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An Exact Solution for the Description of the Gradient Flow of a Vortex Fluid

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Abstract. An isothermal nonlinear gradient flow of a horizontal layer of a vertically vortex fluid is considered. The Navier-Stokes equation uses a solution describing the velocity and pressure fields. This solution is a linear function of the longitudinal (horizontal) coordinates with coefficients depending on the transverse (vertical) coordinate. For the obtained general exact solution, the boundary-value problem is solved with the no-slip conditions, nonzero tangential stresses, and constant pressure gradients set at the boundaries of the infinite fluid layer. It is shown that, for the considered boundary conditions, up to three stagnation points can arise in the fluid layer. The velocity or its components change their direction to the opposite at these stagnation points.

PROBLEM STATEMENT

The Navier-Stokes equation describing the isothermal motion of a viscous incompressible fluid, supplemented by the incompressibility equation [1], is written in the vector form as

$$\frac{\partial \mathbf{V}}{\partial t} + \left(\mathbf{V}\cdot\nabla\right)\mathbf{V} = -\nabla P + \mathbf{F} + \nu\Delta\mathbf{V}\,, \tag{1}$$

$$\nabla\cdot\mathbf{V} = 0. \tag{2}$$

The following notation is introduced in equations (1) and (2): $\mathbf{V}\left(t,x,y,z\right)=\left(V_x,V_y,V_z\right)$ is a velocity vector; P is the deviation of pressure from hydrostatic, taken relative to constant average fluid density ρ; $\mathbf{F}=\left(0;0;g\right)$ is the density of the field of mass forces; ν is kinematic viscosity; $\nabla = \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$ is the three-dimensional Hamilton operator, $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the three-dimensional Laplace operator [1].

The motion of a viscous incompressible fluid occurs in a horizontal infinite layer parallel to the plane Oxy. The coordinate axis Oz is directed vertically upwards. The lower and upper boundaries of the fluid layer are set by the equations of the planes $z=0$ and $z=h$, respectively. The problem assumes that the flow is stratified, i.e. that the vertical velocity component is zero, $V_z=0$.

The exact solution to system (1), (2) is considered in the following form [2–6]:

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$$V_x(y,z) = U(z) + yu(z);$$
$$V_y = V(z);$$
$$P(x,y,z) = P_0(z) + xP_1 + yP_2. \tag{3}$$

The uniform pressure term is defined as follows:

$$P_0(z) = g(z-h) + S, \tag{4}$$

where S is the atmospheric pressure specified on the upper free surface of the considered fluid layer. Formula (4) defines the pressure field that is used when the hydrostatic approximation of the Navier-Stokes equations are considered.

The components P_1 and P_2 are constant coefficients, pressure gradients along the longitudinal coordinates x and y, respectively. The fluid motion characterized by the pressure function (3) is a generalization of the classical Poiseuille flow [7–11].

We substitute the exact solutions class (3) into the nonlinear system (1), (2) projected on the axis of the Cartesian coordinate system. The primes mark the derivatives of the functions with respect to the z coordinate. We obtain a system of ordinary differential equations for determining three unknown functions U, u, and V written in the order of integration as

$$u'' = 0;$$
$$\nu V'' = P_2;$$
$$\nu U'' = Vu + P_1. \tag{5}$$

AN EXACT SOLUTION TO THE BOUNDARY VALUE PROBLEM

The horizontal (longitudinal) pressure gradients P_1 and P_2 are constant and set at the upper boundary of the fluid layer determined by the equation $z = h$.

Let the no-slip condition be satisfied at the lower boundary of the horizontal layer of a viscous incompressible fluid defined by the plane equation $z = 0$. When studying the flow properties, the tangential stresses at the upper boundary defined by the plane equation $z = h$ are assumed to be non-constant, but spatially inhomogeneous, by analogy with the boundary conditions considered in [12, 3, 13, 14]. Thus, taking into account the form of solutions (3), the boundary conditions are written in the form

$$U(0) = 0; \qquad u(0) = 0; \qquad V(0) = 0;$$
$$\nu \left.\frac{\partial U}{\partial z}\right|_{z=h} = -\tau_1; \quad \nu \left.\frac{\partial u}{\partial z}\right|_{z=h} = -\tau_2; \quad \nu \left.\frac{\partial V}{\partial z}\right|_{z=h} = -\tau_3. \tag{6}$$

The exact particular solution to the boundary value problem (5), (6) has the form

$$u = -\frac{\tau_2}{\nu} z; \qquad V = \frac{z}{2\nu}\left[P_2 z - 2(P_2 h + \tau_3)\right];$$
$$U = -\frac{P_2\tau_2}{40\nu^3} z^5 + \frac{\tau_2 z^4}{12\nu^3}(P_2 h + \tau_3) + \frac{P_1 z^2}{2\nu} -$$
$$-\frac{z}{24\nu^3}\left[24\nu^2(P_1 h + \tau_1) + \tau_2 h^3 (5P_2 h + 8\tau_3)\right]. \tag{7}$$

ANALYSIS OF THE EXACT SOLUTION FOR THE VELOCITY FIELD

Let us analyze the obtained solution (7). Extensional acceleration u (parallel to the abscissa axis) is a monotonic function that increases or decreases depending on the sign of the horizontal gradient of the tangential stress τ_2 at the free boundary of the fluid layer specified by the equation $z = h$.

The velocity component V can take on zero values for $z = 0$ and $z = 2(P_2 h + \tau_3)/P_2$. Thus, for the function V to vanish on the interval $(0;h)$, the following double inequality must be satisfied:

$$-1 < \frac{\tau_3}{hP_2} < -\frac{1}{2}.$$

We can conclude that, for the stagnation point of the velocity component V to exist, it is necessary that the pressure gradient along the ordinate axis $\partial P / \partial y$ and the tangential stress $\partial V / \partial z$ at the upper boundary of the fluid layer be simultaneously either positive or negative, i.e. either compressive or tensile. Figure 1 shows the profile of the velocity component V for the case of the stagnation point existing inside the thickness of the considered fluid layer.

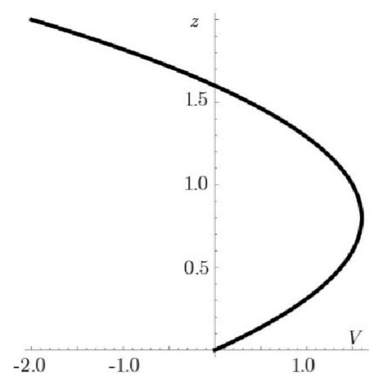

FIGURE 1. The profile of the velocity component V for the case of the stagnation point existing in the bulk of the fluid layer ($h = 2$ m, $P_2 = -5 \cdot 10^{-6}$ m^2/s^2, $\tau_3 = 6 \cdot 10^{-6}$ m/s^2)

The analysis of the velocity component U in the interval $(0;h)$ reduces to the determination of the zero points of the function U, i.e. to finding the roots of the equation $U = 0$. We introduce the dimensionless coordinate $q = z/h$; thus, the study area is reduced to the interval $q \in (0;1]$. Figure 2 shows the profile of the velocity component U in the case of the existence of three zero points in the interval $(0;1)$. Figure 3 shows the corresponding streamlines in the case that the velocity component U has three zero points in the interval $(0;1)$ and the velocity component V has one.

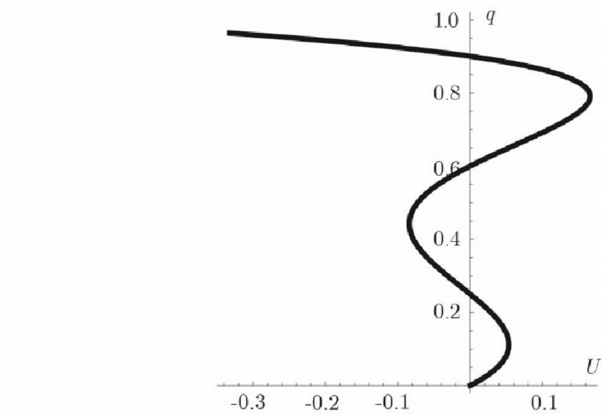

FIGURE 2. The profile of the velocity component U in the case of the existence of three zero points in the interval $(0;1)$

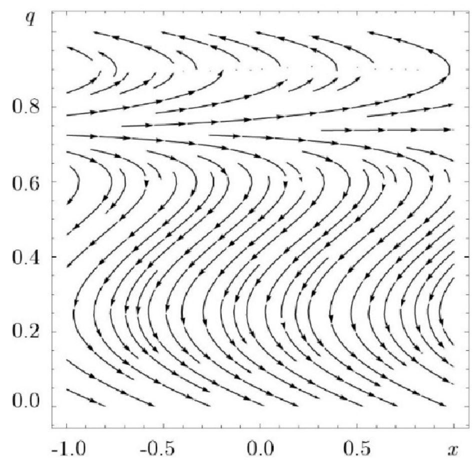

FIGURE 3. The streamlines for the following parameter values: $\nu = 10^{-6}$ m^2/s, h=100 m, $\tau_1 = 1.464 \cdot 10^{-6}$ m^2/s^2, $\tau_2 = 10^{-15}$ m/s^2, $\tau_3 = -2.86 \cdot 10^{-8}$ m^2/s^2, $P_1 = -2.82 \cdot 10^{-9}$ m/s^2, $P_2 = 4 \cdot 10^{-10}$ m/s^2.

Taking into account the dimensionless coordinate q, the specific kinetic energy for the obtained type of velocity has the form

$$T = \frac{h^2 q^2}{4\nu^2}\left[hP_2 q - 2\left(hP_2 + \tau_3\right)\right]^2 + \left\{-\frac{hqy\tau_2}{\nu} + \right.$$
$$\left. + q\left[\frac{h^2 P_1 q}{2\nu} - \frac{h^5 P_2 q^4 \tau_2}{40\nu^3} + \frac{h^4 q^3 \tau_2 \left(hP_2 + \tau_3\right)}{12\nu^3} - \frac{h^2 P_1 + \tau_1}{\nu} - \frac{h^4 \tau_2 \left(5hP_2 + 8\tau_3\right)}{24\nu^3}\right]\right\}^2 .$$

The total velocity of the fluid flows vanishes at the point where the specific kinetic energy T is zero. Figure 4 shows a graph of the specific kinetic energy function with one zero point in the plane $y = 0$. Figure 5 shows the streamlines for the same parameter values.

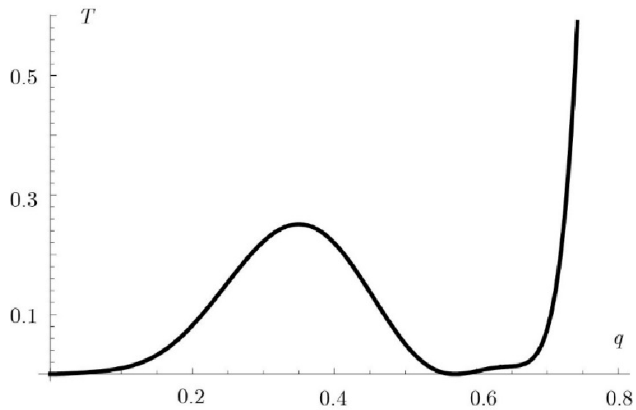

FIGURE 4. The graph of the specific kinetic energy function T

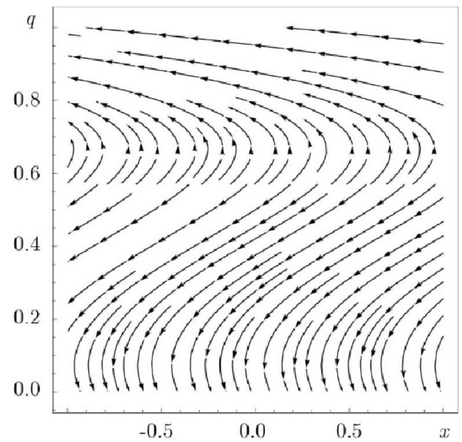

FIGURE 5. The streamlines for the following parameter values: $\nu = 10^{-6}$ m^2/s, h=100 m, $\tau_1 = 1.472 \cdot 10^{-6}$ m^2/s^2, $\tau_2 = 10^{-15}$ m/s^2, $\tau_3 = -2.86 \cdot 10^{-8}$ m^2/s^2, $P_1 = -2.82 \cdot 10^{-9}$ m/s^2, $P_2 = 4 \cdot 10^{-10}$ m/s^2

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CONCLUSION

An exact solution has been constructed for the isothermal gradient Poiseuille flow, which describes the motion of a viscous incompressible fluid in a horizontal infinite layer. It has been shown that, for the considered boundary conditions, up to three counterflow regions can arise in the bulk of the fluid layer, where the velocity changes its direction to the opposite. Streamline solutions corresponding to various cases have been demonstrated.

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