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SOLVING EQUATIONS OF WATER DISTRIBUTION NETWORK BY THE CONTINUATION METHOD

Abstract. Now engineers have a large number of specialized programs for design and simulation of water distribution networks in different situations. Regardless of the type of problem it is necessary to solve a nonlinear system of network equations. The equations are solved using iterative methods. In some cases, application of the above methods involves the convergence problem. To resolve it, a correct choice of the initial values of calculated parameters is required. This article describes solving the system of network equations by the continuation method. The method allows calculation any of the network parameters to a given accuracy with nearly all initial values of the variables.

Keywords: pipeline network, water distribution system, calculation methods.

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РЕШЕНИЕ УРАВНЕНИЙ РАСПРЕДЕЛИТЕЛЬНОЙ СЕТИ ВОДОСНАБЖЕНИЯ МЕТОДОМ ПРОДОЛЖЕНИЯ

Аннотация. Сейчас в распоряжении инженеров имеется большое количество специализированных программ для проектирования и моделирования работы распределительных сетей водоснабжения в различных ситуациях. Независимо от вида поставленной задачи возникает необходимость решения нелинейной системы уравнений сети. Решение осуществляется теми или иными итерационными методами. Иногда при их использовании возникает проблема сходимости. Для ее решения необходимо правильно выбирать начальные значения рассчитываемых параметров. В данной статье рассматривается решение системы уравнений сети методом продолжения. Метод позволяет рассчитывать любые параметры сети с заданной точностью при практически любых начальных значениях переменных.

Ключевые слова: трубопроводная сеть, распределительная сеть водоснабжения, методы расчета.

Introduction

Municipal water utilities are by far the most common users of water distribution mathematical models. Water distribution network simulation is used for a variety of purposes, such as: long-range master planning, including both new development and rehabilitation, fire protection studies, water quality investigation, energy management, system design, daily operation uses including operator training etc. These models are based on the continuity and energy equations. For real systems, these equations can be numbered in the hundreds.

Note that nowadays solving the network equations system is often a part of other algorithms. For example, they are used in the genetic algorithms for optimal design of a network [1], when the system of equations has to be solved many times.

Many modern computer programs for modeling of water supply need water demand of consumers to be specified prior to the calculation. This is due to the fact that the system of equations is solved on the set of network flows. At the same time, theoretically it's possible to solve it on the set of heads (hydraulic grade). However, in the latter case use of the iterative methods leads to emergence of the problem of convergence, and the solution cannot be obtained [4–7, 9].

Below, a method of solving the network equations will be described, allowing calculating any of its parameters with any desired accuracy and with a minimum amount of calculations.

Network equations system

The concept of network is fundamental to a water distribution model. The network contains all the components of the system and defines the way in which those elements are interconnected. Networks comprise nodes which represent the features at specific locations within the system, and links which define the relationships between the nodes.

The node with index i is an element which is characterized by head H_i (m). In the node water is removed (demand) or added (inflow) from (to) the system in an amount of q_i (m^3/s).

In general, a functional dependence may exist between H and q, for example when the node is an orifice. Thus the nodes are such elements as junctions, tanks, reservoirs, orifices, hydropneumatic accumulators, etc.

At each node there is a mass balance. The balance indicates that the mass going into a node Q_{inflow} must be equal to the mass which comes out of the node $Q_{outflow}$:

$$q = Q_{inflow} - Q_{outflow}. \quad (1)$$

The link between nodes i and j is characterized by water flow Q_{ij}. Experience shows that for most types of the links [1, 9]:

$$H_i - H_j = a + b\left|Q^{c-1}\right|Q, \quad (2)$$

where a, b and c are some coefficients depending on the type of the link and the head change model used.

The main types of the links are pipelines. For them in expression (2) $a = 0$, and b has the meaning of resistance. If a link is a pump $H_i < H_j$ and $a < 0$, $b > 0$.

Consider a simple network (Fig). It consists of five nodes, and five links. Four of the links are tubes, one link is a pump. Node 2 is a fountain, so the discharge rate is a function of the head [7]:

$$q_2 = C_d D_f^2 \sqrt{H_2}, \quad (3)$$

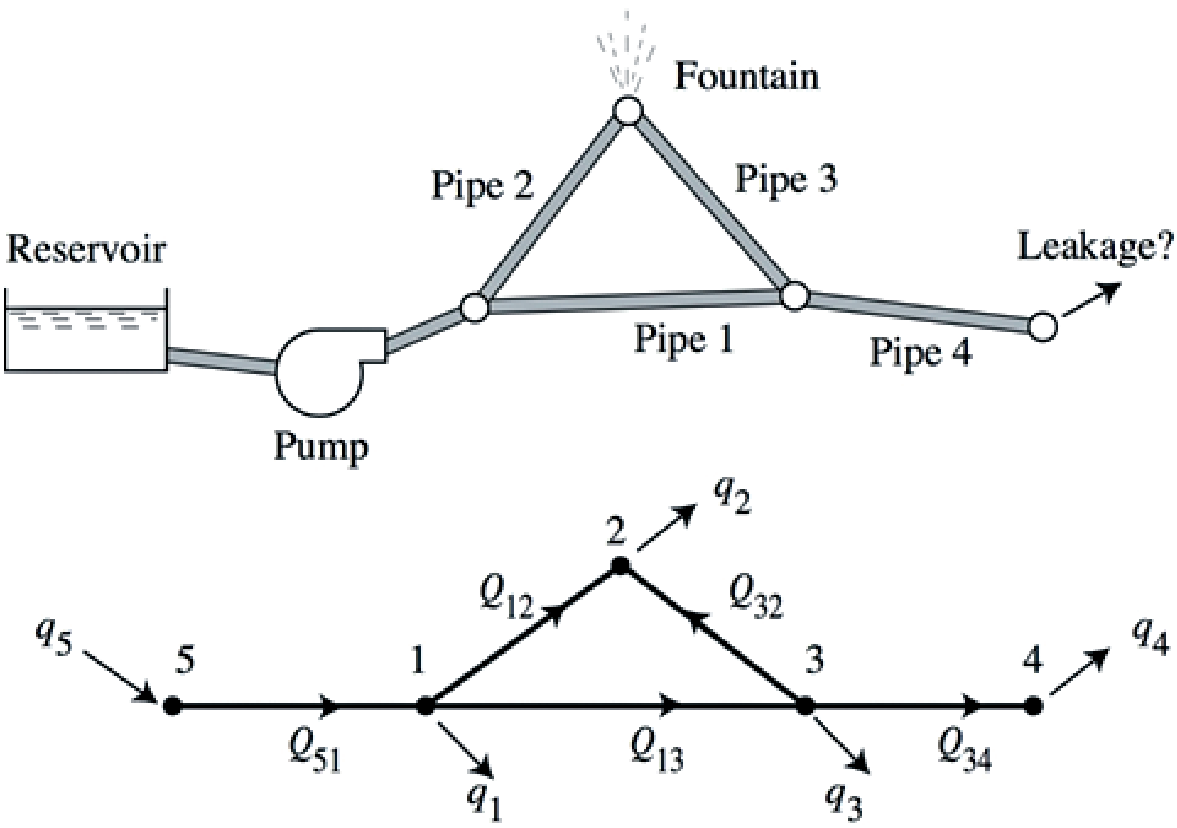
where C_d is the discharge coefficient, D_f is the orifice diameter (hereinafter, to be more specific, we take $C_d D_f^2 = a_2 = 0.0025$).

To be specific, the head loss ΔH calculation will be carried out using the Hazen–Williams formula [1]:

$$\Delta H = \frac{10.7L}{C^{1.852} D^{4.87}} Q^{1.852}, \quad (4)$$

where L — distance, m; C — Hazen-Williams C-factor; D — diameter, m; Q — pipeline flow rate, m^3/s.

The relationship between the pump head and the pump discharge is set as (2), where $a = a_{51} = 24$, $b = b_{51} = -2.145\times10^4$, $c = c_{51} = 3$.



An example of water distribution network and its graph

Further on, $H_5 = H_4 = 0$, $q_1 = q_3 = 0$. All the pipes are of equal length $L = 100$ m, and diameter $D = 150$ mm, Hazen-Williams coefficient $C = 130$ (cast iron). In this case, the coefficients $b_{ij} = 1340$ (except b_{51}).

The system of equations has the form:

$$
\begin{gathered}
Q_{51} - q_5 = 0 \\
Q_{51} - Q_{13} - Q_{12} = 0 \\
Q_{13} - Q_{32} - Q_{34} = 0 \\
Q_{12} + Q_{32} - 0.0025 H_2^{0.5} = 0 \\
Q_{34} - q_4 = 0 \qquad (5) \\
H_1 - 24 + 2.145 \cdot 10^4 Q_{51}^3 \\
H_1 - H_3 - 1340 Q_{13} |Q_{31}|^{0.852} = 0 \\
H_3 - H_2 - 1340 Q_{32} |Q_{32}|^{0.852} = 0 \\
H_3 - 1340 Q_{34} |Q_{34}|^{0.852} = 0 \\
H_1 - H_2 - 1340 Q_{12} |Q_{12}|^{0.852} = 0 .
\end{gathered}
$$

Thus, we have ten equations. The unknown values are the heads H_1, H_2, H_3, outflow q_4, q_5 and all the pipeflows Q_{ij}.

Solving the system of equations

Usually, the continuation method is used to solve of differential equations with a parameter. Kane [2] proposed to use this method for solving systems of nonlinear equations. The method essence consists in building and solving a system of ordinary differential equations of the first order. Some numerical examples of applications of the method are described by Na [3].

As an illustration of the method, let us consider the solution of a single equation

$$f(x) = 0. \qquad (6)$$

As the initial approximation we assume an arbitrary $x = x^*$ and consider the equation

$$f(x) = f(x^*)(1-t) , \qquad (7)$$

where t is a new independent variable changing from 0 to 1.

If $t = 0$ then $f(x) = f(x^*)$. If $t = 1$ then $f(x) = 0$, i.e. x is the root of the equation (6). Thus, one should expect that when t varies from 0 to 1, x will change from the arbitrary initial value x^* to the root of the original equation.

To find a solution, let us differentiate equation (7) by t

$$\frac{df(x)}{dx}\frac{dx}{dt} = -f(x^*) \qquad (8)$$

or

$$\frac{dx}{dt} = -\frac{f(x^*)}{\frac{df(x)}{dx}} \qquad (9)$$

The boundary condition: if $t = 0$ then $x = x^*$.

As a result of the integration of the equation (9) to $t = 1$ we obtain the solution of the original equation (6).

The method of integration can be arbitrary. However, in any case it is necessary to choose the integration step Δt allowing the predetermined accuracy to be achieved.

Of course, there are algorithms with automatic step selection. However, when it is necessary to solve a system of several hundreds of differential equations, these algorithms require use of powerful computers.

Our research has shown that considerable gain in the amount of computations can be obtained if the integration is carried out by the Euler method with a step equal to the value of the interval, i.e. when $\Delta t = 1$.

After a single calculation we obtain

$$x^{**} = x^* + \frac{dx}{dt} . \qquad (10)$$

Apparently, this value will be very far from the roots of the equation. Therefore, x^{**} should be used as a new boundary condition for (9) with the new value $f(x^*) = f(x^{**})$ at the right side, and the integration should be repeated once again.

It is important that in this case the derivative value $\frac{dx}{dt}$ has the meaning of the accuracy of the solution and it can be considered as a criterion of completion of the calculations.

Let us return to the system of equations (5). All the unknown variables are considered as functions of the parameter t. We set their initial values (indicated by *) and transform the equations to the required form (8). As the integration step $\Delta t = 1$, the result will have the following form:

$$
\begin{gathered}
\frac{dQ_{51}}{dt} - \frac{dq_5}{dt} = -\left(Q_{51}^* - q_5^*\right) \\
\frac{dQ_{51}}{dt} - \frac{dQ_{13}}{dt} - \frac{dQ_{12}}{dt} = -\left(Q_{51}^* - Q_{13}^* - Q_{12}^*\right) \\
\frac{dQ_{13}}{dt} - \frac{dQ_{32}}{dt} - \frac{dQ_{34}}{dt} = -\left(Q_{13}^* - Q_{32}^* - Q_{34}^*\right) \\
\frac{dQ_{32}}{dt} + \frac{dQ_{12}}{dt} - 0.00125 H_2^{*-0.5} \frac{dH_2}{dt} = -\left(Q_{32}^* + Q_{12}^* - 0.00125 H_2^{*0.5}\right) \\
\frac{dQ_{34}}{dt} - \frac{dq_4}{dt} = -\left(Q_{34}^* - q_4^*\right) \qquad (11) \\
\frac{dH_1}{dt} + 6.435 \cdot 10^4 Q_{51}^{*2} \frac{dQ_{51}}{dt} = -\left(H_1^* - 24 + 2.145 \cdot 10^4 Q_{51}^{*3}\right) \\
\frac{dH_1}{dt} - \frac{dH_3}{dt} - 2482 |Q_{13}^*|^{0.852} \frac{dQ_{13}}{dt} = -\left(H_1^* - H_3^* - 1340 Q_{13}^* |Q_{13}^*|^{0.852}\right) \\
\frac{dH_3}{dt} - \frac{dH_2}{dt} - 2482 |Q_{32}^*|^{0.852} \frac{dQ_{32}}{dt} = -\left(H_3^* - H_2^* - 1340 Q_{32}^* |Q_{32}^*|^{0.852}\right)
\end{gathered}
$$

$$\frac{dH_3}{dt} - 2482\left|Q_{34}^*\right|^{0.852}\frac{dQ_{34}}{dt} = -\left(H_3^* - 1340Q_{34}^*\left|Q_{34}^*\right|^{0.852}\right)$$

$$\frac{dH_1}{dt} - \frac{dH_2}{dt} - 2482\left|Q_{12}^*\right|^{0.852}\frac{dQ_{12}}{dt} = -\left(H_1^* - H_2^* + 1340Q_{12}^*\left|Q_{12}^*\right|^{0.852}\right)$$

This system is linear with respect to the derivatives, and its solution can be found by any method (e. g., using matrix operations). After calculation of the derivatives boundary conditions are corrected in the same way as (10), and the calculation is repeated.

The calculations were performed using the spreadsheet and matrix operations. The progress of solving of the system (11) is shown in Table (the inflow q_5 is not shown in table as $q_5 = Q_{51}$). The initial values of the parameters are presented in row 0. The solution accuracy equal to 0.0001 is attained already after 5 steps. According to the formula (3) outflow $q_2 = 0.084$. After the sixth step, the error is reduced to $10^{-6.}$

The solution of this system can be found under any other (even fantastic) boundary conditions.

Conclusions

A modification of the continuation method for solving systems of nonlinear algebraic equations is proposed. It involves adjusting the boundary conditions at each step of the computations. Such approach allows moving from solving the system of differential equations to solving the system of linear algebraic equations. This substantially simplifies the process of finding the solution and increases its efficiency.

Application of the continuation method for the calculation of water distribution networks eliminates most of the problems arising from the application of the traditional iteration methods:

- there is no need to solve the problem of preliminary water distribution in the pipes to determine the initial values of the variables;
- the network topology analysis is minimized;
- the solution can be found quickly and with any desired accuracy.

The progress of solving the system (11)

No	Pipeline flow rate, m³/s					Head, m			Outflow, m³/s
	Q_{12}	Q_{13}	Q_{32}	Q_{34}	Q_{51}	H_1	H_2	H_3	q_4
0	0.2	0.2	0.2	0.2	0.2	1	1	1	0.2
1	0.0341	0.1063	0.0198	0.0866	0.1404	5.62	42.07	–3.41	0.0866
2	0.0338	0.0633	–0.0225	0.0858	0.0972	19.47	16.94	14.18	0.0858
3	0.0344	0.0466	–0.0258	0.0723	0.0810	14.16	11.55	10.05	0.0723
4	0.0342	0.0440	–0.0258	0.0698	0.0782	13.78	11.20	9.67	0.0698
5	0.0342	0.0440	–0.0258	0.0698	0.0782	13.77	11.20	9.66	0.0698

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