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On the Nonlinear Problem of the Three-Axis Reorientation of a Three-Rotor Gyrostat in the Game Noise Model

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Abstract—The nonlinear game problem of the three-axis reorientation of an asymmetric solid body with three flywheels (rotors) has been solved. Acceptable levels of uncontrollable noise depending on given constraints of control moments have been estimated.

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INTRODUCTION

Investigations [1–4] of the dynamics of spacecraft that carry rotating bodies (flywheels, gyrodynes) showed that the application of rotors strongly expands the family of possible steady-state motions of the system and the region of their stability, as well as makes it possible to compensate the destabilizing effect of elastic elements. The use of rotors also has a number of advantages for controlling the rotational motion of spacecraft [5, 6].

In this work, we consider the problem of the reorientation of a spacecraft using three engine flywheels [5–9], taking into account external uncontrollable noise, whose statistical description is absent.

We propose to form controlling moments applied to flywheels through feedback as nonlinear functions of the phase variables of the conflict-controlled system including the Euler dynamic equations and kinematic equations in Rodrigues–Hamilton variables. As a result, the solution of the initial nonlinear game control problem is reduced to the solution of simple linear game problems and reorientation is achieved by one spatial turn without additional constraints on the character of the resulting motion. Admissible levels of noise depending on given constraints on controlling moments are estimated.

This work develops the results obtained in [10, 11], where control was ensured by external force moments. It is noteworthy that the proposed approach in the absence of noise makes it possible to obtain speed-suboptimal laws for reorientation control [12, 13].

1. FORMULATION OF THE PROBLEM

We consider an asymmetric solid body with homogeneous symmetric flywheels whose rotation axes are fixed along the principal central axes of inertia of the body. The rotation of this system (gyrostat) about the center of mass is described by the differential equations [7]

$$\begin{gathered}
(A_1 - J_1)x_1' \\
= (A_2 - A_3)x_2x_3 + J_2x_3\varphi_2' - J_3x_2\varphi_3' - u_1 + v_1, \\
(A_2 - J_2)x_2' \\
= (A_3 - A_1)x_1x_3 + J_3x_1\varphi_3' - J_1x_3\varphi_1' - u_2 + v_2, \\
(A_3 - J_2)x_3' \\
= (A_1 - A_2)x_1x_2 + J_1x_2\varphi_1' - J_2x_1\varphi_2' - u_3 + v_3, \\
J_i\left(\varphi_i'' + x_i'\right) = u_i,
\end{gathered} \tag{1.1}$$

Here, A_i are the principal central moments of inertia of the gyrostat; x_i are the projections of the angular velocity of the main body on the respective principal central axes $\mathbf{k}_i$ of the ellipsoid of inertia of the gyrostat; J_i and φ_i are the axial moments of inertia and the rotation angles of flywheels (rotors), respectively, whose rotation axes are fixed along the axes $\mathbf{k}_i$; the control moments u_i (moments of internal forces) are applied to flywheels and are created by special engines; and moments v_i characterize external forces and uncontrollable external perturbations acting on the main body.

The vectors with the components $\mathbf{x}$, $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\varphi}'$, and x_i, $u_i = 1, 2, 3$, are denoted as v_i, φ_i', respectively.

In addition to Eqs. (1.1), we consider the following kinematic equations in the Rodrigues–Hamilton variables [14]:

$$\begin{gathered}
2\eta_1' = \eta_4x_1 + \eta_2x_3 - \eta_3x_2, \\
2\eta_2' = \eta_4x_2 + \eta_3x_1 - \eta_1x_3, \\
2\eta_3' = \eta_4x_3 + \eta_1x_2 - \eta_2x_1, \quad \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 1.
\end{gathered} \tag{1.2}$$

They determine the orientation of the solid body. The vector with the components (η_i, and η_4) is denoted as $\boldsymbol{\eta}$. The control moments $u_i = u_i(\mathbf{x}, \boldsymbol{\eta}, \boldsymbol{\varphi}')$ are sought according to the feedback principle in the class K of functions discontinuous in $\mathbf{x}$ and $\boldsymbol{\eta}$. The realizations $u_i[t]$ are measurable functions satisfying the given constraints

$$|u_i| \le \alpha_i = \text{const} > 0. \tag{1.3}$$

Noise $v_i \in K_1$ can be implemented in the form of any measurable functions $v_i = v_i[t]$ with the constraints

$$|v_i| \le \beta_i = \text{const} > 0. \tag{1.4}$$

For any admissible noise implementation $v_i[t]$, the solutions of the system of Eqs. (1.1) and (1.2) for $u_i \in K$ are treated in A.F. Filippov's sense [15] as absolutely continuous functions $\mathbf{x}[t]$ and $\boldsymbol{\eta}[t]$ satisfying the corresponding system of differential inclusions.

Problem of three-axis reorientation. It is necessary to determine the control moments $u_i \in K$ that should be applied to flywheels at any admissible values $v_i \in K_1$ in order to transform the solid body in a finite time from an arbitrary initial position $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}_0$ to a given position $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}_1$. Both states are states of rest $\mathbf{x}(t_0) = \mathbf{x}(t_1) = \mathbf{0}$. In addition, $\boldsymbol{\varphi}'(t_0) = \mathbf{0}$. The time $t_1 > t_0$ is not fixed.

Without loss of generality, we accept $\boldsymbol{\eta}(t_1) = (0, 0, 0, 1)$. In this case, the coordinate system associated with the body is matched with the given coordinate system in the process of reorientation.

2. AUXILIARY LINEAR CONFLICT-CONTROLLED SYSTEM

We consider nonlinear control moments of the form (expressions for u_1, u_2 and u_3 are obtained from Eq. (2.1) by the cyclic permutation of indices $1 \to 2 \to 3$)

$$\begin{aligned} u_1 = -\frac{2(A_1 - J_1)}{\eta_4}\Big[& u_1^*\left(\eta_1^2 + \eta_4^2\right) \\ & + u_2^*(\eta_1\eta_2 + \eta_3\eta_4) + u_3^*(\eta_1\eta_3 - \eta_2\eta_4) \\ & + 1/4\,\eta_1\left(x_1^2 + x_2^2 + x_3^2\right)\Big] \\ & + \left(A_2 x_2 + J_2\varphi_2'\right)x_3 - \left(A_3 x_3 + J_3\varphi_3'\right)x_2 \\ & (1 \to 2 \to 3), \end{aligned} \tag{2.1}$$

where u_i^* are certain auxiliary constraints, which will be discussed below.

Control moments (2.1) make it possible to separate the following linear conflict-controlled system of differential equations from the closed nonlinear conflict-controlled system of Eqs. (1.1), (1.2), and (2.1):

$$\eta_i'' = u_i^* + v_i^*. \tag{2.2}$$

"Noise" v_i^* in Eqs. (2.2) have the form

$$\begin{gathered} v_1^* = 1/2\big[\eta_4 v_1/(A_1 - J_1) \\ + \eta_2 v_3/(A_3 - J_3) - \eta_3 v_2/(A_2 - J_2)\big] \\ (1 \to 2 \to 3). \end{gathered}$$

Noise v_i^* can be estimated using the Cauchy–Bunyakovsky inequality and inequality (1.4):

$$\left|v_i^*\right| \le \beta^*, \tag{2.3}$$

$$\begin{gathered} \beta^* = 1/2\Big[\left(\beta_1/(A_1 - J_1)\right)^2 + \left(\beta_3/(A_3 - J_3)\right)^2 \\ + \left(\beta_2/(A_2 - J_2)\right)^2\Big]^{1/2}. \end{gathered}$$

For system (2.2), we solve the control problem of the fastest transfer to the position u_i^*.

$$\eta_i = \eta_i' = 0. \tag{2.4}$$

Control is achieved through u_i^* at any acceptable implementations v_i^*, satisfying inequality (2.3). To solve this game control problem, the acceptable levels u_i^* should be higher than the levels v_i^*. The corresponding constraints are accepted in the form

$$\left|u_i^*\right| \le \alpha_i^*, \quad \left|v_i^*\right| \le \beta^* = \rho_i\alpha_i^*, \quad 0 < \rho_i < 1. \tag{2.5}$$

The procedure of setting the levels α_i^* will be discussed below. Here, they are treated as given, so that conditions (2.5) are satisfied.

The solution of the indicated linear game problem for system (2.2) involves the solution of the optimal speed problem for the system [16]

$$\eta_i'' = (1 - \rho_i)u_j^*. \tag{2.6}$$

The boundary conditions are the same as for system (2.2). The solution of the optimal speed problem for system (2.6) has the form [17]

$$u_i^*\left(\eta_i, \eta_i'\right) = \begin{cases} \alpha_i^* \operatorname{sign}\psi_i^\rho\left(\eta_i, \eta_i'\right), & \psi_i^\rho \ne 0 \\ \alpha_i^* \operatorname{sign}\eta_i = -\alpha_i^* \operatorname{sign}\eta_i', & \psi_i^\rho = 0 \end{cases}, \tag{2.7}$$

where $\psi_i^\rho\left(\eta_i, \eta_i'\right) = -\eta_i - [2(1 - \rho_i)\alpha_i^*]^{-1}\eta_i'\left|\eta_i'\right| =$ are the switching functions.

At $v_i^* \ne -\rho_i u_i^*$, motions described by the system of Eqs. (2.2) and (2.7) on the phase planes of the variables η_i and η_i' first (until achieving switching curves) occur between the branches of parabolas that are trajectories of the systems $\eta_i'' = (1 \pm \rho_i)u_i^*$ with u_i^* of form (2.7). Further, when switching curves $\psi_i^\rho(\eta_i, \eta_i') = 0$ are achieved, motions occur along them in the sliding mode until achieving the required final values $\eta_i = \eta_i' = 0$.

The indicated motions described by the system of differential equations (2.2) and (2.7) with the discontinuous right-hand side are treated as solutions in Filippov's sense as absolutely continuous functions satisfying almost everywhere the corresponding system of differential inclusions

$$\eta_i'' \in F_i = U_i^* + v_i^*,$$

$$U_i^*\left(\eta_i, \eta_i'\right) = \begin{cases} \alpha_i^* \operatorname{sign} \psi_i^\rho\left(\eta_i, \eta_i'\right), & \psi_i^\rho \neq 0, \\ \left[-\alpha_i^*, \alpha_i^*\right], & \psi_i^\rho = 0. \end{cases}$$

On the sections of the solutions that correspond to the sliding modes, auxiliary controls u_i^* have the values $\pm\alpha_i^*$ with infinitely frequent changes in sign. The quantity

$$\tau = \max(\tau_i),\ \tau_i = 2\{|\eta_{i0}|(1-\rho_i)\alpha_i^*]^{-1}\}^{1/2} \tag{2.8}$$

determines the minimum guaranteed time τ of achieving the position $\eta_i = \eta_i' = 0$ in the auxiliary linear game problem.

It is noteworthy that subsystems of system (2.2) that achieve the required position earlier than the last subsystem will stay in this position. In this case, the corresponding control u_i^* in these subsystems will block noise v_i^*.

3. ALGORITHM FOR THE SOLUTION OF THE PROBLEM OF THREE-AXIS REORIENTATION

Solving the equations of system (1.2) as algebraic equations with respect to x_i, we obtain the equalities

$$x_1 = \frac{2}{\eta_4}\Big[\eta_1'\left(\eta_1^2 + \eta_4^2\right) + \eta_2'(\eta_1\eta_2 + \eta_3\eta_4) + \eta_3'(\eta_1\eta_3 - \eta_2\eta_4)\Big] \tag{3.1}$$

$$(1 \to 2 \to 3).$$

Therefore, the solution of the considered linear game problem of the fastest transfer to the position $\eta_i = \eta_i' = 0$ means the solution of the initial nonlinear problem of reorientation through control moments (2.1). The guaranteed reorientation time is τ.

The iterative algorithm for solving the formulated nonlinear reorientation problem includes the following stages.

(i) Choice of structure (2.1) of the control moments u_i with u_i^* of form (2.7). In the case $\boldsymbol{\eta}(t_1) \neq (0, 0, 0, 1)$, it is sufficient to pass to control moments obtained from Eq. (2.1) by the permutation of the indices.

(ii) Estimate of the level β^* of "auxiliary" noise v_i^* by formulas (2.3).

(iii) Setting the levels α_i^* of auxiliary controls u_i^*. In this case, α_i^*, β^* predetermine the corresponding guaranteed time $\tau = t_1 - t_0$ of the reorientation of the solid body.

(iv) Test of the feasibility of specified constraints (1.3) for the control moments u_i. In view of Eqs. (3.1), this test can be performed on a set of possible states of the auxiliary linear system of differential equations (2.2) and (2.7), as well as the linear inhomogeneous system of differential equations for the determination of φ_i'.

If constraints (1.3) are not satisfied or, on the contrary, they are satisfied excessively, it is necessary to continue to seek the appropriate values α_j^*. Otherwise, reorientation occurs in the time τ.

The system of differential equations for the determination of φ_i' is obtained after the substitution of first three equations of system (1.1), where Eqs. (2.1) are substituted for u_i, into the remaining three equations of system (1.1). The coefficients in the indicated linear inhomogeneous system of differential equations depend on x_i and η_i, the free terms depend on x_i, η_i, and v_i, and can be estimated on the set of possible states of the auxiliary linear system of Eqs. (2.2) and (2.7).

However, in view of the structure of control moments (2.1), they can be estimated having only estimates of expressions $A_i x_i + J_i\varphi_i'$.

Such estimates can be obtained using the function

$$M^2(t) = \sum\Big[A_i x_i(t) + J_i\varphi_i'(t)\Big]^2.$$

Using the Cauchy–Bunyakovsky inequality, inequalities (1.4), and system (1.1), we obtain $M'(t) \leq \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}$. As a result, the desired estimates have the form

$$\left|A_i x_i(t) + J_i\varphi_i'(t)\right| \leq t\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}. \tag{3.2}$$

4. ESTIMATES OF ACCEPTABLE NOISE LEVELS

We present direct estimates of acceptable noise levels v_i, that determine the possibility of solving the initial nonlinear reorientation problem through control moments (2.1) applied to flywheel.

Let us designate

$$\gamma = \min\,[(A_i - J_i)^{-1}(1 + \eta_{i0}^2/\eta_{40}^2)^{-1/2}\alpha_i].$$

Theorem. *If the noise range is determined by the inequality*

$$\sqrt{3}\Big[\sum\big(\beta_i^2/(A_i - J_i)^2\big)\Big]^{1/2} + 8\sqrt{2}\big(1-\eta_{40}^2\big)^{1/2}\Big[\sum\beta_i^2\Big]^{1/2} < \gamma,$$

the reorientation problem can be solved using control moments given by Eqs. (2.1) and (2.7) satisfying given constraints (1.3).

5. COMPUTER SIMULATION RESULTS

For a gyrostat with $A_1 = 4 \cdot 10^4$, $A_2 = 8 \cdot 10^4$, $A_3 = 5 \cdot 10^4$ kg m^2; $J_1 = 4 \cdot 10^3$, $J_2 = 8 \cdot 10^3$, $J_3 = 5 \cdot 10^3$ kg m^2, we consider the problem of three-axis reorientation from the initial position with $\eta_{10} = 0.353$, $\eta_{20} = 0.434$, $\eta_{30} = 0.432$, $\eta_{40} = 0.707$ rad, and $\varphi'_{i0} = 0$ to the given equilibrium position ($x_i = 0$) with $\eta_i = 0$ and $\eta_4 = 1$. Complicating the problem, we set $x_{10} = 1 \cdot 10^{-3}$, $x_{20} = 1.55 \cdot 10^{-3}$, $x_{30} = 1.15 \cdot 10^{-3}$ rad/s.

The acceptable bounds of noise v_i are specified by the equality $\beta* = 10^{-3}$ rad/s^2, which is valid, e.g., for the case with $\beta_1 = 41.57$ N m, $\beta_2 = 83.14$ N m, and $\beta_3 = 51.96$ N m.

According to the requirements for the flexibility of a spacecraft, we set $\tau = 70$ s. We estimate resources required in this case for the reorientation using control moments specified by Eqs. (2.1) and (2.7).

The given τ value predetermines the levels α_i^* of auxiliary controls u_i^*. We use the equalities $T_i = \tau$

Table 1

t	u_i^*	η'_i	η_i
$[0, T_i^*]$	$-\alpha_i^*$	$-(1-\rho_i)\alpha_i^* t + \eta'_{i0}$	$-0.5(1-\rho_i)\alpha_i^* t^2 + \eta'_{i0}t + \eta_{i0}$
$(T_i^*, T_i]$	α_i^*	$(1-\rho_i)\alpha_i^*(t - T_i)$	$0.5(1-\rho_i)\alpha_i^*(t - T_i)^2$

meaning that the guaranteed reorientation times in all variables η_i are the same. Since the points (η_{i0}, η'_{i0}) of the phase plane in the case under consideration are above the switching curves $\eta_i = [2(1-\rho_i)\alpha_i^*]^{-1}(\eta'_i)^2$, values are determined from the relations

$$\left(\eta'_{i0} + 2S_i\right)P_i^{-1} = \tau,\quad S_i = \left[\eta_{i0}P_i + 0.5\left(\eta'_{i0}\right)^2\right]^{1/2}, \tag{5.1}$$

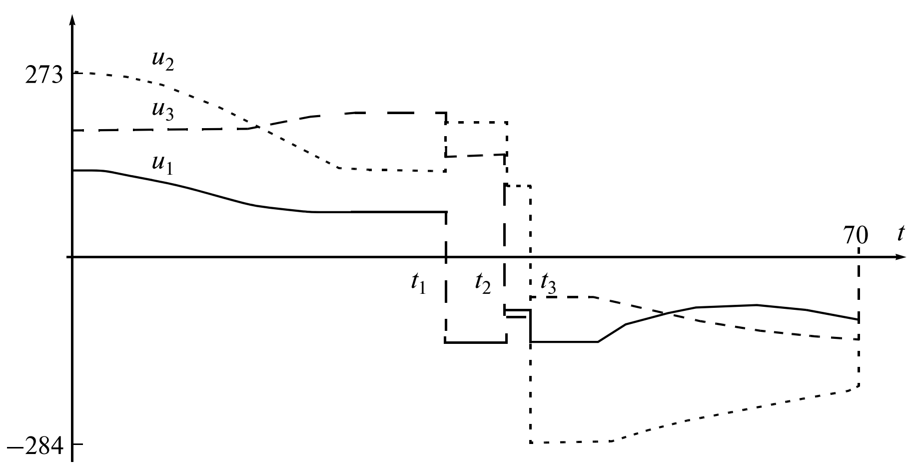
$$P_i = (1-\rho_i)\alpha_i^*.$$

From Eqs. (5.1),

$$\alpha_1^* = 1.295 \cdot 10^{-3},\ \alpha_2^* = 1.369 \cdot 10^{-3},\quad \alpha_3^* = 1.368 \cdot 10^{-3}\ (\text{rad/s}^2). \tag{5.2}$$

(i) *Case of the worst values* $v_i^* = -\rho_i u_i^*$. The values $\alpha_i = \max|u_i|$ can be determined along the optimal trajectories of system (2.6), where α_i^* are given by Eqs. (5.2), as well as by numerically integrating the linear inhomogeneous system of differential equations for the determination of φ'_i. The expressions necessary for the calculations are summarized in Table 1, where $T_i^* = \left(\eta'_{i0} + S_i\right)P_i^{-1}$, $T_i = T_i^* + S_iP_i^{-1}$. The calculation gives $\alpha_1 = 131.25$ N m, $\alpha_2 = 283.55$ N m, $\alpha_3 = 210.55$ N m.

Control moments u_1, u_2, and u_3 given by Eqs. (2.1) and (2.7) are piecewise continuous functions with three switching moments at $t_1 = 35.43$ s, $t_2 = 36.66$ s,



Control moments in case 1.

Table 2

t	u_i^*	η_i'	η_i
$[0, T_i^*]$	$-\alpha_i^*$	$-\alpha_i^* t + \eta_{i0}'$	$-0.5(1-\rho_i)\alpha_i^* t^2 + \eta_{i0}' t + \eta_{i0}$
$(T_i^*, T_i]$	$\pm\alpha_i^*$	$(1-\rho_i)\alpha_i^*(t - T_i)$	$0.5(1-\rho_i)\alpha_i^*(t-T_i)^2$
$(T_i, \max T_i]$	0	0	0

and $t_3 = 36.75$ s (see figure, where a different time scale is introduced in the interval $[t_1, t_3]$ for convenience).

(ii) *Case of the absence of noise* $v_i^* \equiv 0$. In this case, the sliding mode occurs along switching curves and, on average, $u_i^* = (1-\rho_i)\alpha_i^*$ или $u_i^* = -(1-{}_i)\,\alpha_i^*$ on the corresponding branches of the switching curves. To estimate the levels of the control moments u_i, we assume that (i) sliding occurs strictly along the switching curves in the time corresponding to the indicated average u_i^*; values and (ii) u_i^* along the switching curves have the values $\pm\alpha_i^*$ with a finite number of changes in sign. The expressions necessary for the estimate are summarized in Table 2, where

$$T_i^* = \left\{\eta_{i0}' + \left[\left(2\eta_{i0}\alpha_j^* + \left(\eta_{i0}'\right)^2\right)(1-\rho_i)\right]^{1/2}\right\}\left[(2-\rho_i)^{1/2}\alpha_i^*\right]^{-1},$$

$$T_i = \left\{\eta_{i0}' + \left[\left(2\eta_{i0}\alpha_i^* + \left(\eta_{i0}'\right)^2\right)(2-\rho_i)\right]^{1/2}\right\} \times \left[(1-\rho_i)^{1/2}\alpha_i^*\right]^{-1}.$$

The calculation gives

$\alpha_1 = 198.86$, $\alpha_2 = 310.66$, $\alpha_3 = 184.69$ (N m).

CONCLUSIONS

(i) A constructive method has been proposed for the solution of the nonlinear problem of the three-axis reorientation of an asymmetric solid body using the internal force moments applied to attached flywheels with the game noise model. Acceptable levels of noise depending on constraints on control moments have been estimated.

The result is overestimated because of the use of a number of inequalities for their justification, as well as because of the assumption of the worst behavior of the system under the accepted constraints. However, the computer simulation indicates that the proposed structure of control moments (2.1) is efficient even for the cases where the noise levels are beyond the indicated constraints.

The proposed method for solution is generally similar to the decomposition method for controlled systems [18–20]. In contrast to [18–20], the initial nonlinear conflict-controlled system is decomposed into a set of the simplest independent linear conflict controlled subsystems owing to the specially chosen structural form of the control moments. Owing to the indicated difference in the choice of the structural form of the control moments, the functions presenting their implementations are also different. Controls in [18–20] are relay and reach the extremal allowable values, whereas the control moments specified by Eqs. (2.1) and (2.7) generally reach their extremal values only at certain times and the specified constraints on the control resources are tested in the iterative regime on the set of possible states of the auxiliary linear conflict-controlled system of differential equations.

(ii) The proposed structure of the control moments can be efficiently used when the initial perturbations of the angular velocity of the body (initial values of the variables x_i) are fairly small, whereas the initial angular deviation of the fixed axis associated with the body from a given direction in space can be quite large. In view of this circumstance, it is worth noting that the required initial angular velocity of the body can be always ensured through its preliminary reduction.

(iii) The problem considered in the paper belongs to the class of partial control problems (over a part of variables determining the state of a system under analysis) [12, 13, 21]. Also solved [22] is the problem of uniaxial reorientation of a solid body in the game noise model.

APPENDIX. PROOF OF THE THEOREM

The solvability conditions for the auxiliary linear game control problem and, consequently, the possibility of the solution of the initial nonlinear reorientation problem using control moments given by Eqs. (2.1), where u_i^* are specified by Eqs. (2.7), are determined by the inequalities

$$\beta_i^* < \alpha_i^*. \qquad \text{(A.1)}$$

We set $\alpha_i^* - \beta_i^* = \varepsilon > 0$ (where ε is a fairly small number). For convenience, we represent Eqs. (2.1) for the moments u_i in the form of the sum of two terms $u_i = u_i^{(1)} + u_i^{(2)}$, where

$$u_1^{(1)} = -\frac{A_1 - J_1}{\eta_4}\Big[u_1^*\left(\eta_1^2 + \eta_4^2\right) + u_2^*(\eta_1\eta_2 + \eta_3\eta_4) + u_3^*(\eta_1\eta_3 - \eta_2\eta_4)\Big] + \left(A_2x_2 + J_2\varphi_2'\right)x_3 - \left(A_3x_3 + J_3\varphi_3'\right)x_2 \quad (1 \to 2 \to 3).$$

The analysis of the phase portrait indicates that the following inequalities are valid for the set Ω of the states of the system of Eqs. (2.2) and (2.7):

$$\eta_i^2 \le \eta_{i0}^2, \quad \eta_{40}^2 \le \eta_4^2,$$
$$\max\left(\eta_i'\right)^2 = |\eta_{i0}|\left[\left(\alpha_i^*\right)^2 - \left(\beta_i^*\right)^2\right]\left(1/\alpha_i^*\right). \quad (A.2)$$

Taking into account estimate (3.2), Eqs. (2.8) for τ_i, the phase portrait of the system of Eqs. (2.2) and (2.7), and the Cauchy–Bunyakovsky inequality, we arrive at the relations

$$\left|\left(A_2x_2 + J_2\varphi_2'\right)x_3\right| \le \frac{2}{|\eta_4|}\left[\sum\beta_i^2\right]^{1/2} \times\left|\tau_3\eta_3'\left(\eta_3^2 + \eta_4^2\right) + \tau_1\eta_1'(\eta_3\eta_1 + \eta_2\eta_4) + \tau_2\eta_2'(\eta_3\eta_2 - \eta_1\eta_4)\right| \le \frac{2}{|\eta_4|}\left(\eta_1^2 + \eta_4^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2}$$
$$\times\left[\sum\left(\tau_i\eta_i'\right)^2\right]^{1/2} \le 4\left(1 + \eta_{10}^2/\eta_{40}^2\right)^{1/2} \le \Big(\left(\alpha_3^* + \beta^*\right) \times \eta_{30}^2/\alpha_3^* + \left(\alpha_1^* + \beta^*\right)\eta_{10}^2/\alpha_1^* + \left(\alpha_2^* + \beta^*\right)\eta_{20}^2/\alpha_2^*\Big)^{1/2}\left[\sum\beta_i^2\right]^{1/2}$$
$$\times\left[\sum\left(\eta_{i0}^2\right)\right]^{1/2} \le 4\sqrt{2}\left(1 + \eta_{10}^2/\eta_{40}^2\right)^{1/2} \times\left(1 - \eta_{40}^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2} \quad (1 \to 2 \to 3).$$

Then, taking into account inequalities (A.1) and (A.2) and the Cauchy–Bunyakovsky inequality, we obtain the chain of inequalities

$$\left|u_1^{(1)}\right| \le \frac{2(A_1 - J_1)}{|\eta_{40}|}\left(\eta_1^2 + \eta_4^2\right)^{1/2}\left[\sum\left(\alpha_i^{*2}\right)\right]^{1/2} + \left|\left(A_2x_2 + J_2\varphi_2'\right)x_3\right| + \left|\left(A_3x_3 + J_3\varphi_3'\right)x_2\right|$$
$$\le (A_1 - J_1)\left(1 + \eta_{10}^2/\eta_{40}^2\right)^{1/2}\left[\sum\left(\alpha_i^{*2}\right)\right]^{1/2} + 8\sqrt{2}\left(1 + \eta_{10}^2/\eta_{40}^2\right)^{1/2}\left(1 - \eta_{40}^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2}$$
$$(1 \to 2 \to 3).$$

Under the condition $\alpha_i^* - \beta^* = \varepsilon$, we arrive at the following estimates for $u_i^{(2)}$:

$$\left|u_1^{(2)}\right| \le \frac{6(A_1 - J_1)|\eta_{10}|}{|\eta_4|^3}\left(\eta_1^2 + \eta_4^2\right)^{1/2}\left[\sum\left(\eta_i'\right)^2\right]^{1/2}$$
$$\le \frac{6(A_1 - J_1)|\eta_{10}|}{|\eta_{40}|^3}\left\{\sum\left[|\eta_{i0}|\left(\alpha_i^{*2} - \beta^{*2}\right)/\alpha_i^*\right]\right\}^{1/2}$$
$$\le \frac{6\sqrt{2\varepsilon}(A_1 - J_1)|\eta_{10}|}{|\eta_{40}|^3}\left[\sum|\eta_{i0}|\right]^{1/2} \quad (1 \to 2 \to 3).$$

At a sufficiently small value $\varepsilon > 0$, the values $\left|u_i^{(2)}\right|$ are negligibly small. As a result, using Eqs. (1.3) and (A.1), we obtain the relations

$$2(A_1 - J_1)\left(1 + \eta_{10}^2/\eta_{40}^2\right)^{1/2}\left[\sum\left(\alpha_i^{*2}\right)\right]^{1/2} + 8\sqrt{2}\left(1 + \eta_{10}^2/\eta_{40}^2\right)^{1/2}\left(1 - \eta_{40}^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2} \le \alpha_1$$
$$(1 \to 2 \to 3),$$

From these relations, taking into account inequalities (2.3), we arrive at the following estimates for the acceptable noise levels:

$$\sqrt{3}(A_1 - J_1)\left[\sum\left(\beta_i^2/(A_i - J_i)^2\right)\right]^{1/2} + 8\sqrt{2}\left(1 - \eta_{40}^2\right)^{1/2}\left[\sum\beta_i^2\right]^{1/2} < \left(1 + \eta_{10}^2/\eta_{40}^2\right)^{-1/2}\alpha_1 \quad (A.3)$$
$$(1 \to 2 \to 3).$$

If the noise levels v_i satisfy inequalities (A.3), the formulated nonlinear reorientation problem can be solved using the control moments specified by Eqs. (2.1) and (2.7) satisfying the constraints given by Eq. (1.3). The theorem has been proven.

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