(X) = \mathbb{B}(X) = C_p(X_{\aleph_0})$. Then, by Theorem 6 in [16], X is scattered. □

It is well-known that for a compact space X, $C_p(X)$ is Fréchet-Urysohn if and only if $C_p(X)$ is a k-space if and only if X is scattered [15, 19].

Corollary 6.16. For a compact space X and $\alpha > 0$, $B_\alpha(X)$ is Fréchet-Urysohn if and only if $B_\alpha(X)$ is a k-space if and only if X is scattered.

Thus we have that if a compact space X is not scattered, then $t(B_\alpha(X)) \geq l(X_{\aleph_0}) \geq \mathfrak{c}$.

Note that there exists a scattered space Z such that $t(B_1(Z)) > \omega_0$.

Example 6.17. *Let Z be the set of all countable ordinals endowed with the interval topology. Then Z is scattered pseudocompact and $t(B_1(Z)) > \omega_0$.*

A.V. Arhangel'skii [3] (see also [24]) asked the question: For what compact spaces X does the inequality $l(X_{\aleph_0}) \leq \mathfrak{c}$ hold ?

It is well-known that the answer is positive in the following cases:

1. X is a finite product of ordered compact spaces [24].
2. X is a compact space of countable tightness [20].
3. X is a weakly Corson compact space [21].

This implies, in particular, $t(B_\alpha(X)) \leq \mathfrak{c}$ for any space X in these classes of spaces.

In [3, 24], it was shown that the Lindelöf number of $X_{\aleph_0}$ for a compact space X can be arbitrary large (for example, the Stone-Čech compactification $\beta(D)$ of a discrete space D). Therefore, the tightness of $B_\alpha(X)$ for compact spaces X is not bounded. E.G. Pytkeev proved the following remarkable result (Theorem 1.1. in [21]).

Theorem 6.18. *(Pytkeev) Let X be a Tychonoff space. Then* $t(C_p(X)) \leq t(B_\alpha(X)) \leq exp(t(C_p(X)) \cdot t(X))$.

7. Density

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.

Theorem 7.1. *(Noble [11]) $d(C_p(X)) = iw(X)$.*

Let $A \subset Y$. Put $[A]'_\tau = \bigcup\{\overline{B} : B \subset A, |B| \leq \tau\}$, $T(x, A, Y) = \min\{\tau : x \in [A]'_\tau\}$, $T(A, Y) = \sup\{T(x, A, Y) : x \in \overline{A}\}$. Then $T(C_p(X), B_\alpha(X)) = \omega_0$. Since $C_p(X)$ is dense in $B_\alpha(X)$, $d(B_\alpha(X)) \leq d(C_p(X)) = iw(X)$.

Let $\mu = d(B_\alpha(X))$. Then there is $D \subset B_\alpha(X)$ such that $|D| = \mu$ and $\overline{D} = B_\alpha(X)$. The equality $T(C_p(X), B_\alpha(X)) = \omega_0$ means that $[C_p(X)]'_{\omega_0} = B_\alpha(X)$. For each $d \in D$, fix a set $C_d \subset C_p(X)$ such that $|C_d| \leq \omega_0$ and $d \in \overline{C_d}$. Then the set $S = \bigcup\{C_d : d \in D\}$ is dense in $C_p(X)$ and $|S| \leq \mu$. Hence, $d(B_\alpha(X)) \geq d(C_p(X))$. Thus, we have the Theorem P6 of Pestryakov that $d(B_\alpha(X)) = iw(X)$ $(0 < \alpha \leq \omega_1)$.

Example 7.2. *Let X be a first-countable space such that $|X| \leq \mathfrak{c}$ and $iw(X) > \omega_0$. Then $d(B_\alpha(X)) = iw(X) > iw(X_{\aleph_0}) = d(C_p(X_{\aleph_0}))$.*

For example, if Z is the set of all countable ordinals endowed with the interval topology, then $d(B_\alpha(Z)) > d(C_p(Z_{\aleph_0}))$.

Note also that if $\mathfrak{c} < 2^{\omega_1}$ then $|B_{\omega_1}(Z)| = \mathfrak{c} < 2^{\omega_1} = |C_p(Z_{\aleph_0})|$, otherwise $|B_{\omega_1}(Z)| = |C_p(Z_{\aleph_0})|$.

8. Pseudocharacter, pseudoweight

It is well-known that $\psi(C_p(X)) = iw(C_p(X)) = d(X)$ [2].

Theorem 8.1. *$\psi(\mathbb{B}(X)) = \psi w(\mathbb{B}(X)) = i\chi(\mathbb{B}(X)) = iw(\mathbb{B}(X)) = d(X_{\aleph_0})$.*

Proof. Note that if there exists a condensation (one-to-one continuous map) $f : Y \to Z$ of a space Y onto a space Z then $\psi(Y) \leq \psi(Z) \leq \chi(Z) \leq w(Z)$ and $\psi(Y) \leq \psi w(Z) \leq w(Z)$. Since the space Z is arbitrary, we get that $\psi(Y) \leq i\chi(Y) \leq iw(Y)$ and $\psi(Y) \leq \psi w(Y) \leq iw(Y)$.

Since $iw(C_p(X)) = d(X)$ (Theorem 7.1) and $\mathbb{B}(X) \subset C_p(X_{\aleph_0})$, it is enough to prove that $d(X_{\aleph_0}) \leq \psi(\mathbb{B}(X))$.

Assume that $d(X_{\aleph_0}) > \psi(\mathbb{B}(X))$. Let $\{\mathbf{0}\} = \bigcap\{U_\xi : \xi \in M\}$, $|M| = \psi(\mathbb{B}(X))$. We can assume that $U_\xi = (x_1(\xi), ..., x_n(\xi), \epsilon(\xi)) = \{f : f \in \mathbb{B}(X), |f(x_i(\xi))| < \epsilon(\xi)\}$. Let $A = \{x_i(\xi) : \xi \in M, 1 \leq i \leq n(\xi)\}$. Since $|A| < d((X_{\aleph_0})$, there exists a zero-set D in X such that $D \cap A = \emptyset$. Note that the characteristic function χ_D of the set D is in $\mathbb{B}(X)$, $\chi_D \neq \mathbf{0}$ and $\chi_D \in \bigcap\{U_\xi : \xi \in M\}$, a contradiction. ☐

9. Network weight

Lemma 9.1. *Define the function $\varphi : X_{\aleph_0} \to C_p(\mathbb{B}(X))$ by the rule: $\varphi(x)(f) = f(x)$ for each $f \in \mathbb{B}(X)$. Then $X_{\aleph_0}$ is homeomorphic to $\varphi(X_{\aleph_0}) \subset C_p(\mathbb{B}(X))$.*

Proof. Obviously, φ is bijection from $X_{\aleph_0}$ onto $\varphi(X_{\aleph_0})$.

Note that $\mathbb{B}(X) \subset C_p(X_{\aleph_0})$. The equality $\varphi^{-1}(\{h : h \in \varphi(X_{\aleph_0}), |h(f_i) - \varphi(x)(f_i)| < \epsilon, 1 \leq i \leq n, f_i \in \mathbb{B}(X)\}) = \bigcap\limits_{i=1}^{n} f_i^{-1}(f_i(x) - \epsilon, f_i(x) + \epsilon)$ implies that φ is a continuous map.

The set $\varphi(M) = \{h : h \in \varphi(X), |h(\chi_M) - 1| < 1\}$ for a characteristic function χ_M of the zero-set M is an open set in $\varphi(X)$. Thus, φ^{-1} is a continuous map.

☐

Theorem 9.2. *$nw(\mathbb{B}(X)) = nw(X_{\aleph_0})$.*

Proof. Since $nw(C_p(Y)) = nw(Y)$ for a Tychonoff space Y [2] and $\mathbb{B}(X) \subseteq C(X_{\aleph_0})$ we get that $nw(\mathbb{B}(X)) \leq nw(X_{\aleph_0})$. By Lemma 9.1, $nw(X_{\aleph_0}) \leq nw(C_p(\mathbb{B}(X))$. Thus, $nw(X_{\aleph_0}) \leq nw(\mathbb{B}(X))$. □

Note that $nw(X) \leq nw(X_{\aleph_0}) \leq nw(X)^{\omega_0}$. Then we have the following result.

Corollary 9.3. If $\kappa = \kappa^{\omega_0}$, then $nw(\mathbb{B}(X)) = nw(C_p(X_{\aleph_0})) = nw(X) = \kappa$.

10. The Lindelöf number

The following result is well known in C_p-theory [5].

Theorem 10.1. *(Asanov)* $l(C_p(X)) \geq t^*(X)$.

For a space $\mathbb{B}(X) \in \mathbb{B}$, we have the following result.

Theorem 10.2. $l(\mathbb{B}(X)) \geq t^*(X_{\aleph_0})$.

Proof. Denote as usually $[Y]^{<\omega}$ the set of all non-empty finite subsets of a space Y. Consider the topological space $Y_p = ([Y_{\aleph_0}]^{<\omega}, \tau)$ where the topology τ generated by the base $\beta = \{H^* : H^* = \{F \in [Y_{\aleph_0}]^{<\omega} : F \subset H\}$ for any open H in $Y\}$. Since $t(Y^n) \leq t(Y_p)$ for every $n \in \omega$ [5] it is enough to prove that $t(X_{\aleph_0 p}) \leq l(\mathbb{B}(X))$.

Let $M \subset X_{\aleph_0 p}$ and $S \in \overline{M} \setminus M$. Note that the family $\{< p, (-1,1) >: p \in M\}$ is a cover of the set $\{f : f \in \mathbb{B}(X), f(S) = 0\}$ where $< p, (-1,1) >= \{f : f \in \mathbb{B}(X), f(p) \subset (-1,1)\}$. Since $\{f : f \in \mathbb{B}(X), f(S) = 0\}$ is closed in $\mathbb{B}(X)$, choose $M' \subset M$ such that $|M'| \leq l(\mathbb{B}(X))$ and $\{< p, (-1,1) >: p \in M'\}$ is a cover of $\{f : f \in \mathbb{B}(X), f(S) = 0\}$. Then $S \in \overline{M'}$.

□

Note that $l(B_1([0,1])) = \mathfrak{c} > \omega_0 = t^*([0,1]_{\aleph_0})$.

Question. Is it possible to replace $X_{\aleph_0}$ by X in Theorem 10.2 ?

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