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Asymptotic Representation of a Solution to a Singular Perturbation Linear Time-Optimal Problem

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Abstract—A time-optimal control problem is considered for a linear system with fast and slow variables and smooth geometric constraints on the control. An asymptotic expansion of the optimal time up to the second order of smallness is constructed and validated.
Keywords: optimal control, time-optimal control problem, asymptotic expansion, singular perturbation problems, small parameter.

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Dedicated to A.M. Il'in,
our Teacher

INTRODUCTION

We consider the time-optimal problem [1–4] for a linear system with fast and slow variables [5–9] and smooth geometric constraints on the control. An asymptotic representation of the optimal time is constructed and validated. In the present paper, we use the methods developed in [10–13].

Other statements of singularly perturbed control problems are given in [6,7,14–16]. The systems investigated in the present paper and also in [14, 15] do not belong to the class of systems studied in [16], since they do not satisfy the assumptions made in [16].

1. PROBLEM STATEMENT

Consider the time-optimal problem for a linear autonomous system with fast and slow variables in the class of piecewise continuous controls with smooth geometric constraints:

$$\begin{cases} \dot{y} = A_0 y + A_{12} z + B_1 u, \\ \varepsilon \dot{z} = \quad - \alpha z + B_2 u, \qquad \alpha > 0, \end{cases} \tag{1.1}$$

$$U \colon \|u\| \le 1. \tag{1.2}$$

Here and elsewhere, $\|\cdot\|$ is the Euclidean norm,

$$y(0) = y^0, \qquad z(0) = z^0, \tag{1.3}$$

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$$y(\Theta) = 0, \qquad z(\Theta) = 0, \qquad \Theta \longrightarrow \min, \tag{1.4}$$

$\varepsilon > 0$ is a small parameter, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $u \in \mathbb{R}^r$, and $m \le r < n$.

Condition 1. $\mathrm{Ker} B_2^* = \{0\}$ (or $\mathrm{Im} B_2 = \mathbb{R}^m$).

Degenerate problem (for $\varepsilon = 0$):

$$\dot y = A_0 y + B_0 u, \qquad U\colon \|u\| \le 1, \tag{1.5}$$

$$y(0) = y^0, \qquad y(\Theta_0) = 0, \qquad \Theta_0 \longrightarrow \min, \tag{1.6}$$

where $B_0 = B_1 + \alpha^{-1} A_{12} B_2$.

Condition 2. $\mathrm{rank}\,[B_0, A_0 B_0, \ldots, A_0^{n-1} B_0] = n$.

Conditions 1 and 2 provide the complete controllability of the pairs $(-\alpha I, B_2)$ and (A_0, B_0), respectively.

Condition 3. $\mathrm{rank}\, B_0 = r \in [2, n-1]$.

Condition 4. Let a pair (A_0, B_0) be such that, if $B_0^* e^{A_0^* t} r_1 \parallel B_0^* e^{A_0^* t} r_2$ on some interval, then $r_1 \parallel r_2$. Here, * means transposition.

Note that Condition 2 follows from Condition 4; nevertheless, Condition 2 is given here because it is a known condition, under which the maximum principle is a sufficient condition for the optimality of the control. Condition 4, as shown in [15], provides the uniqueness of the representation of an optimal control in the degenerate problem in terms of the initial vector of the adjoint system.

Condition 5. The initial vector y^0 is such that problem (1.5), (1.6) is solvable.

Introduce the notation

$$x = \begin{pmatrix} y \\ z \end{pmatrix}, \qquad x^0 = \begin{pmatrix} y^0 \\ z^0 \end{pmatrix}, \qquad A_\varepsilon = \begin{pmatrix} A_0 & A_{12} \\ 0 & -\dfrac{\alpha}{\varepsilon} I \end{pmatrix}, \qquad B_\varepsilon = \begin{pmatrix} B_1 \\ \dfrac{1}{\varepsilon} B_2 \end{pmatrix},$$

where I is the identity matrix.

As shown in [5], if Conditions 1 and 2 are satisfied, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \le \varepsilon_0$, the pair $(A_\varepsilon, B_\varepsilon)$ is completely controllable.

In [15], it is proved that, for any x^0, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, problem (1.1)–(1.4) is solvable and

$$\Theta(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \Theta_0, \tag{1.7}$$

where Θ_0 is the optimal time in degenerate problem (1.5), (1.6).

Define $\vartheta = \vartheta(\varepsilon) = \Theta_\varepsilon - \Theta_0$. In view of (1.7), we have

$$\vartheta(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{1.8}$$

In view of the complete controllability of the system from (1.5), which is equivalent to Condition 2, and of the form of constraints on the control, Pontryagin's maximum principle [1] is a necessary and sufficient condition for the optimality of the control. The adjoint system has the form $\dot\psi_0 = -A_0^* \psi_0$. Therefore, $\psi_0(t) = e^{A_0^*(\Theta_0 - t)} \lambda_0$, where λ_0 is a constant vector. According to the maximum principle, the optimal control $u_0(t)$ satisfies the relation

$$\langle \psi_0(t), B_0 u_0(t) \rangle = \max_{\|u\| \le 1} \langle \psi_0(t), B_0 u \rangle = \max_{\|u\| \le 1} \langle B_0^* e^{A_0^*(\Theta_0 - t)} \lambda_0, u \rangle = \|B_0^* e^{A_0^*(\Theta_0 - t)} \lambda_0\|.$$

Here and elsewhere, $\langle\cdot,\cdot\rangle$ is the scalar product in the corresponding finite-dimensional space. Then, for t such that $B_0^* e^{A_0^*(\Theta_0-t)}\lambda_0 \neq 0$, the optimal control in the degenerate problem has the form

$$u_0(t) = \frac{B_0^* e^{A_0^*(\Theta_0-t)}\lambda_0}{\|B_0^* e^{A_0^*(\Theta_0-t)}\lambda_0\|}. \tag{1.9}$$

Definition. Any vector λ_0 satisfying (1.9) will be called a vector *generating the optimal control*.

It is known [4, p. 171] that the optimal control in problem (1.5), (1.6) is unique. From (1.5), (1.6), and (1.9), we obtain

$$0 = e^{A_0\Theta_0}y^0 + \int\limits_0^{\Theta_0} \frac{e^{A_0(\Theta_0-t)}B_0\,B_0^*\,e^{A_0^*(\Theta_0-t)}\lambda_0}{\|B_0^* e^{A_0^*(\Theta_0-t)}\lambda_0\|}dt. \tag{1.10}$$

Changing the integration variable by the formula $\tau = \Theta_0 - t$, we come to the relation

$$0 = e^{A_0\Theta_0}y^0 + \int\limits_0^{\Theta_0} \frac{C_0(\tau)\lambda_0}{\langle C_0(\tau)\lambda_0,\lambda_0\rangle^{1/2}}d\tau, \qquad C_0(\tau) = e^{A_0\tau}B_0\,B_0^*\,e^{A_0^*\tau}, \tag{1.11}$$

which is equivalent to (1.10). Thus, the vector λ_0 is a vector generating the optimal control if and only if λ_0 satisfies (1.11).

Define also $\psi(\tau) = \langle C_0(\tau)\lambda_0,\lambda_0\rangle$. As shown in [10], there exists a vector y^0 such that the corresponding optimal control in the degenerate problem $u_0(t)$ has a unique discontinuity point $\overline{t}\in(0,\Theta_0)$, where

$$B_0^* e^{A_0^*(\Theta_0-\overline{t})}\lambda_0 = 0, \qquad B_0^*\,A_0^*\,e^{A_0^*(\Theta_0-\overline{t})}\lambda_0 \neq 0, \qquad \text{and} \qquad \forall t\neq\overline{t}\quad B_0^* e^{A_0^*(\Theta_0-t)}\lambda_0 \neq 0. \tag{1.12}$$

In what follows, we assume that the initial vector y^0 satisfies conditions (1.12). Note that the condition of a unique discontinuity point of the optimal control in the degenerate problem is not essential. Since the function $B_0^* e^{A_0^*(\Theta_0-t)}\lambda_0$ is analytic, it can have only a finite number of zeros. The analysis of an optimal control with a finite number of discontinuity points is similar to the case of one discontinuity point but is more complicated from the technical point of view.

Define $\overline{\tau} = \Theta_0 - \overline{t}$. Without loss of generality, assume that

$$e^{A_0\overline{\tau}}B_0 = \begin{pmatrix} I_r \\ 0 \end{pmatrix}_{n\times r}; \qquad \text{thus,} \qquad Q = C_0(\overline{\tau}) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n\times n}. \tag{1.13}$$

2. BASIC RELATIONS

Consider original problem (1.1)–(1.4). Relations (1.9), (1.10) are also satisfied for the perturbed problem with A_0, B_0, Θ_0, y^0, $u_0(t)$, and λ_0 replaced by A_ε, B_ε, Θ_ε, x^0, $u_\varepsilon(t)$, and r_ε, respectively: $0 = e^{A_\varepsilon\Theta_\varepsilon}x^0 + \int_0^{\Theta_\varepsilon} \dfrac{e^{A_\varepsilon(\Theta_\varepsilon-t)}B_\varepsilon\,B_\varepsilon^*\,e^{A_\varepsilon^*(\Theta_\varepsilon-t)}r_\varepsilon}{\|B_\varepsilon^* e^{A_\varepsilon^*(\Theta_\varepsilon-t)}r_\varepsilon\|}\,dt$. Changing the integration variable by the formula $\tau = \Theta_\varepsilon - t$, we get

$$0 = e^{A_\varepsilon\Theta_\varepsilon}x^0 + \int\limits_0^{\Theta_\varepsilon} \frac{e^{A_\varepsilon\tau}B_\varepsilon\,B_\varepsilon^*\,e^{A_\varepsilon^*\tau}r_\varepsilon}{\|B_\varepsilon^* e^{A_\varepsilon^*\tau}r_\varepsilon\|}\,d\tau. \tag{2.1}$$

Equation (2.1) is positively homogeneous with respect to the vector r_ε; hence, we assume that $\|r_\varepsilon\| = 1$. The following lemma was proved in [15].

Lemma 1 [15]**.** *Let the pair* (A_0, B_0) *satisfy Condition* 4. *If* l_1 *and* l_2 *are vectors generating the optimal control in problem* (1.5), (1.6) *and* $\|l_1\| = \|l_2\| = 1$, *then* $l_1 = l_2$. □

Thus, if Condition 4 is satisfied, the normed vector r_0 generating the optimal control in limit problem (1.5), (1.6) is unique and $\|r_\varepsilon\| = 1 \implies r_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} r_0$, $\|r_0\| = 1$. In addition (see [15]), if $r_\varepsilon = \begin{pmatrix} r_\varepsilon^{(1)} \\ r_\varepsilon^{(2)} \end{pmatrix}$ and $r_0 = \begin{pmatrix} r_0^{(1)} \\ r_0^{(2)} \end{pmatrix}$, we have

$$r_0^{(2)} = 0, \qquad \|r_0^{(1)}\| = 1, \tag{2.2}$$

and $r_0^{(1)}$ is a vector generating the optimal control in degenerate problem (1.5), (1.6). Thus, $u_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} u_0(t)$ uniformly on all closed sets not containing discontinuity points of the limit control.

Write equations (2.1) in the form

$$0 = \overline{y}_\varepsilon + \overline{z}_\varepsilon + \int\limits_0^{\Theta_\varepsilon} \frac{D_\varepsilon(\tau)\left(D_\varepsilon^*(\tau) r_\varepsilon^{(1)} + \frac{1}{\varepsilon}\, e^{-\alpha\frac{\tau}{\epsilon}} B_2^* r_\varepsilon^{(2)}\right)}{\left\|D_\varepsilon^*(\tau) r_\varepsilon^{(1)} + \frac{1}{\varepsilon}\, e^{-\alpha\frac{\tau}{\epsilon}} B_2^* r_\varepsilon^{(2)}\right\|}\, d\tau, \tag{2.3a}$$

$$0 = e^{-\alpha\frac{\Theta_\varepsilon}{\epsilon}} z^0 + \frac{1}{\varepsilon}\int\limits_0^{\Theta_\varepsilon} e^{-\alpha\frac{\tau}{\epsilon}} B_2\, \frac{D_\varepsilon^*(\tau) r_\varepsilon^{(1)} + \frac{1}{\varepsilon}\, e^{-\alpha\frac{\tau}{\epsilon}} B_2^* r_\varepsilon^{(2)}}{\left\|D_\varepsilon^*(\tau) r_\varepsilon^{(1)} + \frac{1}{\varepsilon}\, e^{-\alpha\frac{\tau}{\epsilon}} B_2^* r_\varepsilon^{(2)}\right\|}\, d\tau, \tag{2.3b}$$

where

$$\overline{y}_\varepsilon = e^{A_0\Theta_\varepsilon} y^0, \qquad \overline{z}_\varepsilon = \varepsilon(\alpha I + \varepsilon A_0)^{-1}\left(e^{A_0\Theta_\varepsilon} - e^{-\alpha\frac{\Theta_\varepsilon}{\epsilon}} I\right) A_{12}\, z^0,$$

$$D_\varepsilon(\tau) = e^{A_0\tau} B_1 + (\alpha I + \varepsilon A_0)^{-1}\left(e^{A_0\tau} - e^{-\alpha\frac{\tau}{\epsilon}} I\right) A_{12}\, B_2. \tag{2.4}$$

We will seek the vector $r_\varepsilon^{(2)}$ in the form

$$r_\varepsilon^{(2)} = \varepsilon(\, r_0^{(2)} + r^{(2)}(\varepsilon)), \qquad r^{(2)}(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{2.5}$$

Using (2.4), transform the expression

$$D_\varepsilon^*(\tau) r_\varepsilon^{(1)} + \frac{1}{\varepsilon}\, e^{-\alpha\frac{\tau}{\epsilon}} B_2^* r_\varepsilon^{(2)} = B_0^* e^{A_0^*\tau}\lambda_\varepsilon - \frac{\varepsilon}{\alpha} B_2^* A_{12}^* (\alpha I + \varepsilon A_0^*)^{-1} A_0^* e^{A_0^*\tau}\lambda_\varepsilon + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu_\varepsilon, \tag{2.6}$$

where $\lambda_\varepsilon = r_\varepsilon^{(1)}$ and $\nu_\varepsilon = -A_{12}^*(\alpha I + \varepsilon A_0^*)^{-1} r_\varepsilon^{(1)} + 1/\varepsilon\, r_\varepsilon^{(2)}$. Note that the vectors $r_\varepsilon^{(1)}$ and $r_\varepsilon^{(2)}$ are uniquely recovered from λ_ε and ν_ε. Now, let λ_ε and ν_ε be new unknown vectors. Define

$$\lambda(\varepsilon) = \lambda_\varepsilon - \lambda_0, \qquad \nu(\varepsilon) = \nu_\varepsilon - \nu_0, \tag{2.7}$$

where $\nu_0 = r_0^{(2)} - 1/\alpha\, A_{12}^*\lambda_0$. By relations (2.2), (2.5), we have $\lambda(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$ and $\nu(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$. Thus, the problem reduces to finding the asymptotics of $\lambda(\varepsilon)$, $\nu(\varepsilon)$ from (2.7) and of $\vartheta(\varepsilon)$ from (1.8) as $\varepsilon \to 0$.

For this, we first study the asymptotics of the integrals in (2.3) as $\varepsilon \to 0$. In expanding the corresponding integrands, we take into account the boundary-layer character of the values in these functions as well as the appearance of singularities for $\tau = \overline{\tau}$ in (2.3a) because of conditions (1.12). We have

$$D_\varepsilon^*(\tau) r_\varepsilon^{(1)} + \frac{1}{\varepsilon} e^{-\alpha\frac{\tau}{\epsilon}} B_2^* r_\varepsilon^{(2)} = B_0^* e^{A_0^*\tau}\lambda_0 + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu_0 + B_0^* e^{A_0^*\tau}\lambda(\varepsilon) + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu(\varepsilon) + R(\tau,\varepsilon,\lambda(\varepsilon)),$$

where $R(\tau,\varepsilon,\lambda(\varepsilon)) = -\varepsilon/\alpha\, B_2^* A_{12}^*(\alpha I + \varepsilon A_0^*)^{-1} A_0^* e^{A_0^*\tau}(\lambda_0 + \lambda(\varepsilon)) = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$ uniformly in $\tau \in [0, \Theta_1]$ for $\Theta_1 > \Theta_0$. Write equation (2.3b) in the form

$$0 = e^{-\alpha\frac{\Theta_\varepsilon}{\epsilon}} z^0$$
$$+ \frac{1}{\varepsilon}\int\limits_0^{\Theta_\varepsilon} e^{-\alpha\frac{\tau}{\epsilon}} B_2 \frac{B_0^* e^{A_0^*\tau}\lambda_0 + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu_0 + B_0^* e^{A_0^*\tau}\lambda(\varepsilon) + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu(\varepsilon) + R(\tau,\varepsilon,\lambda(\varepsilon))}{\left\| B_0^* e^{A_0^*\tau}\lambda_0 + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu_0 + B_0^* e^{A_0^*\tau}\lambda(\varepsilon) + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu(\varepsilon) + R(\tau,\varepsilon,\lambda(\varepsilon))\right\|}\, d\tau. \quad (2.8)$$

To find the asymptotics of the integral in (2.8), we apply the method of an auxiliary parameter, which was described in [12, 17]. Decompose the integral from (2.8) into the sum of two terms

$$\int\limits_0^{\Theta_\varepsilon} \cdot = \int\limits_0^{\mu} \cdot + \int\limits_{\mu}^{\Theta_\varepsilon} \cdot\,,$$

where μ is a small auxiliary parameter. Let $\mu = \varepsilon^q$ for $q \in (0,1)$. Then, $\int_\mu^{\Theta_\varepsilon} \cdot = \mathbb{O}$, and equation (2.8) yields

$$I_1(\varepsilon,\mu)$$
$$:= \frac{1}{\varepsilon}\int\limits_0^{\mu} e^{-\alpha\frac{\tau}{\epsilon}} B_2 \frac{B_0^* e^{A_0^*\tau}\lambda_0 + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu_0 + B_0^* e^{A_0^*\tau}\lambda(\varepsilon) + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu(\varepsilon) + R(\tau,\varepsilon,\lambda(\varepsilon))}{\left\| B_0^* e^{A_0^*\tau}\lambda_0 + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu_0 + B_0^* e^{A_0^*\tau}\lambda(\varepsilon) + e^{-\alpha\frac{\tau}{\epsilon}} B_2^*\nu(\varepsilon) + R(\tau,\varepsilon,\lambda(\varepsilon))\right\|}\, d\tau = \mathbb{O}.$$

Here, $\mathbb{O}$ is an asymptotic zero with respect to the power asymptotic sequence; i.e., $\mathbb{O} = o(\varepsilon^\gamma)$ $\forall\gamma > 0$ $(\varepsilon \to 0)$.

Consider the asymptotic expansion of the integral $I_1(\varepsilon,\mu)$. Changing $\eta = \tau/\varepsilon$, we obtain

$$I_1(\varepsilon,\mu) = \int\limits_0^{\mu/\varepsilon} e^{-\alpha\eta} B_2 \frac{B_0^*\lambda_0 + e^{-\alpha\eta} B_2^*\nu_0 + \widetilde{R}(\eta,\varepsilon,\lambda(\varepsilon),\nu(\varepsilon))}{\left\| B_0^*\lambda_0 + e^{-\alpha\eta} B_2^*\nu_0 + \widetilde{R}(\eta,\varepsilon,\lambda(\varepsilon),\nu(\varepsilon))\right\|}\, d\eta = \mathbb{O}, \quad (2.9)$$

where

$$\widetilde{R}(\eta,\varepsilon,\lambda(\varepsilon),\nu(\varepsilon)) = B_0^*\left(e^{A_0^*\varepsilon\eta} - I\right)\lambda_0 + B_0^* e^{A_0^*\varepsilon\eta}\lambda(\varepsilon) + e^{-\alpha\eta} B_2^*\nu(\varepsilon) + R(\varepsilon\eta,\varepsilon,\lambda(\varepsilon)).$$

Define $\delta(\varepsilon) = \|(\lambda(\varepsilon)^T, \nu(\varepsilon)^T, \vartheta(\varepsilon))^T\|$. Then, $\widetilde{R}(\eta,\varepsilon,\lambda(\varepsilon),\nu(\varepsilon)) = \mathcal{O}(\varepsilon\eta + \delta(\varepsilon) + \varepsilon)$ as $\varepsilon \to 0$ for $\eta \in [0, \mu/\varepsilon]$.

We will seek a solution of system (2.3) among the vectors $(\lambda(\varepsilon)^T, \nu(\varepsilon)^T, \vartheta(\varepsilon))^T$ for which

$$\delta(\varepsilon) = \mathcal{O}(\varepsilon), \qquad \varepsilon \to +0. \quad (2.10)$$

First, we find ν_0. In the limit as $\varepsilon \to 0$, equality (2.9) yields

$$0 = \int\limits_0^{+\infty} e^{-\alpha\eta} B_2 \frac{B_0^*\lambda_0 + e^{-\alpha\eta} B_2^*\nu_0}{\|B_0^*\lambda_0 + e^{-\alpha\eta} B_2^*\nu_0\|}\, d\eta \quad (2.11)$$

or

$$0=\int\limits_0^{+\infty}\frac{e^{-\alpha\eta}\,d\eta}{\|B_0^*\lambda_0+e^{-\alpha\eta}B_2^*\nu_0\|}\,B_2B_0^*\lambda_0+\int\limits_0^{+\infty}\frac{e^{-2\alpha\eta}\,d\eta}{\|B_0^*\lambda_0+e^{-\alpha\eta}B_2^*\nu_0\|}\,B_2B_2^*\nu_0.$$

Therefore, there exists $\sigma>0$ such that

$$B_2B_2^*\nu_0=-\sigma B_2B_0^*\lambda_0. \tag{2.12}$$

In particular, multiplying (2.12) scalarly by the vector ν_0, we get

$$\|B_2^*\nu_0\|^2=-\sigma\langle B_2B_0^*\lambda_0,\nu_0\rangle. \tag{2.13}$$

On the other hand, multiplying scalarly equality (2.11) by the vector $-\alpha\nu_0$, we derive

$$0=\int\limits_0^{+\infty}\frac{\langle B_0^*\lambda_0+e^{-\alpha\eta}B_2^*\nu_0,\;-\alpha e^{-\alpha\eta}B_2^*\nu_0\rangle}{\|B_0^*\lambda_0+e^{-\alpha\eta}B_2^*\nu_0\|}\,d\eta. \tag{2.14}$$

Changing $\chi=\|B_0^*\lambda_0+e^{-\alpha\eta}B_2^*\nu_0\|^2$ in the integral on the left-hand side of (2.14), we come to the relation

$$0=\|B_0^*\lambda_0\|-\|B_0^*\lambda_0+B_2^*\nu_0\|\,, \tag{2.15}$$

which implies

$$0=2\langle B_2B_0^*\lambda_0,\nu_0\rangle+\|B_2^*\nu_0\|^2. \tag{2.16}$$

Thus, from relations (2.13) and (2.16), we find that

$$\|B_0^*\lambda_0\|^2=-2\langle B_2B_0^*\lambda_0,\nu_0\rangle=\frac{2}{\sigma}\,\|B_0^*\lambda_0+B_2^*\nu_0\|\,. \tag{2.17}$$

It follows from (2.17) that either

$$B_2^*\nu_0=0,\qquad B_2B_0^*\lambda_0=0 \tag{2.18}$$

or

$$\sigma=2,\qquad \|B_2^*\nu_0\|\neq 0. \tag{2.19}$$

Note that equality (2.12) has the form $B_2(B_2^*\nu_0+\sigma B_0^*\lambda_0)=0$. If $B_2^*\nu_0+\sigma B_0^*\lambda_0=0$, then we have case (2.19) and $B_2^*\nu_0=-2B_0^*\lambda_0$. Then, the function $B_0^*\lambda_0+e^{-\alpha\eta}B_2^*\nu_0=B_0^*\lambda_0(1-2e^{-\alpha\eta})$ vanishes for $e^{-\alpha\eta}=1/2$. In what follows, we will consider the case when

$$B_0^*\lambda_0+e^{-\alpha\eta}B_2^*\nu_0\neq 0 \tag{2.20}$$

for any $\eta\geq 0$. A sufficient condition for (2.20) is $B_2^*\nu_0=0$; i.e., in view of (2.18) and Condition 1, we have $B_2B_0^*\lambda_0=0$ and $\nu_0=0$. Note that $B_0^*\lambda_0\neq 0$ in view of assumption (1.12). Equation (2.9) now takes the form

$$\int\limits_0^{\mu/\varepsilon}e^{-\alpha\eta}B_2\,\frac{\widetilde{R}(\eta,\varepsilon,\lambda(\varepsilon),\nu(\varepsilon))}{\left\|B_0^*\lambda_0+\widetilde{R}(\eta,\varepsilon,\lambda(\varepsilon),\nu(\varepsilon))\right\|}\,d\eta=\mathbb{O}.$$

We have

$$\left\|B_0^*\lambda_0+\widetilde{R}\right\|^{-1}=\left(\|B_0^*\lambda_0\|^2+2\langle B_0^*\lambda_0,\widetilde{R}\rangle+\|\widetilde{R}\|^2\right)^{-1/2}=\|B_0^*\lambda_0\|^{-1}\left(1-\frac{\langle B_0^*\lambda_0,\widetilde{R}\rangle}{\|B_0^*\lambda_0\|^2}+\mathcal{O}\left(\|\widetilde{R}\|^2\right)\right),$$

where

$$\widetilde{R}(\eta,\varepsilon,\lambda(\varepsilon),\nu(\varepsilon)) = B_0^*\left(e^{A_0^*\varepsilon\eta}-I\right)\lambda_0 + B_0^* e^{A_0^*\varepsilon\eta}\lambda(\varepsilon) + e^{-\alpha\eta}B_2^*\nu(\varepsilon)$$
$$-\frac{\varepsilon}{\alpha}B_2^*A_{12}^*(\alpha I+\varepsilon A_0^*)^{-1}A_0^*e^{A_0^*\varepsilon\eta}(\lambda_0+\lambda(\varepsilon))$$
$$= \varepsilon\eta\, B_0^*A_0^*\lambda_0 + B_0^*\lambda(\varepsilon) + e^{-\alpha\eta}B_2^*\nu(\varepsilon) - \frac{\varepsilon}{\alpha^2}B_2^*A_{12}^*A_0^*\lambda_0 + \overset{1}{R}(\eta,\varepsilon,\lambda(\varepsilon)).$$

Here, $\overset{1}{R}(\eta,\varepsilon,\lambda(\varepsilon)) = \mathcal{O}\left(\varepsilon^2\eta^2+\varepsilon\eta\delta+\varepsilon\delta+\varepsilon^2\right)$ as $\varepsilon\to 0$ for $\eta\in[0,\mu/\varepsilon]$. Then,

$$I_1(\varepsilon,\mu) \sim \frac{\varepsilon}{\beta}B_2B_0^*A_0^*\lambda_0\int\limits_0^{\mu/\varepsilon}\eta\, e^{-\alpha\eta}\,d\eta + \frac{1}{\beta}B_2B_0^*\lambda(\varepsilon)\int\limits_0^{\mu/\varepsilon}e^{-\alpha\eta}\,d\eta + \frac{1}{\beta}B_2B_2^*\nu(\varepsilon)\int\limits_0^{\mu/\varepsilon}e^{-\alpha\eta}\,d\eta$$
$$-\frac{\varepsilon}{\alpha^2\beta}B_2B_2^*A_{12}^*A_0^*\lambda_0\int\limits_0^{\mu/\varepsilon}e^{-\alpha\eta}\,d\eta + \frac{1}{\beta}B_2\int\limits_0^{\mu/\varepsilon}e^{-\alpha\eta}\overset{2}{R}(\eta,\varepsilon,\lambda(\varepsilon))\,d\eta,$$

where $\beta=\|B_0^*\lambda_0\|$ and $\overset{2}{R}(\eta,\varepsilon,\lambda(\varepsilon)) = \mathcal{O}\left(\varepsilon^2\eta^2+\delta^2+\varepsilon^2\right)$.

Thus,

$$\frac{\varepsilon}{\alpha^2\beta}B_2B_0^*A_0^*\lambda_0 + \frac{1}{\alpha\beta}B_2B_0^*\lambda(\varepsilon) + \frac{1}{2\alpha\beta}B_2B_2^*\nu(\varepsilon) - \frac{\varepsilon}{\alpha^3\beta}B_2B_2^*A_{12}^*A_0^*\lambda_0 = \mathcal{O}\left(\varepsilon^2+\delta^2\right)$$

or, since $B_0^* = B_1^* + \alpha^{-1}B_2^*A_{12}^*$,

$$\frac{\varepsilon}{\alpha^2\beta}B_2B_1^*A_0^*\lambda_0 + \frac{1}{\alpha\beta}B_2B_0^*\lambda(\varepsilon) + \frac{1}{2\alpha\beta}B_2B_2^*\nu(\varepsilon) = \mathcal{O}\left(\varepsilon^2+\delta^2\right).$$

Hence, in accordance with Condition 1 and equality (2.10), we have

$$\nu(\varepsilon) = -\frac{2\varepsilon}{\alpha}(B_2B_2^*)^{-1}B_2B_1^*A_0^*\lambda_0 - 2(B_2B_2^*)^{-1}B_2B_0^*\lambda(\varepsilon) + \mathcal{O}\left(\varepsilon^2\right). \tag{2.21}$$

Consider equation (2.3a) and, in view of relation (2.6), write it in the form

$$0 = \overline{y}_\varepsilon + \overline{z}_\varepsilon + \int\limits_0^{\Theta_\varepsilon}\frac{D_\varepsilon(\tau)\left(\Lambda_\varepsilon^*(\tau)\lambda_\varepsilon + e^{-\alpha\frac{\tau}{\epsilon}}B_2^*\nu_\varepsilon\right)}{\left\|\Lambda_\varepsilon^*(\tau)\lambda_\varepsilon + e^{-\alpha\frac{\tau}{\epsilon}}B_2^*\nu_\varepsilon\right\|}\,d\tau, \tag{2.22}$$

where $\Lambda_\varepsilon(\tau) := e^{A_0\tau}B_0 - \frac{\varepsilon}{\alpha}e^{A_0\tau}A_0(\alpha I+\varepsilon A_0)^{-1}A_{12}B_2$ and $D_\varepsilon(\tau)$ is defined by (2.4).

Assume that vectors $\lambda(\varepsilon)$ and $\nu(\varepsilon)$ are such that $\Lambda_\varepsilon^*(\tau)(\lambda_0+\lambda(\varepsilon)) + e^{-\alpha\frac{\tau}{\epsilon}}B_2^*\nu(\varepsilon)\neq 0$ for any $\tau\in[0,\Theta_1]$, $\Theta_1>\Theta_0$.

In view of (1.12), the expansion of the integrand in (2.22) has singularities for $\tau=\overline{\tau}$, and, in a sufficiently small neighborhood of zero, we must take into account the influence of the exponents, which have the boundary-layer character. Decompose the integral in (2.22) into the sum of two terms containing the points $\tau=0$ and $\tau=\overline{\tau}$:

$$\int\limits_0^{\Theta_\varepsilon}\cdot = \int\limits_0^{\tau_1}\cdot + \int\limits_{\tau_1}^{\Theta_\varepsilon}\cdot =: J_1(\varepsilon)+J_2(\varepsilon),$$

where $0 < \tau_1 < \overline{\tau}$. Let us investigate the asymptotic representation of the integrals $J_1(\varepsilon)$ and $J_2(\varepsilon)$ by the method of an auxiliary parameter, which was described in [12, 13, 17]. Let

$$J_1(\varepsilon) = \int\limits_0^{\tau_1} \cdot = \int\limits_0^{\mu} \cdot + \int\limits_{\mu}^{\tau_1} \cdot =: J_{1,1}(\varepsilon, \mu) + J_{1,2}(\varepsilon, \mu),$$

where μ is a small auxiliary parameter: $\mu = \varepsilon^q$ for $q \in (0, 1)$. Changing $\eta = \tau/\varepsilon$ in the integral $J_{1,1}(\varepsilon)$, we obtain

$$J_{1,1}(\varepsilon, \mu) = \overset{1}{\mathcal{F}}(\varepsilon, \delta, \mu) + \mathcal{O}\left(\varepsilon^2\right). \tag{2.23}$$

Here and elsewhere, we denote by $\overset{1}{\mathcal{F}}(\varepsilon, \delta, \mu)$ sums of a finite number of terms of the form $\varphi(\varepsilon)\mu^a \ln^b \mu$ or $\varphi_1(\delta)\mu^a \ln^b \mu$, where $a^2 + b^2 \neq 0$; $\varphi(\varepsilon) = \mathcal{O}^*(\varepsilon^\gamma)$ for some $\gamma \geq 0$; and $\varphi_1(\delta) = \mathcal{O}(\delta)$, which, according to assumption (2.10), yields $\varphi_1(\delta) = \mathcal{O}(\varepsilon)$. In this case, the expression $\varphi(\varepsilon) = \mathcal{O}^*(\varepsilon^\gamma)$ as $\varepsilon \to 0$ means that $\varphi(\varepsilon) = o(\varepsilon^\sigma)$ $\forall \sigma < \gamma$ [13, 17]. By the lemma from [17], terms of this form can be neglected in the expansion of an integral by the method of an auxiliary parameter.

Note that, for $\mu = \varepsilon^q$, $0 < q < 1$, in view of (2.4), we have

$$\int\limits_{\mu}^{\Theta_\varepsilon} \frac{D_\varepsilon(\tau)\left(\Lambda_\varepsilon^*(\tau)\lambda_\varepsilon + e^{-\alpha\frac{\tau}{\epsilon}} B_2^* \nu_\varepsilon\right)}{\left\|\Lambda_\varepsilon^*(\tau)\lambda_\varepsilon + e^{-\alpha\frac{\tau}{\epsilon}} B_2^* \nu_\varepsilon\right\|}\, d\tau = \int\limits_{\mu}^{\Theta_\varepsilon} \frac{\Lambda_\varepsilon(\tau)\Lambda_\varepsilon^*(\tau)\lambda_\varepsilon}{\|\Lambda_\varepsilon^*(\tau)\lambda_\varepsilon\|}\, d\tau + \mathbb{O}. \tag{2.24}$$

For the integral $J_{1,2}(\varepsilon, \mu)$, using representation (2.24), we obtain

$$J_{1,2}(\varepsilon, \mu) = \int\limits_0^{\tau_1} \frac{C_0(\tau)\lambda_0}{\sqrt{\psi(\tau)}}\, d\tau + \int\limits_0^{\tau_1} \frac{L_1(\tau; \varepsilon, \lambda(\varepsilon))}{\sqrt{\psi^3(\tau)}}\, d\tau + \overset{2}{\mathcal{F}}(\varepsilon, \delta, \mu) + \mathcal{O}\left(\varepsilon^2\right), \qquad \varepsilon \to 0, \tag{2.25}$$

where

$$L_1(\tau; \varepsilon, \lambda(\varepsilon)) = \psi(\tau)\left(C_0(\tau)\lambda(\varepsilon) + \varepsilon C_1(\tau)\lambda_0\right) - \frac{\varepsilon}{2}\langle C_1(\tau)\lambda_0, \lambda_0\rangle\, C_0(\tau)\lambda_0 - \langle C_0(\tau)\lambda_0, \lambda(\varepsilon)\rangle\, C_0(\tau)\lambda_0,$$

$$C_1(\tau) = -\frac{1}{\alpha^2}\, e^{A_0\tau}\left(A_0 A_{12} B_2 B_0^* + B_0 B_2^* A_{12}^* A_0^*\right) e^{A_0^*\tau}.$$

Note that $\psi(\tau) > 0$ for $\tau \in [0, \tau_1]$ because of conditions (1.12).

In view of the lemma from [13], for $\mu = \varepsilon^q$, $0 < q < 1$, relations (2.23) and (2.25) yield

$$J_1(\varepsilon) = \int\limits_0^{\tau_1} \frac{C_0(\tau)\lambda_0}{\sqrt{\psi(\tau)}}\, d\tau + \int\limits_0^{\tau_1} \frac{L_1(\tau; \varepsilon, \lambda(\varepsilon))}{\sqrt{\psi^3(\tau)}}\, d\tau + \mathcal{O}\left(\varepsilon^2\right). \tag{2.26}$$

Further, let

$$rJ_2(\varepsilon) = \int\limits_{\tau_1}^{\Theta_\varepsilon} \cdot = \int\limits_{\tau_1}^{\overline{\tau}-\mu} \cdot + \int\limits_{\overline{\tau}-\mu}^{\overline{\tau}+\mu} \cdot + \int\limits_{\overline{\tau}+\mu}^{\Theta_\varepsilon} \cdot =: J_{2,1}(\varepsilon, \mu) + J_{2,2}(\varepsilon, \mu) + J_{2,3}(\varepsilon, \mu),$$

where μ is a new small auxiliary parameter: $\mu = \varepsilon^q$ for $q \in (0, 1)$.

According to conditions (1.12), we have $\psi(\tau) = (\tau - \overline{\tau})^2 \|B_0^* e^{A_0^* \overline{\tau}} A_0^* \lambda_0\|^2 \left(1 + (\tau - \overline{\tau})\psi_1(\tau)\right)$, where $\psi_1(\tau) \in C^\infty[\tau_1, \Theta_1]$. Using assumption (1.13), we derive

$$\|B_0^* e^{A_0^* \overline{\tau}} A_0^* \lambda_0\|^2 = \|e^{A_0 \overline{\tau}} B_0 B_0^* e^{A_0^* \overline{\tau}} A_0^* \lambda_0\|^2 = \|Q A_0^* \lambda_0\|^2.$$

Introduce the notation

$$\widetilde{\lambda}_0 = Q A_0^* \lambda_0, \qquad \widehat{\lambda}_0 = \frac{1}{\alpha^2} e^{A_0 \overline{\tau}} B_0 B_2^* A_{12}^* e^{A_0^* \overline{\tau}} A_0^* \lambda_0, \qquad \widetilde{\lambda}(\varepsilon) = Q\, \lambda(\varepsilon),$$

$$\widetilde{\lambda}(\varepsilon) - \varepsilon \widehat{\lambda}_0 = \overset{1}{\rho}(\varepsilon) + l(\varepsilon)\widetilde{\lambda}_0, \qquad \overset{1}{\rho}(\varepsilon) \perp \widetilde{\lambda}_0, \qquad l(\varepsilon) \in \mathbb{R}. \tag{2.27}$$

Here, $\overset{1}{\rho} \in Q\,\mathbb{R}^n$ and $\widetilde{\lambda}_0 \in Q\,\mathbb{R}^n$. Consequently, $\dfrac{1}{\sqrt{\psi(\tau)}} = \dfrac{1}{|\tau - \overline{\tau}|\,\|\widetilde{\lambda}_0\|} + \dfrac{\tau - \overline{\tau}}{|\tau - \overline{\tau}|}\, \psi_2(\tau)$, where $\psi_2(\tau) \in C^\infty[\tau_1, \Theta_1]$.

Let us first consider the integral $J_{2,1}(\varepsilon, \mu)$. By analogy with [13], we apply the integral regularization method and obtain

$$J_{2,1}(\varepsilon, \mu) = \int_{\tau_1}^{\overline{\tau}} \frac{C_0(\tau)\lambda_0}{\sqrt{\psi(\tau)}} d\tau + \int_{\tau_1}^{\overline{\tau}} \frac{L_1(\tau; \varepsilon, \lambda(\varepsilon)) - \mathcal{A}_2(L_1(\tau; \varepsilon, \lambda(\varepsilon)))}{\sqrt{\psi^3(\tau)}} d\tau$$

$$+ \|\widetilde{\lambda}_0\|^2 \overset{1}{\rho}(\varepsilon) \int_{\tau_1}^{\overline{\tau} - \mu} \frac{(\tau - \overline{\tau})^2}{\sqrt{\psi^3(\tau)}} d\tau + \overset{3}{\mathcal{F}}(\varepsilon, \delta, \mu) + \mathcal{O}\left(\varepsilon^2\right), \qquad \varepsilon \to 0, \tag{2.28}$$

for $\mu = \varepsilon^q$, $q \in (0, 1)$. Here, $\mathcal{A}_2(L_1(\tau; \varepsilon, \lambda(\varepsilon)))$ is the segment of the Taylor series of the function $L_1(\tau; \varepsilon, \lambda(\varepsilon))$ with respect to τ at the point $\overline{\tau}$ that contains powers $(\tau - \overline{\tau})^i$ for $i \le 2$:

$$\mathcal{A}_2(L_1(\tau; \varepsilon, \lambda(\varepsilon)))$$

$$= (\tau - \overline{\tau})^2 \left(\|\widetilde{\lambda}_0\|^2 \overset{1}{\rho}(\varepsilon) + \|\widetilde{\lambda}_0\|^2 l(\varepsilon) \widetilde{\lambda}_0 - \langle \widetilde{\lambda}_0, \overset{1}{\rho}(\varepsilon) \rangle \widetilde{\lambda}_0 - l(\varepsilon) \langle \widetilde{\lambda}_0, \widetilde{\lambda}_0 \rangle \widetilde{\lambda}_0 \right) = (\tau - \overline{\tau})^2 \|\widetilde{\lambda}_0\|^2 \overset{1}{\rho}(\varepsilon).$$

For the integral $J_{2,3}(\varepsilon, \mu)$, we have

$$J_{2,3}(\varepsilon, \mu) = \int_{\overline{\tau}}^{\Theta_0} \frac{C_0(\tau)\lambda_0}{\sqrt{\psi(\tau)}} d\tau + \vartheta(\varepsilon) \frac{C_0(\Theta_0)\lambda_0}{\sqrt{\psi(\Theta_0)}} + \int_{\overline{\tau}}^{\Theta_0} \frac{L_1(\tau; \varepsilon, \lambda(\varepsilon)) - \mathcal{A}_2(L_1(\tau; \varepsilon, \lambda(\varepsilon)))}{\sqrt{\psi^3(\tau)}} d\tau$$

$$+ \|\widetilde{\lambda}_0\|^2 \overset{1}{\rho}(\varepsilon) \int_{\overline{\tau} + \mu}^{\Theta_0} \frac{(\tau - \overline{\tau})^2}{\sqrt{\psi^3(\tau)}} d\tau + \overset{5}{\mathcal{F}}(\varepsilon, \delta, \mu) + \mathcal{O}\left(\varepsilon^2\right), \qquad \varepsilon \to 0. \tag{2.29}$$

Further, we seek a solution satisfying for sufficiently small ε the conditions

$$\overset{1}{\rho}(\varepsilon) = o(\varepsilon), \qquad \overset{1}{\rho}(\varepsilon) \ne 0, \qquad \varepsilon^{1+\gamma} = o(\|\overset{1}{\rho}(\varepsilon)\|) \text{ for all } \gamma > 0. \tag{2.30}$$

Under these conditions, we find the asymptotic representation of the integral $J_{2,2}(\varepsilon, \mu)$ similarly to [13]:

$$J_{2,2}(\varepsilon, \mu) = 2\, l(\varepsilon) \frac{\widetilde{\lambda}_0}{\|\widetilde{\lambda}_0\|} + \frac{2\overset{1}{\rho}(\varepsilon)}{\|\widetilde{\lambda}_0\|} \ln \frac{2\|\widetilde{\lambda}_0\|}{\|\overset{1}{\rho}(\varepsilon)\|} + \overset{4}{\mathcal{F}}(\varepsilon, \delta, \mu) + \mathcal{O}^*(\varepsilon^2), \qquad \varepsilon \to 0. \tag{2.31}$$

Using relations (2.28), (2.31), and (2.29), we obtain

$$J_2(\varepsilon) = \int\limits_{\tau_1}^{\Theta_0} \frac{C_0(\tau)\lambda_0}{\sqrt{\psi(\tau)}} d\tau + \lim_{\gamma\to+0}\Bigg[\Bigg(\int\limits_{\tau_1}^{\overline{\tau}-\gamma} + \int\limits_{\overline{\tau}+\gamma}^{\Theta_0}\Bigg)\frac{L_1(\tau;\varepsilon,\lambda(\varepsilon))}{\sqrt{\psi^3(\tau)}} d\tau + \frac{2\ln\gamma}{\|\widetilde{\lambda}_0\|}\overset{1}{\rho}(\varepsilon)\Bigg]$$

$$+ \vartheta(\varepsilon)\frac{C_0(\Theta_0)\lambda_0}{\sqrt{\psi(\Theta_0)}} + 2\,l(\varepsilon)\frac{\widetilde{\lambda}_0}{\|\widetilde{\lambda}_0\|} + \frac{2\overset{1}{\rho}(\varepsilon)}{\|\widetilde{\lambda}_0\|}\ln\frac{2\|\widetilde{\lambda}_0\|}{\|\overset{1}{\rho}(\varepsilon)\|} + \overset{8}{\mathcal{F}}(\varepsilon,\delta,\mu) + \mathcal{O}^*(\varepsilon^2). \qquad (2.32)$$

By the lemma from [17], the term $\overset{8}{\mathcal{F}}(\varepsilon,\delta,\mu)$ can be neglected.

3. FINDING THE FIRST TERMS OF THE ASYMPTOTICS OF THE SOLUTION

Substituting relations (2.26) and (2.32) into (2.22) and using identity (1.11), we derive

$$0 = \frac{\varepsilon}{\alpha}e^{A_0\Theta_0}A_{12}z^0 + \lim_{\gamma\to+0}\Bigg[\Bigg(\int\limits_{0}^{\overline{\tau}-\gamma} + \int\limits_{\overline{\tau}+\gamma}^{\Theta_0}\Bigg)\frac{L_1(\tau;\varepsilon,\lambda(\varepsilon))}{\sqrt{\psi^3(\tau)}} d\tau + \frac{2\ln\gamma}{\|\widetilde{\lambda}_0\|}\overset{1}{\rho}(\varepsilon)\Bigg]$$

$$+ 2\,l(\varepsilon)\frac{\widetilde{\lambda}_0}{\|\widetilde{\lambda}_0\|} + \frac{2\overset{1}{\rho}(\varepsilon)}{\|\widetilde{\lambda}_0\|}\times\ln\frac{2\|\widetilde{\lambda}_0\|}{\|\overset{1}{\rho}(\varepsilon)\|} + \vartheta(\varepsilon)\Bigg(\frac{C_0(\Theta_0)\lambda_0}{\sqrt{\psi(\Theta_0)}} + A_0e^{A_0\Theta_0}y^0\Bigg) + \mathcal{O}^*(\varepsilon^2), \qquad (3.1)$$

$$\widetilde{\lambda}_0^*\overset{1}{\rho}(\varepsilon) = 0.$$

The vector $\lambda(\varepsilon)$ can be represented in the form $\lambda(\varepsilon) = \widetilde{\lambda}(\varepsilon) + \overset{2}{\rho}(\varepsilon)$, where $Q\overset{2}{\rho}(\varepsilon) = 0$. Further, $\lambda(\varepsilon) = \widetilde{\lambda}(\varepsilon) - \varepsilon\widehat{\lambda}_0 + \overset{2}{\rho}(\varepsilon) + \varepsilon\widehat{\lambda}_0 = \overset{1}{\rho}(\varepsilon) + l(\varepsilon)\widetilde{\lambda}_0 + \overset{2}{\rho}(\varepsilon) + \varepsilon\widehat{\lambda}_0$. Consider the new unknown vector $\rho(\varepsilon) := \lambda(\varepsilon) - \varepsilon\widehat{\lambda}_0$ and represent $\rho(\varepsilon)$ in the form $\rho(\varepsilon) := Q\rho(\varepsilon) + \overset{2}{\rho}(\varepsilon)$. By analogy with [11], we can prove the following lemma.

Lemma 2. *There exists $\varepsilon_0 > 0$ such that*

$$\Lambda_\varepsilon^*(\tau)(\lambda_0 + \rho(\varepsilon) + \varepsilon\widehat{\lambda}_0) + e^{-\alpha\frac{\tau}{\epsilon}}B_2^*\nu(\varepsilon) \neq 0$$

for all $\tau \in [0,\Theta_1]$, $\Theta_1 > \Theta_0$, $0 < \varepsilon \le \varepsilon_0$, and $\rho(\varepsilon)$ satisfying the conditions

$$\langle\widetilde{\lambda}_0, Q\rho(\varepsilon)\rangle \le \widetilde{\beta}\|\widetilde{\lambda}_0\|\|Q\rho(\varepsilon)\|, \quad 0 < \widetilde{\beta} < 1, \qquad (3.2a)$$

$$Q\rho(\varepsilon) = o(\varepsilon), \quad \varepsilon^{1+\gamma} = o(\|Q\rho(\varepsilon)\|), \quad \overset{2}{\rho}(\varepsilon) = \mathcal{O}(\|Q\rho(\varepsilon)\|), \quad \varepsilon\to+0 \quad \forall\gamma>0. \qquad (3.2b)$$

In what follows, we will consider the values of ρ that satisfy conditions (3.2). Note that these conditions do not contradict conditions (2.30), which were introduced earlier, and it follows from (3.2) that $\|Q\rho(\varepsilon)\| \neq 0$. According to the adopted notation, $\rho(\varepsilon) = \overset{1}{\rho}(\varepsilon) + l(\varepsilon)\widetilde{\lambda}_0 + \overset{2}{\rho}(\varepsilon)$. Then, equation (3.1) takes the form

$$\varepsilon V_1 = \lim_{\gamma\to+0}\Bigg[\Bigg(\int\limits_{0}^{\overline{\tau}-\gamma} + \int\limits_{\overline{\tau}+\gamma}^{\Theta_0}\Bigg)\frac{\psi(\tau)C_0(\tau)\rho(\varepsilon) - \langle C_0(\tau)\lambda_0,\rho(\varepsilon)\rangle C_0(\tau)\lambda_0}{\sqrt{\psi^3(\tau)}}\|\widetilde{\lambda}_0\|\,d\tau + 2\ln\gamma\overset{1}{\rho}(\varepsilon)\Bigg]$$

$$+ 2\,l(\varepsilon)\,\widetilde{\lambda}_0 + 2\overset{1}{\rho}(\varepsilon)\ln\frac{2\|\widetilde{\lambda}_0\|}{\|\overset{1}{\rho}(\varepsilon)\|} + \vartheta(\varepsilon)\Bigg(\frac{C_0(\Theta_0)\lambda_0}{\sqrt{\psi(\Theta_0)}} + A_0e^{A_0\Theta_0}y^0\Bigg)\|\widetilde{\lambda}_0\| + \mathcal{O}^*(\varepsilon^2), \qquad (3.3)$$

where

$$V_1 = -\frac{\|\widetilde{\lambda}_0\|}{\alpha}\, e^{A_0\Theta_0} A_{12} z^0$$

$$-\,\|\widetilde{\lambda}_0\| \int\limits_0^{\Theta_0} \frac{\psi(\tau)(C_1(\tau)\lambda_0 + C_0(\tau)\widehat{\lambda}_0) - \frac{1}{2}\langle C_1(\tau)\lambda_0, \lambda_0\rangle\, C_0(\tau)\lambda_0 - \langle C_0(\tau)\lambda_0, \widehat{\lambda}_0\rangle\, C_0(\tau)\lambda_0}{\sqrt{\psi^3(\tau)}}\, d\tau. \quad (3.4)$$

Note that the integrand in (3.4) is bounded on $[0, \Theta_0]$.

An additional equation equivalent to the condition that equation (2.1) is homogeneous with respect to the vector r_ε can be specified by analogy with [11] so that the matrix of the linear part of system (3.3) remains symmetric:

$$\|\widetilde{\lambda}_0\| \left(\frac{C_0(\Theta_0)\lambda_0}{\sqrt{\psi(\Theta_0)}} + e^{A_0\Theta_0} A_0 y^0 \right)^* \rho(\varepsilon) = 0.$$

Note that $l(\varepsilon) = \dfrac{\langle \widetilde{\lambda}_0, \rho(\varepsilon)\rangle}{\|\widetilde{\lambda}_0\|^2}$ and $\overset{1}{\rho}(\varepsilon) = Q\rho(\varepsilon) - l(\varepsilon)\widetilde{\lambda}_0 = Q\rho(\varepsilon) - \dfrac{\langle \widetilde{\lambda}_0, \rho(\varepsilon)\rangle \widetilde{\lambda}_0}{\|\widetilde{\lambda}_0\|^2}$. Write the system for finding the first approximations ρ_1, ϑ_1, and ν_1 of the unknowns ρ, ϑ, and ν:

$$\begin{pmatrix} \varepsilon V_1^{(1)} \\ \varepsilon V_1^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} \\ \mathcal{B}_{12}^* & \mathcal{B}_{22} & \mathcal{B}_{23} \\ \mathcal{B}_{13}^* & \mathcal{B}_{23}^* & 0 \end{pmatrix} \begin{pmatrix} \rho_1^{(1)} \\ \rho_1^{(2)} \\ \vartheta_1 \end{pmatrix} + \begin{pmatrix} \mathcal{H}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1^{(1)} \\ \rho_1^{(2)} \\ \vartheta_1 \end{pmatrix} \ln \frac{\gamma_0}{g(\rho_1^{(1)})}, \quad (3.5)$$

$$\nu_1(\varepsilon) = -2(B_2 B_2^*)^{-1} B_2 B_0^*\, \rho_1(\varepsilon) + \varepsilon S_1. \quad (3.6)$$

Here, $\rho_1^{(1)} \in \mathbb{R}^r$, $\rho_1^{(2)} \in \mathbb{R}^{n-r}$, $\gamma_0 = 4\|\widetilde{\lambda}_0\|^2$,

$$\rho_1(\varepsilon) = \begin{pmatrix} \rho_1^{(1)} \\ \rho_1^{(2)} \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{B}_{13} \\ \mathcal{B}_{23} \end{pmatrix} = \left(\frac{C_0(\Theta_0)\lambda_0}{\sqrt{\psi(\Theta_0)}} + e^{A_0\Theta_0} A_0 y^0 \right) \|\widetilde{\lambda}_0\|,$$

$$\begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{12}^* & \mathcal{B}_{22} \end{pmatrix} \rho_1 = \lim_{\gamma \to +0} \left[\left(\int\limits_0^{\overline{\tau}-\gamma} + \int\limits_{\overline{\tau}+\gamma}^{\Theta_0} \right) \frac{\psi(\tau) C_0(\tau)\rho_1(\varepsilon) - \langle C_0(\tau)\lambda_0, \rho_1(\varepsilon)\rangle\, C_0(\tau)\lambda_0}{\sqrt{\psi^3(\tau)}} \|\widetilde{\lambda}_0\| d\tau \right.$$

$$\left. + 2\ln\gamma \left(Q\rho_1(\varepsilon) - \frac{\langle \widetilde{\lambda}_0, \rho_1(\varepsilon)\rangle \widetilde{\lambda}_0}{\|\widetilde{\lambda}_0\|^2} \right) \right] + 2\, \frac{\langle \widetilde{\lambda}_0, \rho_1(\varepsilon)\rangle \widetilde{\lambda}_0}{\|\widetilde{\lambda}_0\|^2},$$

$$\mathcal{B}_{11} = \mathcal{B}_{11}^*, \quad \mathcal{B}_{22} = \mathcal{B}_{22}^*, \qquad \begin{pmatrix} \mathcal{H}_{11} & 0 \\ 0 & 0 \end{pmatrix} \rho_1 = Q\rho_1(\varepsilon) - \frac{\langle \widetilde{\lambda}_0, \rho_1(\varepsilon)\rangle \widetilde{\lambda}_0}{\|\widetilde{\lambda}_0\|^2},$$

$$g(\rho_1^{(1)}) = \rho_1^*(\varepsilon) Q \rho_1(\varepsilon) - \frac{\langle \widetilde{\lambda}_0, \rho_1(\varepsilon)\rangle^2}{\|\widetilde{\lambda}_0\|^2} = \|\overset{1}{\rho}_1(\varepsilon)\|^2, \quad (3.7)$$

$$S_1 = -\frac{2}{\alpha} (B_2 B_2^*)^{-1} B_2 B_1^* A_0^* \lambda_0 - 2(B_2 B_2^*)^{-1} B_2 B_0^*\, \widehat{\lambda}_0.$$

Note that $g(\rho_1^{(1)}) = 0$ if and only if $\rho_1^{(1)} \parallel Pr_r\,\widetilde{\lambda}_0$; here, the vector $Pr_r\,\widetilde{\lambda}_0$ is obtained from the first r coordinates of $\widetilde{\lambda}_0$.

Similarly to [11], we can prove that system (3.5), (3.6) has a unique solution

$$\begin{pmatrix} \rho_1 \\ \vartheta_1 \end{pmatrix} = \varepsilon \mathcal{R}_1 V_1 + \mathcal{R}_2 \times \operatorname{diag}\Big(\varepsilon \ln^{-1} \frac{\gamma_i}{W(\varepsilon)}\Big) \mathcal{R}_3 V_1, \tag{3.8}$$

$$\nu_1 = \varepsilon \mathcal{R}_4 V_1 + \varepsilon S_1 + \mathcal{R}_5 \times \operatorname{diag}\Big(\varepsilon \ln^{-1} \frac{\gamma_i}{W(\varepsilon)}\Big) \mathcal{R}_6 V_1 \tag{3.9}$$

for all sufficiently small ε. The function $W(\varepsilon)$ is a solution of some scalar equation such that

$$W(\varepsilon) = o(\varepsilon), \qquad W(\varepsilon) \ln \frac{1}{W(\varepsilon)} \sim \varepsilon K_0 \quad \text{as } \varepsilon \to +0,$$

$$\varepsilon^{1+\gamma} = o(W(\varepsilon)) \quad \text{as } \varepsilon \to +0 \quad \text{for all } \gamma > 0.$$

Here, $K_0 > 0$; γ_i $(i = \overline{1, r-1})$ are constants depending only on the data of the problem; and $\mathcal{R}_k$ $(k = \overline{1,6})$ are known constant matrices of size $(n+1) \times n$, $(n+1) \times (r-1)$, $(r-1) \times n$, $m \times n$, $m \times (r-1)$, and $(r-1) \times n$, respectively. Class (3.8), (3.9) contains a solution satisfying conditions (3.2) for some $\widetilde{\beta} \in (0,1)$. For example, if the vector V_1 (3.4) satisfies the relations

$$QV_1 = V_1, \qquad \widetilde{\lambda}_0^* V_1 = 0, \tag{3.10}$$

then this solution of first approximation system (3.5), (3.6) takes the form

$$\omega_1 := \begin{pmatrix} \rho_1 \\ \vartheta_1 \end{pmatrix} = \mathcal{R}_2 \times \operatorname{diag}\Big(\varepsilon \ln^{-1} \frac{\gamma_i}{W(\varepsilon)}\Big) \mathcal{R}_3 V_1, \tag{3.11}$$

$$\nu_1 = \varepsilon S_1 + \mathcal{R}_5 \times \operatorname{diag}\Big(\varepsilon \ln^{-1} \frac{\gamma_i}{W(\varepsilon)}\Big) \mathcal{R}_6 V_1. \tag{3.12}$$

The validity of conditions (3.10) is provided by an appropriate choice of the problem's data, i.e., the initial vector z^0 and the matrix A_{12}. Note that each component of the vector $\mathbf{v}_1 := (\rho_1^T,\, \vartheta_1,\, \nu_1^T)^T$ is a rational function of ε and $\ln(1/W(\varepsilon))$; moreover, $\|\omega_1\| = \mathcal{O}\left(W(\varepsilon)\right)$ and $\|\nu_1\| = \mathcal{O}\left(\varepsilon\right)$.

Introducing the corresponding notation, we write system (3.3), (2.21) in the form

$$\varepsilon \widetilde{V} = \mathcal{B}\,\widetilde{\rho} + \mathcal{H}\,\widetilde{\rho}\, \ln \frac{\gamma_0}{\widetilde{g}\,(\widetilde{\rho})} + \mathcal{O}^*\!\left(\varepsilon^2\right),$$

$$\nu = b\rho + \varepsilon S_1 + \mathcal{O}\left(\varepsilon^2\right),$$

where $\widetilde{\rho} := (\rho^T, \vartheta)^T$ and $\widetilde{g}\,(\widetilde{\rho}) = \widetilde{\rho}^* \mathcal{H}\,\widetilde{\rho}$. Note that $\widetilde{g}\,(\widetilde{\rho}) = g(\rho^{(1)})$ from (3.7).

Let $\rho = \rho_1 + \rho_2$, $\vartheta = \vartheta_1 + \vartheta_2$, and $\nu = \nu_1 + \nu_2$, where ρ_1, ϑ_1, and ν_1 are the components of solution (3.11), (3.12) of system (3.5), (3.6). Define $\omega = (\rho_2^T,\, \vartheta_2)^T$ and $\mathbf{v} := (\rho_2^T,\, \vartheta_2,\, \nu_2^T)^T$. It follows from the results of [11] that $g(\rho_1^{(1)}) = W^2(\varepsilon)$ (3.7). Then,

$$\ln g^{-1}(\rho^{(1)}) = \ln \frac{1}{W^2(\varepsilon)} - \frac{2\,\omega_1^*\,\mathcal{H}\,\omega}{W^2(\varepsilon)} + \mathcal{O}\left(\frac{\|\omega\|^2}{W^2(\varepsilon)}\right).$$

Since $\mathbf{v}_1$ is a solution of the first approximation system, we obtain the following system of equations for ω and ν_2:

$$0 = \mathcal{B}\,\omega + \mathcal{H}\,\omega\,\ln\frac{\gamma_0}{W^2(\varepsilon)} - 2\,\mathcal{H}\,\omega_1(\varepsilon)\,\frac{\omega_1^*(\varepsilon)\,\mathcal{H}\,\omega}{W^2(\varepsilon)} + \mathcal{O}\left(\frac{\|\omega\|^2}{W(\varepsilon)}\right) + \mathcal{O}^*(\varepsilon^2)\,, \tag{3.13}$$

$$\nu_2 = b\rho_2 + \mathcal{O}\left(\varepsilon^2\right). \tag{3.14}$$

By the linear transformations described in [11], we come to a system equivalent to (3.13), (3.14):

$$\mathbf{v} = F(\varepsilon, \mathbf{v}), \tag{3.15}$$

where

$$F(\varepsilon, \mathbf{v}) = \mathcal{O}\left(\frac{\|\omega\|^2}{W(\varepsilon)}\right) + \mathcal{O}^*(\varepsilon^2) = o(1)\,\|\omega\| + \mathcal{O}^*(\varepsilon^2) = o(\varepsilon^{1+\gamma}), \quad \varepsilon \to 0,$$

for $\mathbf{v} = \mathcal{O}\left(\varepsilon^{1+\gamma}\right)$ and any $0 < \gamma < 1$. Note that the mapping $F(\varepsilon, \mathbf{v})$ is continuous in $\mathbf{v}$ for any $\varepsilon > 0$. Let us find a compact convex set that contains its own image under $F(\varepsilon, \mathbf{v})$. Then, by the Schauder–Tychonoff theorem [18, p. 628], $F(\varepsilon, \mathbf{v})$ will have a fixed point in this set. Consider the ball $B\left[0, K\varepsilon^{1+\gamma}\right]$ of radius $K\varepsilon^{1+\gamma}$ in the space $\mathbb{R}^{m+n+1}$ centered at zero. Let $\|\mathbf{v}\| \le K\varepsilon^{1+\gamma}$. Then, there exists ε_0 such that $\|F(\varepsilon, \mathbf{v})\| \le K\varepsilon^{1+\gamma}$ for all $\varepsilon \in (0, \varepsilon_0)$; hence, equation (3.15) has a solution $\mathbf{v} = \mathcal{O}\left(\varepsilon^{1+\gamma}\right)$ for $0 < \gamma < 1$ as $\varepsilon \to 0$.

Note that any other solution of system (3.15) has the same asymptotic estimate $\mathbf{v} = \mathcal{O}\left(\varepsilon^{1+\gamma}\right)$. Indeed, it follows from equation (3.15) that

$$\mathbf{v} = \mathcal{O}\left(\frac{\|\mathbf{v}\|^2}{W(\varepsilon)}\right) + \mathcal{O}^*(\varepsilon^2)$$

or

$$\|\mathbf{v}\| \le K\frac{\|\mathbf{v}\|^2}{W(\varepsilon)} + \mathcal{O}^*(\varepsilon^2)$$

for some constant $K > 0$. Let us solve this inequality with respect to $\|\mathbf{v}\|$. Using the assumption $\|\mathbf{v}\| = o(\|\mathbf{v}_1\|)$, we obtain $\|\mathbf{v}\| = \mathcal{O}^*(\varepsilon^2) = \mathcal{O}\left(\varepsilon^{1+\gamma}\right)$, $\varepsilon \to 0$. Thus, we have the following theorem.

Theorem. *Suppose that Conditions* 1–5 *and relations* (1.12) *and* (3.10) *are satisfied. Then, the optimal time* Θ_ε *and the vector of initial conditions of the adjoint system have asymptotic representation* $R_0 + R_1(\varepsilon, \ln(1/W(\varepsilon)) + \mathcal{O}\left(\varepsilon^{1+\gamma}\right)$ *for* $0 < \gamma < 1$ *as* $\varepsilon \to 0$, *where* R_1 *is a rational vector function of its arguments and* $R_1(\varepsilon, \ln(1/W(\varepsilon)) = \mathcal{O}\,(\varepsilon)$.

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