Cybernetics and Systems Analysis, Vol. 49, No. 1, January, 2013

SPECTRAL CRITERION OF STOCHASTIC STABILITY FOR INVARIANT MANIFOLDS[1]

L. B. Ryashko[a†] and I. A. Bashkirtseva[a‡]

UDC 531.36

Abstract. *The mean square stability for invariant manifolds of nonlinear stochastic differential equations is considered. The stochastic stability analysis is reduced to the estimation of the spectral radius of some positive operator. For the important case of manifolds with codimension one, a constructive spectral analysis of this operator is carried out. On the basis of this spectral technique, parametrical criteria of the stochastic stability of limit cycle and 2-torus are developed.*

Keywords: *invariant manifolds, stochastic stability, spectral criterion.*

INTRODUCTION

Many phenomena observed in nonlinear dynamics when passing from order to chaos are often related to a chain of bifurcation modes: stationary (equilibrium point) – periodic (limit cycle) – quasiperiodic (torus) – chaotic (strange attractor). Compact invariant manifolds are a common mathematical model to analyze various dynamic conditions and passages in between. Each such passage is accompanied by buckling of a simple manifold and by the birth of a new, more complicated, stable manifold. The stability analysis of the corresponding invariant manifolds is a key to understanding the complicated behavior of nonlinear dynamic systems.

Studying compact invariant manifolds of dynamic systems draws attention of many scientists. The results of the stability analysis of manifolds under small perturbations and topological equivalence of two dynamic systems coinciding on a manifold are presented in [1–5]. Theory of random systems addresses various types of stochastic stability. One of the major fields in the stability analysis is the technique of Lyapunov functions, widely used by many authors to analyze the stochastic stability of equilibria (see, for example [6–11]).

The method of orbital Lyapunov functions with the use of quasipotential is applied in the stability and sensitivity analysis of stochastically perturbed limit cycles [12, 13]. This method was used to describe in detail the stochastic attractors in period-doubling bifurcations zone and to analyze the passages caused by the noise [14, 15].

The purpose of our study is to present a new general spectral criterion of exponential mean square stability (EMS stability) for stochastically perturbed invariant manifolds of nonlinear systems.

Below we present the necessary theoretical information from [16, 17] and introduce the concept of systems of linear expansion for invariant manifolds of deterministic systems and the term of *P*-stability.

It is rather difficult to analyze the stability by analyzing the solvability of the respective Lyapunov matrix equation, especially in near-critical cases. Efficient criteria for the stochastic stability of equilibrium for linear systems with constant coefficients are obtained in [18, 19] with the use of the spectral theory of positive operators [20] for the analysis of the EMS stability of general invariant manifolds. A parametric criterion of the EMS stability is found. The stochastic stability analysis is reduced to estimating the spectral radius of some positive operator.

A detailed spectral analysis of this operator for the important case of a manifold of codimension one is given. The developed general theory was applied in the EMS stability analysis of stochastic limit cycles of limit and invariant tori. Explicit parametric criteria are presented.

[1]The study was partially sponsored by AVTsP (Grant 1.1099.2011) and FTsP (Grant 14. A18.21.0364).

[a]Ural Federal University, Ekaterinburg, Russia, [†]*lev.ryashko@usu.ru*; [‡]*irina.bashkirtseva@usu.ru*. Translated from Kibernetika i Sistemnyi Analiz, No. 1, January–February, 2013, pp. 82–90. Original article submitted November 7, 2011.

EXPONENTIAL STABILITY OF INVARIANT MANIFOLDS

Let us consider an autonomous nonlinear system of ordinary differential equations

$$dx = f(x)dt,\ x \in \mathbf{R}^n, \tag{1}$$

where $f(x)$ is a sufficiently smooth vector function. System (1) is assumed to have a smooth compact invariant manifold $M \subset \mathbf{R}^n$ [1, 2, 5]. For some neighborhood U of the manifold M the function has the form

$$\gamma(x) = \arg\min_{y\in M} ||x-y||,\ \Delta(x) = x - \gamma(x),$$

where $||\cdot||$ is a Euclidean norm, $\gamma(x)$ is the nearest (to x) point of the manifold M, $\Delta(x) = x - \gamma(x)$ is the vector of the deviation of x from M. The neighborhood U is assumed to be invariant for (1). Function $\gamma(x)$ can generally be multivalued. Considering the stability problems, we cam assume the neighborhood U sufficiently small, the functions $\gamma(x)$ and $\Delta(x)$ being single-valued and smooth in U.

Definition 1. Invariant manifold M is called exponentially stable (E-stable) for system (1) in U if for some $K > 0$ and $l > 0$ for all $t > 0$ the condition $||\Delta(x(t))||^2 \le Ke^{-lt}||\Delta(x_0)||^2$ is satisfied, where $x(t)$ is the solution of system (1) with the initial condition $x(0) = x_0 \in U$.

The classical deterministic theory of stability of equilibria and limit cycles employs the structures of systems of the first approximation such as linear expansions for general invariant manifolds [3, 4].

For (1) the system of linear expansion has the form

$$\begin{aligned} dx &= f(x)\,dt, \quad x \in M, \\ dz &= F(x)z\,dt, \quad z \in \mathbf{R}^n, \end{aligned} \tag{2}$$

where $F(x) = \dfrac{\partial f}{\partial x}(x)$.

For each $x \in M$ denote by T_x the subspace tangent to M in x, and by N_x the orthogonal complement to T_x in $\mathbf{R}^n$. If $\dim M = s$, then $\dim T_x = s$ and $\dim N_x = n-s$. Let P_x be the operator of orthogonal complement on the subspace N_x.

Let us introduce a space Σ of symmetric $(n\times n)$-matrix functions $V(x)$, defined and sufficiently smooth on M, with the condition

$$\forall\, x \in M,\ \forall\, z \in T_x\ \ V(x)z = 0. \tag{3}$$

Considering (3), for elements $V \in \Sigma$ we have $\operatorname{rank} V(x) \le n-s$.

Definition 2. A matrix function $V(x) \in \Sigma$ is called P-positive definite if

$$\forall\, x \in M,\ \forall\, z \in \mathbf{R}^n\ \ P_x z \ne 0 \Rightarrow (z, V(x)z) > 0.$$

In the space Σ, consider a cone $K = \{V \in \Sigma \,|\, V(x)$ is nonnegative definite for all $x \in M\}$ and a set $K_P = \{V \in \Sigma \,|\, V$ is P-positive definite$\}$.

Definition 3. The system of linear expansion (2) is exponentially P-stable (in what follows, P-stable) if there exist $K > 0$ and $l > 0$ such that $||P_{x(t)}z(t)||^2 \le Ke^{-lt}\,||P_{x_0}z_0||^2$ for any $t > 0$, where $(x(t), z(t))$ is the solution of system (2) with the initial condition $(x(0), z(0)) = (x_0, z_0)$, $x_0 \in M$, $z_0 \in \mathbf{R}^n$.

Consider a Lyapunov matrix operator A defined by the formula

$$A[V] = \left(f, \frac{\partial V}{\partial x}\right) + F^{\mathrm{T}}V + VF. \tag{4}$$

Remark 1. Values of the operator $\left(f, \dfrac{\partial V}{\partial x}\right)$ for $V \in \Sigma$ at points of the manifold M are completely determined by the values of the function $V(x)$ on M. Indeed, for an arbitrary solution $x(t) \in M$ of system (1) we have

$$\left(f(x(t)), \frac{\partial V}{\partial x}(x(t))\right) = \frac{d}{dt}V(x(t)).$$

The following theorem from [16] is true. It reduces the problem of exponential stability of a manifold to the analysis of the problem of resolvability of Lyapunov equation.

THEOREM 1. The following statements are equivalent:

— the compact invariant manifold M of system (1) is E-stable;

— system (2) is P-stable;

— for any $W \in K_P$ there exists $V \in K_P$ satisfying the equation $A[V] = -W$.

Let us consider the stochastic stability.

A standard model in the stability analysis of the deterministic system (1) against the influence of random perturbations is a system of stochastic Ito equations [8]:

$$dx = f(x)\,dt + \sum_{r=1}^{m} \sigma_r(x)\,dw_r(t), \tag{5}$$

where $w_r(t)$ $(r = 1, \dots, m)$ are independent standard Wiener processes, $f(x)$ and $\sigma_r(x)$ are sufficiently smooth vector functions. For the manifold M to remain invariant for system (5) as well, we put

$$\sigma_r|_M = 0. \tag{6}$$

Let the neighborhood U be invariant for stochastic system (5) as well.

Definition 4. An invariant manifold M is called exponentially mean square stable (EMS stable) for system (5) in U if for some $K > 0$ and $l > 0$ the condition $\mathrm{E}||\Delta(x(t))||^2 \le Ke^{-lt}\ \mathrm{E}||\Delta(x_0)||^2$ is satisfied for all $t > 0$, where $x(t)$ is the solution of system (5) with the initial condition $x(0) = x_0 \in U$.

The stochastic linear expansion for the nonlinear stochastic system (5) with the invariant manifold M is

$$\begin{aligned} dx &= f(x)\,dt, & x \in M, \\ dz &= F(x)zdt + \sum_{r=1}^{m} S_r(x)zdw_r(t), & z \in \mathbf{R}^n, \end{aligned} \tag{7}$$

where $F(x) = \dfrac{\partial f(x)}{\partial x}$ and $S_r(x) = \dfrac{\partial \sigma_r(x)}{\partial x}$.

Because of (6) the matrix functions $S_r(x)$ are singular:

$$\forall\ x \in M,\ \forall\ z \in T_x\ \ S_r(x)z = 0. \tag{8}$$

Definition 5. The stochastic linear expansion (7) is called exponentially mean square P-stable (P-stable) if for some $K > 0$ and $l > 0$ the condition

$$\mathrm{E}||P_{x(t)}z(t)||^2 \le Ke^{-lt}\mathrm{E}||P_{x_0}z_0||^2$$

is satisfied for all $t > 0$, where $(x(t), z(t))$ is the solution of (7) with the initial conditions $(x(0), z(0)) = (x_0, z_0)$, $x_0 \in M$, $z_0 \in \mathbf{R}^n$.

Let us consider the Lyapunov operator L for system (7):

$$L[V] = \left(f, \frac{\partial V}{\partial x}\right) + F^{\mathrm{T}}V + VF + \sum_{r=1}^{m} S_r^{\mathrm{T}} V S_r. \tag{9}$$

Let us consider the theorem that establishes the equivalence of the EMS stability of stochastically perturbed manifolds and the problem of solvability of the corresponding Lyapunov equation.

THEOREM 2. The statements below are equivalent:

— the compact invariant manifold M of system (5) is EMS-stable;

— system (7) is P-stable;

— for any $W \in K_P$ there exists $V \in K_P$ satisfying the equation

$$L[V] = -W. \tag{10}$$

The proof of the theorem is presented in [17].

Theorem 2 reduces the stochastic stability analysis of the manifold M to the analysis of the solvability of Eq. (10) in the space K_P of P-positive definite matrices.

It is rather difficult to analyze the stability by analyzing the solvability of this matrix Lyapunov equation, especially in near-critical cases. Let us consider a generalized case of efficient criteria [18, 19], which employs the spectral theory of positive operators [20].

Let us present the operator L from (9) as the sum $L = A + S$, where the operator A is defined in (4) and the operator S is defined on elements of the space Σ by the equality $S[V] = \sum_{r=1}^{m} S_r^{\mathrm{T}} V S_r$. Equation (10) can be written as

$$A[V] + S[V] = -W. \tag{11}$$

By virtue of Theorem 1, for the P-stable deterministic system (2) there exists the inverse operator A^{-1}, which is negative with respect to the cone K. Multiplying (11) by A^{-1}, we obtain

$$V - P[V] = -A^{-1}[W], \tag{12}$$

where the operator $P = -A^{-1}S$ is also positive as the product of two positive operators: $-A^{-1}$ and S.

For the space Σ (see (3)) with the norm $||V|| = \max_{x \in M} \sqrt{\mathrm{tr}(V^2(x))}$ the cone K is normal and bodily.

The P-stability analysis of stochastic system (7) can be reduced to estimating the spectral radius $\rho(P)$ of the operator P.

THEOREM 3. The compact invariant manifold M is EMS-stable for the stochastic system (5) if and only if the manifold M of the deterministic system (1) is E-stable and

$$\rho(P) < 1. \tag{13}$$

Proof. Necessity. Let the manifold M of the stochastic system (5) be EMS-stable. Then according to Theorems 1 and 2 the manifold M of the deterministic system (1) is E-stable and there exists operator A^{-1}, and for any $W \in K_P$ there exists $-A^{-1}[W] \in K_P$ and Eq. (12) is true, which yields

$$V - P[V] \in K_P. \tag{14}$$

The cone $K \subset \Sigma$ is normal and bodily. Then (14) (Theorem 16.7 in [20]) yields inequality (13).

Sufficiency. Let the manifold M of the deterministic system (1) be E-stable. This guarantees the existence of the operators A^{-1} and $P = -A^{-1}S$. Consider the operator $R[V] = V - P[V]$. By virtue of (13), there exists the inverse operator $R^{-1} = \sum_{k=0}^{\infty} P^k$.

The operator R is positive. This means that for any $W \in K_P$ the matrix $V = R^{-1}[-A^{-1}[W]] \in K_P$ is the solution of (12). Since (10) and (12) are equivalent, the matrix $V \in K_P$ satisfies Eq. (10), which means (see Theorem 2) that the manifold M is EMS-stable.

Remark 2. The spectral radius $\rho = \rho(P) \neq 0$ determines the bifurcation value $\varepsilon^* = \sqrt{1/\rho}$ of the intensity $\varepsilon \geq 0$ of random perturbations for the system

$$dx = f(x)dt + \varepsilon \sum_{r=1}^{m} \sigma_r(x) dw_r(t). \tag{15}$$

The manifold M for system (15) is EMS-stable for all $\varepsilon < \varepsilon^*$ and unstable for $\varepsilon \geq \varepsilon^*$. The case $\rho = 0$ means that system (15) is EMS-stable for all $\varepsilon \geq 0$.

Remark 3. If the case where it is difficult to find exactly the spectral radius ρ, its two-sided estimates are of interest: $\rho_1 < \rho < \rho_2$. Indeed, the inequality $\rho_2 < 1$ provides the sufficient, and $\rho_1 < 1$ the necessary stability condition. The difference $\rho_2 - \rho_1$ may be a measure of the robustness of these stability conditions.

SPECTRAL ANALYSIS OF OPERATOR *P* FOR MANIFOLDS OF CODIMENSION ONE

Let us consider the case where the manifold M has dimension $\dim(M) = n-1$ $(\operatorname{codim}(M) = 1)$. Here, $\dim(N_x) = 1$, $\operatorname{rank}(P_x) = 1$, $\operatorname{rank}(S_r(x)) \le 1$ and factorization is true for the matrices P_x and $S_r(x)$: $P_x = p(x)p^{\mathrm{T}}(x)$, $S_r(x) = g_r(x)p^{\mathrm{T}}(x)$, $g_r(x) = S_r(x)p(x)$, where $p(x)$ and $g_r(x)$ are n-vector functions defined on M and $p(x)$ is normalized and orthogonal to M at the point x. Because of this factorization, operator S can be written as

$$S[V] = \sum_{r=1}^{m} pg_r^{\mathrm{T}} Vg_r p^{\mathrm{T}} = \sum_{r=1}^{m} g_r^{\mathrm{T}} Vg_r pp^{\mathrm{T}}. \tag{16}$$

It follows from (16) that

$$S[V] = g[V]P, \tag{17}$$

where $g[V] = \sum_{r=1}^{m} g_r^{\mathrm{T}} Vg_r$ is a scalar function. Operator $P = -A^{-1}S$ can be written as

$$P[V] = -A^{-1}[g[V]P]. \tag{18}$$

Along with P let us consider an operator B:

$$B[\varphi] = -g[A^{-1}[\varphi P]], \tag{19}$$

where $\varphi \in \Sigma^1$ and Σ^1 is the space of scalar functions $\varphi(x)$ defined and sufficiently smooth on the manifold M. In the space Σ^1 consider a cone $K^1 = \{\varphi \in \Sigma^1 \,|\, \varphi(x) \ge 0\}$ and its interior $K_P^1 = \{\varphi \in \Sigma^1 \,|\, \varphi(x) > 0\}$. The operator B is positive on K^1.

LEMMA 1. The operators P and B have identical spectral radii: $\rho(P) = \rho(B)$.

Proof. Let $\rho = \rho(P)$ be the eigenvalue corresponding to the eigenfunction V of the operator P:

$$P[V] = \rho V. \tag{20}$$

The equalities $-A^{-1}[g[V]P] = \rho V$ and $-g[A^{-1}[g[V]P]] = \rho g[V]$ follow from (18) and (20). For $\varphi = g[V]$ we have $-g[A^{-1}[\varphi P]] = \rho\varphi$.

This equality and (19) yield $B[\varphi] = \rho\varphi$. This means that ρ is the eigenvalue, and φ is the eigenfunction of the operator B. Hence, $\rho(B) \ge \rho(P)$.

Let us prove the opposite inequality. Let $\rho = \rho(B)$ be the eigenvalue, and φ be the eigenfunction of the operator B:

$$B[\varphi] = \rho\varphi. \tag{21}$$

As follows from (19) and (21), $-g[A^{-1}[\varphi P]] = \rho\varphi$. Take $V = -A^{-1}[\varphi P]$. Then $P[V] = -A^{-1}S[-A^{-1}[\varphi P]] = \rho[-A^{-1}[\varphi P]] = \rho[V]$. This means that ρ is the eigenvalue, and V is the eigenfunction of the operator P. Hence, $\rho(B) \le \rho(P)$. Thus, $\rho(P) = \rho(B)$. The lemma is proved.

Due to this lemma, in the analysis of the EMS-stability of the manifold M in the case of $\dim(M) = n-1$ the operator P in Theorem 3 can be replaced with a simpler operator B. This makes the spectral analysis of stochastic stability more constructive.

Using the matrix function $V = -A^{-1}[\varphi P]$, we can write the spectral equation $B[\varphi] = \rho\varphi$ as a system

$$g[V] = \rho\varphi,$$

$$A[V] = -\varphi P. \tag{22}$$

If $\dim(M) = n-1$, the matrix $V(x)$ has the following factorization: $V(x) = \mu(x)P_x$, where $\mu(x)$ is a scalar function. For μ and ρ (22) yields

$$\rho A[\mu P] + \mu g[P]P = 0. \tag{23}$$

STABILITY OF LIMIT CYCLES

Let the invariant manifold M be a limit cycle corresponding to the T-periodic solution $\xi(t)$ of system (1). Consider the functions

$$F(t)=\frac{\partial f}{\partial x}(\xi(t)),\ S_r(t)=\frac{\partial \sigma_r}{\partial x}(\xi(t)),\ V(t)=V(\xi(t)),\ P(t)=P_{\xi(t)},\ p(t)=p(\xi(t)),$$

defined on $[0,T]$. In this case, Σ is the space of T-periodic symmetric $(n\times n)$-matrices $V(t)$ with the singularity condition $V(t)f(\xi(t))=0$.

For $n=2$ the EMS-stability analysis of the limit cycle M can be carried out based on Theorem 3 and Lemma 1. In this case, $V(t)=\mu(t)P(t)$. Using Remark 1, we can rearrange the spectral equation (23) as

$$\rho[\dot{\mu}P+\mu\dot{P}+\mu(F^{\mathrm{T}}P+PF)]+\mu\beta P=0, \tag{24}$$

where $\beta=g(P)$. Premultiplying (24) by p^{T}, postmultiplying it by p, and considering the identities $p^{\mathrm{T}}Pp\equiv 1$ and $p^{\mathrm{T}}\dot{P}p\equiv 0$, we obtain the equation

$$\rho(\dot{\mu}+\alpha\mu)+\beta\mu=0, \tag{25}$$

where $\alpha=p^{\mathrm{T}}(F^{\mathrm{T}}+F)p$. Dividing Eq. (25) by $\mu\neq 0$ and integrating from $t=0$ to $t=T$, we obtain $\rho=-\frac{<\beta>}{<\alpha>}$, where $<\varphi>=\int_0^T\varphi(t)dt$. The inequality

$$<\alpha><0 \tag{26}$$

is the necessary and sufficient condition of the E-stability of the limit cycle M for the deterministic system (1). Because of the equality $<\alpha>=2<\mathrm{tr}F>$, condition (26) is equivalent to the Poincaré criterion $\lambda=\frac{1}{T}\int_0^T\mathrm{tr}F(t)dt<0$ for $n=2$, where λ is the Lyapunov exponent. For the two-dimensional case we have $\beta(t)=\mathrm{tr}\left(\sum_{r=1}^{m}S_r(t)S_r^{\mathrm{T}}(t)\right)$.

Thus, the inequality $\rho(P)<1$ from Theorem 3 can be rearranged as

$$<\alpha+\beta>=\int_0^T\mathrm{tr}[2F(t)+\sum_{r=1}^{m}S_r(t)S_r^{\mathrm{T}}(t)]dt<0.$$

This criterion is a natural generalization of the classical Poincare criterion for the stochastic case.

STABILITY OF 2-TORI

Let the invariant manifold M of system (1) for $n=3$ be a two-dimensional toroidal surface. Let us consider the following parametrization.

Let a closed sufficiently smooth curve (equator) θ lie on M (Fig. 1) . The curve is defined by the function $\theta(s)$ on the interval $0\le s\le 1$ with the condition $\theta(0)=\theta(1)$, $a=x(0,s)=\theta(s)$ is the initial point of the solution $x(t,s)$, $b=x(T(x),s)=\theta(\tau(s))$ is the point of the first return of the solution $x(t,s)$ to the curve θ. From each point $\theta(s)$ of the curve θ as the initial one, we have the solution $x(t,s)$ of system (1) with the condition $x(0,s)=\theta(s)$. We assume that the trajectory $x(t,s)$ circles the torus M and then crosses the curve θ again. Let $T(s)=\min\{t>0\,|\,x(t,s)\in\theta\}$ be the time of the first return of the trajectory $x(t,s)$ to the curve θ and $x(T(s),s)$ be the point of return. Let $\tau(s)$ be a point from the interval $[0,1)$ such that $\theta(\tau(s))=x(T(s),s)$, where $\tau(s)$ is the function of tracing the Poincaré sections of the curve θ by the phase trajectories of system (1).

We assume that the phase trajectories of the family of solutions $x(t,s)$ of system (1) completely cover M. The toroidal surface may consist of closed phase trajectories (cycles) and of trajectories converging to them as well as of the family of open trajectories lying everywhere dense on M (quasiperiodic case). Function $x(t,s)$ establishes a biunique correspondence between points of the 2-torus M and points of the set $D=\{(t,s)\,|\,0\le t<T(s),\,0\le s<1\}$. Vector functions $\frac{\partial x(t,s)}{\partial t}$ and $\frac{\partial x(t,s)}{\partial s}$ are linearly independent. For each point $\gamma\in M$, it is possible to find $t=t(\gamma)$ and $s=s(\gamma)$ such that $x(t,s)=\gamma$.

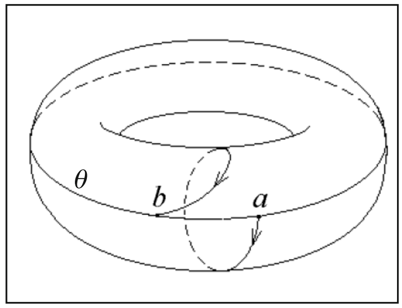

Fig. 1

Using the parametrization of the 2-torus M related to the family of solutions $x(t,s)$, let us introduce functions

$$F(t,s)=\frac{\partial f}{\partial x}(x(t,s)),\ S_r(t,s)=\frac{\partial \sigma_r}{\partial x}(x(t,s)),$$

$$V(t,s)=V(x(t,s)),\quad P(t,s)=P(x(t,s)),\ p(t,s)=p(x(t,s)),$$

defined on D. The equalities $x(t,s+1)=x(t,s)$ and $x(T(s)+t,s)=x(t,\tau(s))$ allow extending these functions to the whole plane $\Pi=\{(t,s)|-\infty<t<+\infty,-\infty<s<+\infty\}$.

In the case under consideration, $V(t,s)=\mu(t,s)P(t,s)$. Using Remark 1, we can rearrange the spectral equation (23) as

$$\rho\left[\frac{\partial \mu}{\partial t}P+\mu\frac{\partial P}{\partial t}+\mu(F^{\mathrm{T}}P+PF)\right]+\mu\beta P=0, \tag{27}$$

where

$$\beta(t,s)=p^{\mathrm{T}}(t,s)\left(\sum_{r=1}^{m}S_r^{\mathrm{T}}(t,s)P(t,s)S_r(t,s)\right)p(t,s).$$

Premultiplying (27) by p^{T}, postmultiplying it by p, and considering $p^{\mathrm{T}}Pp\equiv 1$ and $p^{\mathrm{T}}\frac{\partial P}{\partial t}p\equiv 0$, we obtain the equation

$$\rho\left(\frac{\partial \mu}{\partial t}+\alpha\mu\right)+\beta\mu=0, \tag{28}$$

where $\alpha(t,s)=p^{\mathrm{T}}(t,s)(F^{\mathrm{T}}(t,s)+F(t,s))p(t,s)$.

Let $<\varphi>=\lim_{T\to\infty}\frac{1}{T}\int_0^T\varphi(t)dt$. Dividing Eq. (28) by $\mu\neq 0$ and using the equality

$$<\frac{\partial \mu}{\partial t}/\mu>=\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{\partial \mu}{\partial t}/\mu dt=0,$$

we obtain the explicit formula $\rho=-\frac{<\beta>}{<\alpha>}$. After averaging with respect to t, the value of ρ depends on $s:\rho=\rho_s$. Hence, $\rho(P)=\max_s\left\{\frac{<\beta>}{<\alpha>}\right\}$.

The inequality $\max_s<\alpha>\,<\,0$ is the necessary and sufficient condition of the exponential stability of the 2-torus M for the deterministic system (1).

The criterion $\rho(P)<1$ from Theorem 3 can be written as

$$\max_s<\alpha+\beta>=\max_s\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathrm{tr}[2F(t,s)+\sum_{r=1}^{m}S_r(t,s)S_r^{\mathrm{T}}(t,s)]dt<0.$$

The resultant inequality is the necessary and sufficient condition of the EMS-stability of the 2-torus M for the stochastic system (2) for $n = 3$.

CONCLUSIONS

The exponential stability analysis of the manifold has been reduced to the solvability analysis of the Lyapunov equation. However, a direct solvability analysis is rather toilful, especially in near-critical cases. The key result of the present study is the spectral algebraic criterion of the EMS-stability of general invariant manifolds obtained with the use of the theory of positive operators. A detailed spectral analysis has been carried out for the important case of manifold of codimension one. The developed general theory has been applied to the EMS-stability analysis of stochastic limit cycles and invariant tori.

REFERENCES

1. A. M. Samoilenko, Elements of the Mathematical Theory of Multifrequency Oscillations. Invariant Tori [in Russian], Nauka, Moscow (1987).
2. S. Wiggins, Normally Hyperbolic Invariant Manifolds in Dynamical Systems, Springer-Verlag, Berlin (1994).
3. U. Kirchgaber and K. J. Palmer, Geometry in the Neighborhood of Invariant Manifolds of Maps and Flows and Linearization, Longman, New York (1990).
4. I. U. Bronstein and A. Ya. Kopanskii, Smooth Invariant Manifolds and Normal Forms, World Sci., Singapore (1994).
5. N. Fenichel, "Persistence and smoothness of invariant manifolds for flows," Indiana Univer. Math. J., No. 2, 193–226 (1971).
6. I. Ya. Kats and N. N. Krasovsky, "On the stability of systems with random parameters," Prikl. Mat. Mekh., **24**, Issue 5, 809–823 (1960).
7. H. J. Kushner, Stochastic Stability and Control, Acad. Press, New York (1967).
8. R. Z. Khas'minskii, Stability of Systems of Differential Equations under Random Perturbations of their Parameters [in Russian], Nauka, Moscow (1969).
9. D. G. Korenevskii, Stability of Dynamic Systems under Random Perturbations of Parameters. Algebraic Criteria [in Russian], Naukova Dumka, Kyiv (1989).
10. X. Mao, Exponential Stability of Stochastic Differential Equations, Dekker, Marcel (1994).
11. L. Arnold, Random Dynamical Systems, Springer, Berlin (1998).
12. L. B. Ryashko, "The stability of stochastically perturbed orbital motions," J. Appl. Math. Mech., **60**, Issue 4, 579–590 (1996).
13. I. A. Bashkirtseva and L. B. Ryashko, "Stochastic sensitivity of 3D-cycles," Math. and Comput. in Simulation, No. 66, 55–67 (2004).
14. L. Ryashko, I. Bashkirtseva, A. Gubkin, and P. Stikhin, "Confidence tori in the analysis of stochastic 3D-cycles," Math. and Comput. in Simulation, No. 80, 256–269 (2009).
15. I. Bashkirtseva and L. Ryashko, "Constructive analysis of noise-induced transitions for coexisting periodic attractors of Lorenz model," Phys. Rev. E, No. 79, 041106–041114 (2009).
16. L. B. Ryashko and E. E. Shnol, "On exponentially attracting invariant manifolds of ODEs," Nonlinearity, No. 16, 147–160 (2003).
17. L. B. Ryashko, Exponential mean square stability of stochastically forced invariant manifolds for nonlinear SDE's," Stochastics and Dynamics, No. 7, 389–401 (2007).
18. L. B. Riashko, "Stabilization of linear stochastic systems with state and control dependent perturbations," J. Appl. Math. Mech., No. 43, 655–663 (1979).
19. L. B. Riashko, "Stabilization of linear systems with multiplicative perturbations and incomplete information," J. Appl. Math. Mech., No. 45, 581–587 (1981).
20. M. A. Krasnosel'skii, E. A. Lifshits, and A. V. Sobolev, Pozitive Linear Systems [in Russian], Nauka, Moscow (1985).