

Time-Optimal Control of a Small Mass Point without Environmental Resistance

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We study the problem of the time-optimal control of a small mass point without environmental resistance. The control function in the problem is a magnitude-bounded force. The time required for transferring the point from one given state to another is used as a control criterion. It is shown that the optimal time and the optimal control have complicated asymptotic expansions even in the situation of general position.

1. In the study of various aspects of optimal control problems [1–3] for systems with fast and slow variables [4–8, 11] of the form

$$\begin{aligned}\dot{x} &= A_1x + A_2y + B_1u, & x \in \mathbb{R}^n, & y \in \mathbb{R}^m, \\ \varepsilon\dot{y} &= A_3x + A_4y + B_2u, & u \in \mathbb{R}^k\end{aligned}\quad (1)$$

an important role is played by the following condition.

Condition I. All the eigenvalues of the matrix A_4 have negative real parts.

However, there are problems with a meaningful mechanical interpretation in which Condition I is not satisfied. An example is the following time-optimal control problem in the class of piecewise continuous controls:

$$\begin{aligned}\dot{x} &= y, & x, y \in \mathbb{R}^2, \\ \varepsilon\dot{y} &= u, & u \in \mathbb{R}^2, & \|u(t)\| \leq 1, \\ x(0) &= x_0 \neq 0, & y(0) = y_0, & 0 < \varepsilon \ll 1, \\ x(T_\varepsilon) &= 0, & y(T_\varepsilon) = 0, & T_\varepsilon \rightarrow \min.\end{aligned}\quad (2)$$

It can be interpreted as follows. Given a point particle of small mass $\varepsilon > 0$ moving in a plane under the influence of a bounded control force $u(t)$ lying in this plane, the goal is the minimum-time transfer of the particle

from the given state defined by the position (x_0) and the velocity (y_0) to the origin, where it stops.

Here and below, $\|\cdot\|$ is the Euclidean norm and $\langle \cdot, \cdot \rangle$ is the inner product in the considered finite-dimensional spaces.

The basic difference of this problem from previously considered ones is that $A_4 = 0$. As a result, there is no reduced system and, hence, it is unclear how the reduced problem for (2) looks like (and whether it exists at all).

Note that time-optimal control problems for completely controllable systems of more general form

$$\begin{aligned}\dot{x} &= Ay, & x, y \in \mathbb{R}^n, \\ \varepsilon\dot{y} &= Bu, & u \in \mathbb{R}^m, & \|u(t)\| \leq 1, \\ x(0) &= x_0, & y(0) = y_0, & 0 < \varepsilon \ll 1, \\ x(T_\varepsilon) &= 0, & y(T_\varepsilon) = 0, & T_\varepsilon \rightarrow \min\end{aligned}$$

can be reduced to problem (2) by changing variables.

The following result is straightforward.

Proposition 1. For all sufficiently small ε problem (2) is solvable, and $T_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

As will be shown later, $T_\varepsilon = O(\sqrt{\varepsilon})$ as $\varepsilon \rightarrow 0$, and the characteristic scaled time in problem (2) is $\frac{t}{\sqrt{\varepsilon}}$

rather than $\frac{t}{\varepsilon}$, which occurs in problems satisfying Condition I.

The goal of this study is to construct complete asymptotic expansions in powers of the small parameter ε for the optimal time and the optimal control in problem (2). To this end, we use the methods developed in [9, 10].

It should be stressed that, even in the situation of general position, the asymptotic expansions of these quantities have a complicated character similar to that of the asymptotics obtained in [9, 10].

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2. In problem (2) we pass to the new time $\tau := \frac{t}{\sqrt{\varepsilon}}$

and denote $x(\sqrt{\varepsilon}\tau)$, $\sqrt{\varepsilon}y(\sqrt{\varepsilon}\tau)$, $u(\sqrt{\varepsilon}\tau)$, and $\frac{T_\varepsilon}{\sqrt{\varepsilon}}$ by $X(\tau)$, $Y(\tau)$, $U(\tau)$, and θ_ε , respectively. Then problem (2) becomes

$$\begin{aligned} X' &= Y, \quad X, Y \in \mathbb{R}^2, \\ Y' &= U, \quad U \in \mathbb{R}^2, \quad \|U(\tau)\| \leq 1, \\ X(0) &= x_0, \quad Y(0) = \sqrt{\varepsilon}y_0, \quad 0 < \varepsilon \ll 1, \\ X(\theta_\varepsilon) &= 0, \quad Y(\theta_\varepsilon) = 0, \quad \theta_\varepsilon \rightarrow \min. \end{aligned} \tag{3}$$

By Proposition 1, new problem (3) is solvable at all sufficiently small ε . Moreover, it is well known [1] that problem (3) is also solvable at $\varepsilon = 0$. Note that, for $\varepsilon = 0$, the optimal control is discontinuous.

Theorem 1. *Problem (3) is solvable at all small $\varepsilon \geq 0$ and*

$$\theta_\varepsilon \rightarrow \theta_0 = 2\sqrt{\|x_0\|} \text{ as } \varepsilon \rightarrow 0.$$

3. By Pontryagin's maximum principle [1, 3], which is a necessary and sufficient optimality condition in the case under consideration, the optimal control in problem (3) is given by

$$U_{\text{opt}}(\tau) = \frac{-\tau l_{1,\varepsilon} + l_{2,\varepsilon}}{\|-\tau l_{1,\varepsilon} + l_{2,\varepsilon}\|}, \quad l_{1,\varepsilon}, l_{2,\varepsilon} \in \mathbb{R}^2, \tag{4}$$

and this representation is unique up to the normalization of the vector $(l_{1,\varepsilon}, l_{2,\varepsilon})$.

For problem (3) with $\varepsilon = 0$, these vectors with the normalization condition $\|l_{1,0}\| = 1$ have the form $l_{2,0} = \lambda_0 l_{1,0}$, where $\lambda_0 = \sqrt{\|x_0\|}$, and $l_{1,0} = -\frac{x_0}{\|x_0\|}$.

By virtue of the Cauchy formula, a control of form (4) is optimal for problem (3) if and only if

$$-\begin{pmatrix} x_0 \\ \sqrt{\varepsilon}y_0 \end{pmatrix} = \int_0^{\theta_\varepsilon} \begin{pmatrix} \tau^2 l_{1,\varepsilon} - \tau l_{2,\varepsilon} \\ -\tau l_{1,\varepsilon} + l_{2,\varepsilon} \end{pmatrix} \frac{d\tau}{\|-\tau l_{1,\varepsilon} + l_{2,\varepsilon}\|}. \tag{5}$$

In what follows, we assume that the initial vectors in problem (2) are not parallel:

$$x_0 \not\parallel y_0. \tag{6}$$

Condition (6) corresponds to the situation of general position (i.e., it remains valid under small variations in x_0 and y_0)

A solution of Eq. (5) is sought in the form

$$\begin{aligned} l_{1,\varepsilon} &= l_{1,0} + r, \quad \|l_{1,\varepsilon}\| = 1, \\ l_{2,\varepsilon} &= (\lambda_0 + v)l_{1,\varepsilon} + \tilde{l}_{2,\varepsilon}, \quad \tilde{l}_{2,\varepsilon} \perp l_{1,\varepsilon}, \\ \beta &= \|\tilde{l}_{2,\varepsilon}\|, \quad \theta_\varepsilon = \theta_0 + \vartheta, \end{aligned} \tag{7}$$

where all the quantities introduced are small.

Evaluating the integral in (5) and taking into account (7), we obtain the system of equations

$$\begin{aligned} -x_0 &= l_{1,\varepsilon} \left(\frac{1}{2} F_2(\vartheta, v, \beta) \right. \\ &\quad \left. - \frac{1}{2} \beta^2 F_0(\vartheta, v, \beta) + (\lambda_0 + v) F_1(\vartheta, v, \beta) \right) \\ &\quad - \tilde{l}_{2,\varepsilon} (F_1(\vartheta, v, \beta) + \lambda F_0(\vartheta, v, \beta)), \\ -\sqrt{\varepsilon}y_0 &= -l_{1,\varepsilon} F_1(\vartheta, v, \beta) + \tilde{l}_{2,\varepsilon} F_0(\vartheta, v, \beta), \\ \|l_{1,\varepsilon}\| &= 1, \quad \langle \tilde{l}_{2,\varepsilon}, l_{1,\varepsilon} \rangle = 0, \\ \beta &= \|\tilde{l}_{2,\varepsilon}\|, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &F_0(\vartheta, v, \beta) \\ &= \ln \left(\frac{\theta_0/2 + \vartheta - v + \sqrt{(\theta_0/2 + \vartheta - v)^2 + \beta^2}}{-\theta_0/2 - v + \sqrt{(\theta_0/2 + v)^2 + \beta^2}} \right), \\ &F_1(\vartheta, v, \beta) \\ &= \frac{(\theta_0/2 + \vartheta - v)^2 - (\theta_0/2 + v)^2}{\sqrt{(\theta_0/2 + \vartheta - v)^2 + \beta^2} + \sqrt{(\theta_0/2 + v)^2 + \beta^2}}, \\ &F_2(\vartheta, v, \beta) \\ &= (\theta_0/2 + \vartheta - v) \sqrt{(\theta_0/2 + \vartheta - v)^2 + \beta^2} \\ &\quad + (\theta_0/2 + v) \sqrt{(\theta_0/2 + v)^2 + \beta^2}. \end{aligned}$$

If r , v , β , and ϑ are all sufficiently small, then the functions F_0 , F_1 , and F_2 can be represented as

$$\begin{aligned} F_0(\vartheta, v, \beta) &= 2 \ln \frac{\theta_0}{\beta} + \frac{2\vartheta}{\theta_0} + \frac{2\beta^2}{\theta_0^2} - \frac{2}{\theta_0^2} (v^2 + (\vartheta - v)^2) \\ &\quad + \mathcal{F}_{0,3}(\vartheta, v, \beta), \\ F_1(\vartheta, v, \beta) &= \vartheta - 2v + \mathcal{F}_{1,3}(\vartheta, v, \beta), \\ F_2(\vartheta, v, \beta) &= \frac{\theta_0^2}{2} + \theta_0 \vartheta + \beta^2 + (\vartheta - v)^2 \\ &\quad + v^2 + \mathcal{F}_{2,3}(\vartheta, v, \beta), \end{aligned} \tag{9}$$

where $\mathcal{F}_{i,3}(\vartheta, v, \beta)$ are converging power series beginning with the third power of their arguments with known coefficients.

Solving the first approximation system for (8) in view of expansions (9) yields

$$\begin{aligned} \vartheta_1 &= -\sqrt{\varepsilon} \langle y_0, l_{1,0} \rangle, \quad v_1 = \vartheta_1, \\ r_1 &= -\frac{2}{\theta_0} \sqrt{\varepsilon} y_0 + \frac{2}{\theta_0} \sqrt{\varepsilon} l_{1,0} \langle y_0, l_{1,0} \rangle. \end{aligned}$$

Moreover, $\tilde{l}_{2,1}$ satisfies the equation

$$2\tilde{l}_{2,1} \ln \frac{\theta_0}{\beta_1} = \sqrt{\varepsilon} (l_{1,0} \langle y_0, l_{1,0} \rangle - y_0) =: \sqrt{\varepsilon} v_0^{(6)} \neq 0.$$

Thus, $\beta_1 = \|\tilde{l}_{2,1}\|$ satisfies the equation $2\beta_1 \ln \frac{\theta_0}{\beta_1} = \sqrt{\varepsilon} \|v_0\|$. From this, denoting the inverse of the function $\delta \ln \frac{1}{\delta}$ by $W_0(\cdot)$, we obtain, for small δ ,

$$\beta_1 = W(\varepsilon) := \theta_0 W_0\left(\frac{\sqrt{\varepsilon} \|x_0\|}{2\theta_0}\right) = o(\sqrt{\varepsilon}),$$

$$\varepsilon \rightarrow 0 \text{ (see [9, 10]).}$$

Next, asymptotic expansions for all the indicated quantities are derived following the standard technique (see [9, 10]). They are similar in form to the expansions obtained in [9, 10]. Thus, the following result holds.

Theorem 2. *Under condition (6), the optimal time T_ε and the components of vectors $l_{1,\varepsilon}$ and $l_{2,\varepsilon}$ can be expanded in asymptotic series of the form $\sum_{k=0}^{\infty} R_k\left(\varepsilon, W(\varepsilon), \ln \frac{1}{W(\varepsilon)}\right)$,*

where $R_k(\cdot)$ are rational functions of their arguments and $R_k\left(\varepsilon, W(\varepsilon), \ln \frac{1}{W(\varepsilon)}\right) = O(\varepsilon^{k/2})$.

More specifically,

$$\theta_\varepsilon = \theta_0 - \sqrt{\varepsilon} \langle y_0, l_{1,0} \rangle - \frac{W^2(\varepsilon)}{\theta_0} + \sqrt{\varepsilon} W(\varepsilon) \frac{\|v_0\|}{\theta_0} + P\varepsilon + O(\varepsilon^2),$$

$$l_{1,\varepsilon} = l_{1,0} + \frac{2\sqrt{\varepsilon}}{\theta_0} v_0 + O(\varepsilon),$$

$$l_{2,\varepsilon} = \frac{\theta_0}{2} l_{1,0} - \sqrt{\varepsilon} y_0 + \frac{v_0}{\|v_0\|} W(\varepsilon) + O(\varepsilon), \quad \varepsilon \rightarrow 0,$$

where P is a given constant.

Thus, by virtue of formula (4), the asymptotic representation of the optimal control in problem (3) (and, hence, in problem (2)) has an irregular form up to $O(\varepsilon)$. For example, for $\left|-\tau + \frac{\theta_0}{2}\right| \geq \sqrt{\varepsilon}$, the optimal control has the asymptotic representation

$$U_{\text{opt}}(\tau) = \text{sgn}\left(-\tau + \frac{\theta_0}{2}\right) l_{1,0} + \frac{v_1 W(\varepsilon) - \tau \sqrt{\varepsilon} l_{1,1} - \sqrt{\varepsilon} y_0 + \langle l_{1,0}, \tau \sqrt{\varepsilon} l_{1,1} + \sqrt{\varepsilon} y_0 - v_1 W(\varepsilon) \rangle l_{1,0}}{\left|-\tau + \frac{\theta_0}{2}\right|} + O(\varepsilon)$$

as $\varepsilon \rightarrow 0$. Here, $l_{1,1} = \frac{2v_0}{\theta_0}$ and $a v_1 = \frac{v_0}{\|v_0\|}$.

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