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The Binary Paint Shop Problem

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Abstract

We consider the binary paint shop problem (**PPW**(2,1)). We describe an approach to solve **PPW**(2,1). This approach is based on constructing a logical model for **PPW**(2,1).

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Motivated by the problem of minimizing color changes in a paint shop in an automobile plant in [1] introduced the following problem:

PPW(2,1)[Binary Paint Shop Problem]: Let Σ be a finite alphabet of cardinality n. We call the elements of Σ characters and their occurrence in a word a letter. An instance of **PPW(2,1)** is a word $w = (w_1, \ldots, w_{2n}) \in \Sigma^{2n}$ in which each character occurs exactly twice. A feasible coloring is a coloring $(f_1, \ldots, f_{2n}) \in \{0,1\}^{2n}$ with the property that $i \neq j$ and $w_i = w_j$ implies $\{f_i, f_j\} = \{0,1\}$. The objective is to find a feasible coloring such that the number $\sum_{i=1}^{2n-1} |f_i - f_{i+1}|$ of color changes is minimized.

In [2] and, by a different reduction, in [3] showed that **PPW(2,1)** is **APX**-hard. It is not known whether **PPW(2,1)** is also **APX**-easy. In particular, the following question remains open (see [4], see also [5], Problem 16.): *Is there a constant factor approximation for* **PPW(2,1)***?*

In this paper we describe an approach to solve **PPW(2,1)**. This approach is based on constructing a logical model for **PPW(2,1)**.

In this section we consider an explicit reduction from **PPW(2,1)** to Maximum 2-Satisfiability.

Instance: Set U of variables, 2-CNF φ, i.e. collection C of disjunctive clauses of at most 2 literals, where a literal is a variable or a negated variable in U.

Solution: A truth assignment for U.

Measure: Number $N(\varphi)$ of clauses satisfied by the truth assignment.

Suppose that given a word $w = (w_1, \dots, w_{2n}) \in \Sigma^{2n}$ in which each character occurs exactly twice. Let $\Sigma = \{a_1, a_2, \dots, a_n\}$,

$$\varphi = \wedge_{i=1}^{2n-1}((\neg u_i \vee u_{i+1}) \wedge (u_i \vee \neg u_{i+1}))$$

where if $i < j$ and $w_i = w_j = a_k$, then $u_i = x_k$, $u_j = \neg x_k$.

Theorem. $\min_{(f_1,\dots,f_{2n}) \in \{0,1\}^{2n}} \sum_{i=1}^{2n-1} |f_i - f_{i+1}| = 4n - 2 - \max N(\varphi)$.

Proof. Consider a feasible coloring $(f_1, \dots, f_{2n}) \in \{0,1\}^{2n}$ of w. If $w_i = w_j = a_k$, then suppose that $x_k = f_i$. Since $(f_1, \dots, f_{2n}) \in \{0,1\}^{2n}$ is a feasible coloring, $f_j = \neg f_i$. Therefore, $\neg x_k = f_j$. Thus, $u_l = f_l$ for all $1 \leq l \leq 2n$.

If $|f_i - f_{i+1}| = 0$, then $u_i = u_{i+1}$. It is clear that $\neg u_i \vee u_{i+1} = u_i \vee \neg u_{i+1} = 1$ for such values of u_i and u_{i+1}. Note that either $\neg u_i \vee u_{i+1} = 1$ or $u_i \vee \neg u_{i+1} = 1$ for all values of u_i and u_{i+1}. Suppose that $|f_i - f_{i+1}| > 0$. In this case, $u_i \neq u_{i+1}$. Thus, either $\neg u_i \vee u_{i+1} = 0$ or $u_i \vee \neg u_{i+1} = 0$. So, $\sum_{i=1}^{2n-1} |f_i - f_{i+1}| = 4n - 2 - N(\varphi)$. Therefore,

$$\sum_{i=1}^{2n-1} |f_i - f_{i+1}| \geq 4n - 2 - \max N(\varphi) \tag{1}$$

for any feasible coloring $(f_1, \dots, f_{2n}) \in \{0,1\}^{2n}$ of w.

Now consider a truth assignment for U. Similarly, if $w_i = w_j = a_k$, then suppose that f_i is a truth assignment for x_k. It is easy to check that $\sum_{i=1}^{2n-1} |f_i - f_{i+1}| = 4n - 2 - N(\varphi)$. Therefore,

$$\min_{(f_1,\dots,f_{2n}) \in \{0,1\}^{2n}} \sum_{i=1}^{2n-1} |f_i - f_{i+1}| \leq 4n - 2 - N(\varphi) \tag{2}$$

for any truth assignment for U. In view of (1) and (2), we obtain that $\min_{(f_1,\dots,f_{2n}) \in \{0,1\}^{2n}} \sum_{i=1}^{2n-1} |f_i - f_{i+1}| = 4n - 2 - \max N(\varphi)$. □

In papers [6, 7, 8, 9, 10, 11, 12] the authors considered some algorithms to solve logical models. Our computational experiments have shown that these algorithms can be used to solve the logical model for the paint shop problem.

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