ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2013, Vol. 283, Suppl. 1, pp. S126–S138.

published in Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2012, Vol. 18, No. 3.

On the Application of the Regularization Method for the Correction of Improper Problems of Convex Programming

V. D. Skarin[1]

Received March 10, 2012

Abstract—The residual method, which is one of the standard regularization procedures for ill-posed optimization problems, is applied to a convex programming problem. The connection between this method and the regularized Lagrange function method is investigated in the case of optimal correction of improper problems of convex programming. This approach allows one to decrease the number of impropriety classes to be analyzed. Conditions are formulated and convergence estimates of the method are established.
Keywords: convex programming, improper problem, optimal correction, residual method, regularized Lagrange function.

DOI: 10.1134/S0081543813090137

INTRODUCTION

In [1, 2], methods for the correction of improper problems [3] of convex programming (CP) based on the application of the Lagrange function regularized in both variables were considered. The approach from [1] involved the preliminary reduction of the original problem to a similar intermediate problem with the help of Tikhonov's regularization method [4]. The method of quasi-solutions [4], which is another widespread method for the regularization of ill-posed problems, was used for the same purpose in [2]. In a number of cases, the use of additional regularizing procedures allows one to decrease the number of possible impropriety types of CP problems to be analyzed.

Consider the CP problem

$$\min\{f_0(x) \colon x \in X\}, \tag{1}$$

where $X = \{x \colon f(x) \le 0\}$, $f(x) = [f_1(x), \dots, f_m(x)]$, and $f_i(x)$ $(i = 0, 1, \dots, m)$ are convex functions differentiable on $\mathbb{R}^n$. The problem that is (Lagrange) dual to (1) has the form

$$\sup_{\lambda \ge 0} \inf_x L(x, \lambda), \tag{2}$$

where $L(x, \lambda) = f_0(x) + (\lambda,\, f(x))$ is the Lagrange function for problem (1) with $\lambda \in \mathbb{R}^m_+$.

Problem (1) is called *improper* [3] if it does not satisfy the duality relation $f^* = L^*$, where f^* and L^* are the optimal values of problems (1) and (2), respectively. The presence of the impropriety

[1]Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, ul. S. Kovalevskoi 16, Yekaterinburg, 620990 Russia; Institute of Mathematics and Computer Science, Ural Federal University, pr. Lenina 51, Yekaterinburg, 620000 Russia
email: skavd@imm.uran.ru

property depends to a large extent on the emptiness or nonemptiness of the admissible sets X in problem (1) and $\Lambda = \{\lambda \in \mathbb{R}^m_+ \colon \inf_x L(x,\lambda) > -\infty\}$ in problem (2). If $X = \varnothing$ and $\Lambda \neq \varnothing$, then (1) is called [3] an improper CP problem of the first kind; if $X \neq \varnothing$ and $\Lambda = \varnothing$, then (1) is called an improper CP problem of the second kind; finally, if $X = \varnothing$ and $\Lambda = \varnothing$, then (1) is called an improper CP problem of the third kind.

Improper CP problems of the first kind, which are problems with conflicting constraints, occur most often and have been studied well enough. The interest to inconsistent models is induced both by the needs of the mathematical theory and by the necessity to numerically analyze applied problems with conflicting conditions, first of all, industrial and economic problems. These problems are characterized, on the one hand, by errors in modeling a complicated economic system and, on the other hand, by contradictions inherent in a real object (resource shortage, existence of many criteria, and so on).

Due to the frequency of occurrence of improper problems, it becomes important to develop a theory and methods for their numerical approximation (correction), i.e., objective procedures for the “resolution” of conflicting constraints, transformation of an improper model into a set of solvable problems, and choice of an optimal correction among them.

In the present paper, we propose a method for the optimal correction of improper CP problems based on the application of the regularized Lagrange function

$$L_\sigma(x,\lambda) = L(x,\lambda) + \alpha\,\|x\|^2 - \beta\,\|\lambda\|^2,$$

where $\sigma = [\alpha,\beta] > 0$ and $\|\cdot\|$ denotes the Euclidean norm. For the construction of an intermediate approximating problem, we use the residual method [4], which is a known method for the regularization of ill-posed CP problems.

The application of the residual method to improper problems allows one to decrease the number of impropriety types to be analyzed. First, we deduce estimates characterizing the convergence of the residual method in the cases when the functions in the original problem are given exactly and approximately. Then, we study the connection between finding saddle points of the function $L_\sigma(x,\lambda)$ and solving the approximating problem. Separately, we discuss the work of the proposed correction method for CP problems with conflicting constraints and for problems with consistent system of constraints.

1. THE RESIDUAL METHOD AND A CP PROBLEM

The residual method for the regularization of ill-posed CP problem (1) consists [4] in solving the sequence of problems

$$\min\{\|x\|^2 \colon\ x \in X \cap M_\delta\}, \tag{3}$$

where $M_\delta = \{x \colon f_0(x) \le \delta\}$ and δ is some numerical parameter. If (1) is a solvable CP problem with optimal value f^*, then problem (3) has a unique solution x^*_δ for any $\delta \ge f^*$. Since $M_{\delta_1} \supset M_{\delta_2}$ for $\delta_1 \ge \delta_2$, we have $\|x^*_{\delta_1}\| \le \|x^*_{\delta_2}\| \le \ldots \le \|x^*_0\|$, where x^*_0 is the solution of (1) with minimal norm (the normal solution). Thus, all the points x^*_δ lie in the compact set $\{x \colon \|x\| \le \|x^*_0\|\}$, there exists a limit point $\widetilde{x}$ of the sequence $\{x^*_\delta\}$ as $\delta \to f^*$, $\widetilde{x} \in X$, $f_0(\widetilde{x}) = f^*$, and $\|\widetilde{x}\| \le \|x^*_0\|$. It follows from the uniqueness of x^*_0 that $\widetilde{x} = x^*_0$ and $\lim\limits_{\delta\to f^*} x^*_\delta = x^*_0$.

To establish estimates for the convergence of the method, let us reduce problem (3) to the close problem of minimizing a quadratic penalty function

$$\min F_\delta(x,r), \tag{4}$$

where $F_\delta(x,r) = \|x\|^2 + \sum_{i=1}^m r_i f_i^{+2}(x) + r_0\big(f_0(x) - \delta\big)^{+2}$ and $r = [r_0, r_1, \ldots, r_m] > 0$.

From the theory of the penalty function method, it is known [5, Theorem 25.3] that, under rather weak conditions on problem (3), the solutions x_δ^* and $\widetilde{x}_{r,\delta}$ of problems (3) and (4), respectively, are connected by the estimate $\|\widetilde{x}_{r,\delta} - x_\delta^*\| \le C(1/\sqrt{\overline{r}})$, where C is a constant and $\overline{r} = \min\limits_{0\le i\le m} r_i$. Let us establish a number of relations between solutions of problems (1) and (4).

Theorem 1. *Suppose that problem* (1) *is solvable and satisfies Slater's condition*: *there exists a point* $x^0 \in X$ *such that* $f(x^0) < 0$. *Then, for any* $r > 0$ *and* $\delta \ge f^*$, *the solution* $\widetilde{x}_{r,\delta}$ *of problem* (4) *satisfies the inequalities*

$$f_i^+(\widetilde{x}_{r,\delta}) \le \frac{\|x_0^*\|}{2\sqrt{r_i}}, \qquad i = \overline{1,m}; \tag{5}$$

$$(f_0(\widetilde{x}_{r,\delta}) - \delta)^+ \le \frac{\|x_0^*\|}{2\sqrt{r_0}}; \tag{6}$$

$$|f_0(\widetilde{x}_{r,\delta}) - f^*| \le \max\left\{\frac{\|x_0^*\|}{2}\sum_{i=1}^m \frac{\lambda_i^*}{\sqrt{r_i}},\ \frac{\|x_0^*\|}{2\sqrt{r_0}} + \triangle\right\}, \tag{7}$$

where x_0^* *is the normal solution of* (1), $f^* = f_0(x_0^*)$, $\lambda^* = [\lambda_1^*, \ldots, \lambda_m^*]$ *is the vector of Lagrange multipliers corresponding to* x_0^*, *and* $\triangle = \delta - f^*$.

Proof. Since $(\triangledown_x F_\delta(\widetilde{x}_{r,\delta},\ r),\ x - \widetilde{x}_{r,\delta}) = 0$ and the functions $f_i(x)$ $(i = \overline{1,m})$ are convex, we have

$$\|x - \widetilde{x}_{r,\delta}\|^2 - (x,\ x - \widetilde{x}_{r,\delta}) = -(\widetilde{x}_{r,\delta},\ x - \widetilde{x}_{r,\delta})$$

$$= \sum_{i=1}^m r_i f_i^+(\widetilde{x}_{r,\delta})(\nabla f_i(\widetilde{x}_{r,\delta}), x - \widetilde{x}_{r,\delta}) + r_0(f_0(\widetilde{x}_{r,\delta}) - \delta)^+(\nabla f_0(\widetilde{x}_{r,\delta}),\ x - \widetilde{x}_{r,\delta})$$

$$\le \sum_{i=1}^m r_i f_i^+(\widetilde{x}_{r,\delta})[f_i(x) - f_i(\widetilde{x}_{r,\delta})] + r_0(f_0(\widetilde{x}_{r,\delta}) - \delta)^+[f_0(x) - f_0(\widetilde{x}_{r,\delta})]$$

for any $x \in \mathbb{R}^n$. Hence, for $x = x_0^*$,

$$\sum_{i=1}^m r_i f_i^{+2}(\widetilde{x}_{r,\delta}) + r_0(f_0(\widetilde{x}_{r,\delta}) - \delta)^{+2} \le -\|x_0^* - \widetilde{x}_{r,\delta}\|^2 + (x_0^*,\ x_0^* - \widetilde{x}_{r,\delta})$$

$$= -\big(\|x_0^* - \widetilde{x}_{r,\delta}\| - \frac{1}{2}\|x_0^*\|\big)^2 + \frac{\|x_0^*\|^2}{4} \le \frac{\|x_0^*\|^2}{4}. \tag{8}$$

The latter inequality immediately implies estimates (5) and (6).

By the definition of the points x_0^* and λ^*, we have

$$f^* - f_0(\widetilde{x}_{r,\delta}) = f_0(x_0^*) - f_0(\widetilde{x}_{r,\delta}) \le \sum_{i=1}^m \lambda_i^* f_i^+(\widetilde{x}_{r,\delta}). \tag{9}$$

On the other hand,

$$f_0(\widetilde{x}_{r,\delta}) - f^* = f_0(\widetilde{x}_{r,\delta}) - \delta + \delta - f^* \le (f_0(\widetilde{x}_{r,\delta}) - \delta)^+ + \triangle. \tag{10}$$

Estimating (9) and (10) with the help of relations (5) and (6), respectively, we obtain (7).

The theorem is proved.

The point x_0^* is the unique solution of the inequality $g(x) \le 0$, where $g(x) = \max\{\|x\|^2 - \|x_0^*\|^2,\ f_i(x)\ (i = \overline{1,m}),\ f_0(x) - f^*\}$. The inequality $F_\delta(\widetilde{x}_{r,\delta}, r) \le F_\delta(x_0^*, r)$ implies $\|\widetilde{x}_{r,\delta}\| \le \|x_0^*\|$ for any r and δ. Further, taking into account estimates (5)–(7), we find that $g(\widetilde{x}_{r,\delta}) \to 0$ as $\bar{r} = \min\limits_{0\le i\le m} r_i \to 0$ and $\triangle \to 0$. Since the constraint $g(x) \le 0$ is correct [5], we have $\lim\limits_{\substack{\bar{r}\to 0\\ \delta\to f^*}} \widetilde{x}_{r,\delta} = x_0^*$.

Let us establish a numerical characteristic of the convergence of $\widetilde{x}_{r,\delta}$ to x_0^*. Write the problem of finding the normal solution x_0^* as

$$\min\{\|x\|^2 \colon x \in X,\ f_0(x) \le f^*\}. \tag{11}$$

The Lagrange function for (11) has the form $H(x, u, u_0) = \|x\|^2 + (u, f(x)) + u_0(f_0(x) - f^*)$, where $u \in \mathbb{R}^m_+$ and $u_0 \in \mathbb{R}^1_+$. Denote by $[x_0^*, u^*, u_0^*]$ a saddle point of the function $H(x, u, u_0)$ in the domain $\mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^1_+$. Note that, under the conditions of Theorem 1, this saddle point exists; we can assume that $u_0^* > 0$ (see Section 4 for details).

Theorem 2. *Suppose that the conditions of Theorem* 1 *hold. Then, for any* $r > 0$ *and* $\delta > f^*$, *the following estimate is valid*:

$$\|\widetilde{x}_{r,\delta} - x_0^*\|^2 \le \frac{1}{16}\sum_{i=0}^{m} \frac{u_i^{*2}}{r_i} + \frac{1}{2}u_0^*\,\triangle,$$

where u_i^* *are components of the vector* u^*, $i = \overline{1,m}$.

Proof. The existence of the saddle point $[x_0^*, u^*, u_0^*]$ for the function $H(x, u, u_0)$ is equivalent to the Kuhn–Tucker conditions for (11):

$$\nabla_x H(x_0^*, u^*, u_0^*) = 0, \qquad u_i^* f_i(x_0^*) = 0, \qquad i = \overline{1,m}, \qquad u_0^*(f_0(x_0^*) - f^*) = 0.$$

These conditions imply the relations

$$2(x_0^*,\ x_0^* - \widetilde{x}_{r,\delta}) = \sum_{i=1}^{m} u_i^*(\nabla f_i(x_0^*),\ \widetilde{x}_{r,\delta} - x_0^*) + u_0^*(\nabla f_0(x_0^*),\ \widetilde{x}_{r,\delta} - x_0^*)$$

$$\le \sum_{i=1}^{m} u_i^*[f_i(\widetilde{x}_{r,\delta}) - f_i(x_0^*)] + u_0^*[f_0(\widetilde{x}_{r,\delta}) - f_0(x_0^*)] = \sum_{i=1}^{m} u_i^* f_i(\widetilde{x}_{r,\delta}) + u_0^*(f_0(\widetilde{x}_{r,\delta}) - f^*).$$

Hence,

$$2(x_0^*,\ x_0^* - \widetilde{x}_{r,\delta}) \le \sum_{i=1}^{m} u_i^* f_i^+(\widetilde{x}_{r,\delta}) + u_0^*(f_0(\widetilde{x}_{r,\delta}) - \delta)^+ + u_0^*\ \triangle\,.$$

Substituting this inequality into (8), we obtain

$$\|\widetilde{x}_{r,\delta} - x_0^*\|^2 \le -\sum_{i=1}^{m}\left[r_i f_i^{+2}(\widetilde{x}_{r,\delta}) - \frac{1}{2}u_i^* f_i^+(\widetilde{x}_{r,\delta})\right]$$

$$-\left[r_0(f_0(\widetilde{x}_{r,\delta}) - \delta)^{+2} - \frac{1}{2}u_0^*(f_0(\widetilde{x}_{r,\delta}) - \delta)^+\right] + \frac{1}{2}u_0^*\,\triangle = -\sum_{i=1}^{m}\left[\left(\sqrt{r_i}f_i^+(\widetilde{x}_{r,\delta}) - \frac{u_i^*}{4\sqrt{r_i}}\right)^2 - \frac{u_i^{*2}}{16\,r_i}\right]$$

$$-\left[\left(\sqrt{r_0}(f_0(\widetilde{x}_{r,\delta}) - \delta)^+ - \frac{u_0^*}{4\sqrt{r_0}}\right)^2 - \frac{u_0^{*2}}{16\,r_0}\right] + \frac{1}{2}\,u_0^*\triangle \le \frac{1}{16}\sum_{i=0}^{m}\frac{u_i^{*2}}{r_i} + \frac{1}{2}u_0^*\,\triangle\,.$$

The theorem is proved.

2. A CP PROBLEM WITH INACCURATELY GIVEN INFORMATION

Suppose that continuous functions $f_i^\varepsilon(x)$ defined on $\mathbb{R}^n$ and such that

$$|f_i^\varepsilon(x) - f_i(x)| < \varepsilon \qquad (\forall x \in \mathbb{R}^n,\ i = 0, 1, \ldots, m), \qquad \varepsilon > 0, \tag{12}$$

are known in problem (1) instead of the functions $f_i(x)$. Then, we have the following problem instead of (4):

$$\min_x F_\delta^\varepsilon(x, r), \tag{13}$$

where $F_\delta^\varepsilon(x, r)$ is obtained from (4) by the change of the functions $f_i(x)$ for $f_i^\varepsilon(x)$:

$$F_\delta^\varepsilon(x, r) = \|x\|^2 + \sum_{i=1}^m r_i [f_i^{\varepsilon +}(x)]^2 + r_0 (f_0^\varepsilon(x) - \delta)^{+2}, \qquad r = [r_0, r_1, \ldots, r_m] > 0.$$

.

Lemma 1. *Problem* (13) *is solvable for any* r, δ, *and* ε.

Proof. Inequalities (12) imply

$$F_\delta(x, r) = \|x\|^2 + \sum_{i=1}^m r_i f_i^{+2}(x) + r_0 (f_0(x) - \delta)^{+2}$$

$$= F_\delta^\varepsilon(x, r) + \sum_{i=1}^m r_i \Big[f_i^{+2}(x) - f_i^{\varepsilon +2}(x) \Big] + r_0 \Big[(f_0(x) - \delta)^{+2} - (f_0^\varepsilon(x) - \delta)^{+2} \Big]$$

$$\le F_\delta^\varepsilon(x, r) + \varepsilon \sum_{i=1}^m r_i \big(2 f_i^{\varepsilon +}(x) + \varepsilon \big) + \varepsilon\, r_0 \big[2 (f_0^\varepsilon(x) - \delta)^+ + \varepsilon \big]$$

$$\le F_\delta^\varepsilon(x, r) + \sum_{i=1}^m r_i \Big(f_i^{\varepsilon +2}(x) + \varepsilon^2 \Big) + r_0 \Big[\big(f_0^\varepsilon(x) - \delta \big)^{+2} + \varepsilon^2 \Big] + \varepsilon^2 \sum_{i=0}^m r_i$$

$$= 2 F_\delta^\varepsilon(x, r) - \|x\|^2 + 2\, \varepsilon^2 \sum_{i=0}^m r_i \le 2 F_\delta^\varepsilon(x, r) + 2\, \varepsilon^2 \sum_{i=0}^m r_i.$$

Therefore, if $x' \in M_1^\varepsilon = \{x \colon F_\delta^\varepsilon(x, r) \le C_1\}$, where $C_1 = C_1(\varepsilon, r, \delta)$, then

$$F_\delta(x', r) \le 2\, C_1 + 2\, \varepsilon^2 \sum_{i=0}^m r_i = C_2(\varepsilon, r, \delta).$$

Since the function $F_\delta(x, r)$ is strongly convex in x, the set $M_2 = \{x \colon F_\delta(x, r) \le C_2(\varepsilon, r, \delta)\}$ is bounded for any fixed ε, r, and δ. Consequently, the set M_1^ε is also bounded and the continuous function $F_\delta^\varepsilon(x, r)$ attains its minimum in x on $\mathbb{R}^n$ for any ε, r, and δ.

The lemma is proved.

Lemma 1 implies the existence of a solution $x_{r,\delta}^\varepsilon$ of problem (13):

$$F_\delta^\varepsilon(x_{r,\delta}^\varepsilon, r) = \min_x F_\delta^\varepsilon(x, r).$$

Theorem 3. *If the conditions of Theorem* 1 *hold, then the following estimates are valid for any* $r = [r_0, r_1, \ldots, r_m] > 0$, $\delta > f^*$, *and* $\varepsilon \ge 0$:

(1) $f_i^+(x^\varepsilon_{r,\delta}) \le \dfrac{\sqrt{B_0}}{\sqrt{r_i}} + \varepsilon, \quad i=\overline{1,m};$

(2) $(f_0(x^\varepsilon_{r,\delta}) - \delta)^+ \le \dfrac{\sqrt{B_0}}{\sqrt{r_0}} + \varepsilon;$

(3) $|f_0^+(x^\varepsilon_{r,\delta}) - f^*| \le \max\left\{ \sum\limits_{i=1}^m \lambda_i^*\big(\dfrac{\sqrt{B_0}}{\sqrt{r_i}} + \varepsilon\big),\ \dfrac{\sqrt{B_0}}{\sqrt{r_0}} + \varepsilon + \triangle \right\};$

(4) $\|x^\varepsilon_{r,\delta}\| \le \sqrt{B_0},$ *where* $B_0 = \|x_0^*\|^2 + \varepsilon^2 \sum\limits_{i=0}^m r_i,$ $\triangle = \delta - f^*$, *and the values* λ_i^* *are from Theorem* 1, $i=\overline{1,m}$.

Proof. The definition of the point $x^\varepsilon_{r,\delta}$ implies the inequality $F^\varepsilon_\delta(x^\varepsilon_{r,\delta}, r) \le F^\varepsilon_\delta(x_0^*, r)$. Hence, in view of the inequalities $f_i^\varepsilon(x_0^*) \le f_i(x_0^*) + \varepsilon \le \varepsilon$ and $f_0^\varepsilon(x_0^*) - \delta \le f_0(x_0^*) - \delta + \varepsilon < \varepsilon$, we obtain $\|x^\varepsilon_{r,\delta}\|^2 + \sum_{i=1}^m r_i\big[f_i^{\varepsilon+}(x^\varepsilon_{r,\delta})\big]^2 + r_0\big(f_0^\varepsilon(x^\varepsilon_{r,\delta}) - \delta\big)^{+2} \le \|x_0^*\|^2 + \varepsilon^2\sum_{i=0}^m r_i = B_0$. This relation immediately implies estimates (1), (2), and (4).

To deduce estimate (3), we apply successively inequalities (9) and (10) (with $\widetilde{x}_{r,\delta}$ replaced by $x^\varepsilon_{r,\delta}$) and estimates (1) and (2).

The theorem is proved.

Corollary 1. *Suppose that* $\varepsilon \to 0$, $\triangle \to 0$, $r_i \to +\infty$, $i = 0,1,\dots,m$, *and there exists a constant* K *such that* $0 < r_i \le K \min\limits_{0\le i\le m} r_i$. *Then,* $x^\varepsilon_{r,\delta} \to x_0^*$.

If, in Theorem 3, instead of the point $x^\varepsilon_{r,\delta}$, we take an approximate solution $\bar{x}^\varepsilon_{r,\delta}$ of problem (13) with given accuracy $\xi \ge 0$

$$F^\varepsilon_\delta(\bar{x}^\varepsilon_{r,\delta}, r) \le \min_x F^\varepsilon_\delta(x, r) + \xi,$$

then estimates (1)–(4) change very slightly. The difference is that the value B_0 must be replaced by $B_1 = \|x_0^*\|^2 + \varepsilon^2\sum_{i=0}^m r_i + \xi$.

3. THE REGULARIZED LAGRANGE FUNCTION AND AN IMPROPER CP PROBLEM OF THE FIRST KIND

Using the regularized Lagrange function $L_\sigma(x,\lambda)$, we construct for problem (1) the primal and dual functions

$$\varphi_\sigma(x) = \max_{\lambda\ge 0} L_\sigma(x,\lambda), \qquad \psi_\sigma(\lambda) = \min_x L_\sigma(x,\lambda),$$

which are defined everywhere on $\mathbb{R}^n$ and $\mathbb{R}^m_+$, respectively.

It is easy to see that $\varphi_\sigma(x) = L_\sigma(x, \lambda(x))$, where $\lambda(x) = \dfrac{1}{2\beta} f^+(x)$. For this, it is sufficient to check the inequality $(\triangledown_\lambda L_\sigma(x,\lambda(x)),\ \lambda - \lambda(x)) \le 0$ for all $\lambda \ge 0$. Thus,

$$\varphi_\sigma(x) = f_0(x) + \frac{1}{4\beta}\,\|f^+(x)\|^2 + \alpha\,\|x\|^2.$$

Evidently, the function $\varphi_\sigma(x)$ is strongly convex on $\mathbb{R}^n$. The strong concavity of $\psi_\sigma(\lambda)$ on $\mathbb{R}^m_+$ is also easily verified. Consequently, there exist unique points $x^\sigma = \arg\min\limits_x \varphi_\sigma(x)$ and $\lambda^\sigma = \arg\max\limits_{\lambda\ge 0}\psi_\sigma(\lambda)$ and, according to the minimax theorem (see, for example, [7]),

$$\varphi_\sigma(x^\sigma) = \psi_\sigma(\lambda^\sigma) = L_\sigma(x^\sigma, \lambda^\sigma).$$

Thus, the function $L_\sigma(x,\lambda)$ has a unique saddle point in $\mathbb{R}^n \times \mathbb{R}^m_+$ for any $\sigma > 0$. This is its essential difference from the standard Lagrange function $L(x,\lambda)$, which is known to have no saddle points if $X = \varnothing$. Due to this property, the function $L_\sigma(x,\lambda)$ is applicable in the analysis and correction of improper problems.

Let $X = \varnothing$ in problem (1); i.e., (1) is an improper CP problem of the first or third kind. Let us correct the constraints of problem (1) with respect to the right-hand sides. We use the notation $X_\xi = \{x\colon f(x) \le \xi\}$ and $E = \{\xi \in \mathbb{R}^m_+\colon X_\xi \ne \varnothing\}$. If the set X_ξ is nonempty and bounded for some ξ or the functions $f_i(x)$ $(i = \overline{1,m})$ are affine, then the set E is convex and closed. Then, there exists a unique element $\bar\xi = \arg\min\{\|\xi\|\colon \xi \in E\}$. It is also easy to show that $X_{\bar\xi} = \widetilde{X}$, where $\widetilde{X} = \operatorname{Arg}\min\limits_x \|f^+(x)\|$; moreover, $\bar\xi = f^+(\widetilde{x})$ for $\widetilde{x} \in \widetilde{X}$.

Let us formulate the problem

$$\min\{f_0(x)\colon\ x \in X_{\bar\xi}\}. \tag{14}$$

Problems (1) and (14) coincide for $\bar\xi = 0$; problem (14) is an approximation (an optimal correction) for (1) for $\bar\xi \ne 0$.

Let us formulate a statement about the applicability of the regularized Lagrange function method to the correction of improper CP problems of the first kind.

Assume that the Kuhn–Tucker conditions for problem (14) hold at a point $\bar x \in X_{\bar\xi}$; i.e., there exists a vector $\bar\lambda \in \mathbb{R}^m_+$ such that

$$\nabla_x L(\bar x, \bar\lambda) = 0, \qquad (\bar\lambda, f(\bar x) - \bar\xi) = 0. \tag{15}$$

These conditions hold, for example, for problems of linear and quadratic programming. By (15), problem (14) is solvable and $\bar x$ is one of its solutions. Note also that, if $\bar\xi \ne 0$ and (15) holds, problem (1) is an improper CP problem of the first kind.

Theorem 4 [8]. *Let conditions* (15) *hold for problem* (14). *The following estimates are valid*:

$$\|(f(x^\sigma) - \bar\xi)^+\| \le \sqrt{\beta}\, C_1(\sigma), \tag{16}$$

$$|f_0(x^\sigma) - \bar f\,| \le C_2(\sigma), \tag{17}$$

$$\|\nabla_x L(x^\sigma, \lambda^\sigma)\| \le \sqrt{\alpha}\, C_3(\sigma), \quad 0 \le (\lambda^\sigma,\ f(x^\sigma) - \bar\xi) \le C_4(\sigma), \tag{18}$$

where $\bar f = f_0(\bar x)$, $C_1(\sigma) = 2\left[\sqrt{\beta}\,\|\bar\lambda\| + (\alpha\,\|\bar x\|^2 + \beta\,\|\bar\lambda\|^2)^{1/2}\right]$, $C_2(\sigma) = \max\{\alpha\,\|\bar x\|^2, \sqrt{\beta}\,\|\bar\lambda\|\, C_1(\sigma)\}$, $C_3(\sigma) = 2\left[\sqrt{\alpha}\,\|\bar x\|^2 + \sqrt{\beta}\,\|\bar\lambda\| C_1(\sigma)\right]^{1/2}$, *and* $C_4(\sigma) = \frac{1}{2}C_3^2(\sigma)$.

Estimates (16) and (17) imply that, if $\sigma \to 0$ and $\beta = o(\alpha)$, then $x^\sigma \to \bar x_0$, where $\bar x_0$ is the normal solution of problem (14). Indeed, from the inequality $\varphi_\sigma(x^\sigma) \le \varphi_\sigma(\bar x_0)$, we have

$$\alpha\,\|x^\sigma\|^2 \le \bar f - f_0(x^\sigma) + \frac{1}{4\beta}\left[\|\bar\xi\|^2 - \|f^+(x^\sigma)\|^2\right] + \alpha\,\|\bar x_0\|^2.$$

Hence,

$$\|x^\sigma\|^2 \le \|\bar x_0\|^2 + \frac{1}{\alpha}\left|f_0(x^\sigma) - \bar f\right|. \tag{19}$$

This, by (17), implies the boundedness of the sequence $\{x^\sigma\}$ as $\sigma \to 0$ and $\beta/\alpha \to 0$. According to (16) and (17), the limit points $\{x^\sigma\}$ are optimal in problem (14). Then, (19) and the uniqueness of the normal solution imply the required convergence.

Estimates (18) show that the point $[x^\sigma, \lambda^\sigma]$ satisfies conditions (15) as $\sigma \to 0$.

If original problem (1) is solvable and regular, then the definitions of the saddle points $[x^\sigma, \lambda^\sigma]$ and $[x^*, \lambda^*]$ of the functions $L_\sigma(x, \lambda)$ and $L(x, \lambda)$, respectively, imply [1] the relation

$$f_0(x^*) - \beta\, \|\lambda^*\|^2 \le L_\sigma(x^\sigma, \lambda^\sigma) \le f_0(x^*) + \alpha\, \|x^*\|^2,$$

so that $\lim\limits_{\sigma\to 0} L_\sigma(x^\sigma, \lambda^\sigma) = f_0(x^*) = f^*$. In the case when (1) is an improper CP problem of the first kind, in view of (17), we obtain

$$L_\sigma(x^\sigma, \lambda^\sigma) = \varphi_\sigma(x^\sigma) = f_0(x^\sigma) + \frac{1}{4\beta}\big\|f^+(x^\sigma)\big\|^2 + \alpha\, \|x^\sigma\|^2 \ge \bar f - C_2(\sigma) + \frac{\|\bar\xi\|^2}{4\beta}.$$

Hence, $\lim\limits_{\sigma\to 0} L_\sigma(x^\sigma, \lambda^\sigma) = +\infty$.

Thus, we can judge whether the original problem is proper from the behavior of $\{L_\sigma(x^\sigma, \lambda^\sigma)\}$ as $\sigma \to 0$.

4. A CP PROBLEM WITH CONFLICTING CONSTRAINTS

Suppose that $X = \varnothing$ in problem (1); i.e., (1) is an improper CP problem of the first or third kind. Evidently, (3) is also an improper problem in this case. If we write the Lagrange function for problem (3)

$$H_\delta(x, u, u_0) = \|x\|^2 + (u, f(x)) + u_0(f_0(x) - \delta),$$

then $\inf\limits_x H_\delta(x, u, u_0) > -\infty$ for any $u \in \mathbb{R}^m_+$, $u_0 \in \mathbb{R}^1_+$, and $\delta \in \mathbb{R}^1$; i.e., in contrast to (1), problem (3) can be an improper CP problem of the first kind only.

Assuming $\delta > \inf\limits_{x\in\mathbb{R}^n} f_0(x)$, consider the set $E_\delta = \{\xi \in \mathbb{R}^m_+ \colon X_\xi \cap M_\delta\} \ne \varnothing$. Suppose that there exist a set of indices $I \subset \{0, 1, \dots, m\}$ and a number C such that the set $\bigcap_{i\in I}\{x\colon f_i(x) \le C\}$ is nonempty and bounded. Then, we can specify the unique vector $\bar\xi_\delta = \arg\min\{\|\xi\| \colon \xi \in\ E_\delta\}$. We may obtain the following situations.

1. There exists a value δ_0 of the parameter δ such that $\bar\xi_\delta = 0$ for $\delta \ge \delta_0$. Then, $X \ne \varnothing$ and one of the two cases holds.

1a. Problem (1) is solvable (here, $\delta_0 = f^*$).

1b. The infimum $\widetilde f$ of the function $f_0(x)$ is not attained on X (it is possible that $\widetilde f > -\infty$ and $\widetilde f = -\infty$).

2. For any $\delta > \inf\limits_x f_0(x)$, we have $\bar\xi_\delta \ne 0$ (here, $X = \varnothing$).

First, consider the case when (3) is an improper CP problem of the first kind. This corresponds to above situations 1a (when $\delta < f^*$) and 2.

Let us formulate the following problem:

$$\min\{\|x\|^2 \colon x \in X_{\bar\xi_\delta} \cap M_\delta\}. \tag{20}$$

This problem is always solvable at the unique point $\widetilde x_\delta$ and coincides with (3) for $\bar\xi_\delta = 0$. In our case, $\bar\xi_\delta \ne 0$, and problem (20) can be considered as one of possible approximations for (3).

It is easy to see that the vector $f^+(x_\delta)$, where $x_\delta = \arg\min\limits_{x\in M_\delta} \|f^+(x)\|^2$, can be taken as $\bar\xi_\delta$. The definition of the vectors $\bar\xi_\delta = (\bar\xi^\delta_1, \dots, \bar\xi^\delta_m)$ and x_δ implies the inequality

$$2\sum_{i=1}^m \bar\xi^\delta_i(\bigtriangledown f_i(x_\delta), x - x_\delta) \ge 0 \qquad (\forall x \in M_\delta).$$

Therefore, from the convexity of the functions $f_i(x)$, we obtain

$$\|\bar{\xi}_\delta\|^2 = (\bar{\xi}_\delta, f(x_\delta)) \le (\bar{\xi}_\delta, f(x)) - \sum_{i=1}^{m} \bar{\xi}_i^\delta (\nabla f_i(x_\delta),\ x - x_\delta) \le (\bar{\xi}_\delta, f(x)).$$

Thus,

$$\|\bar{\xi}_\delta\|^2 \le (\bar{\xi}_\delta, f(x)) \qquad (\forall x \in M_\delta). \tag{21}$$

In what follows, we will consider the regularized Lagrange function $L_\sigma(x,\lambda)$ on the set $M_\delta \times \mathbb{R}^m_+$. Similarly to the existence of the saddle point $[x^\sigma, \lambda^\sigma]$ of the function $L_\sigma(x,\lambda)$ in the domain $\mathbb{R}^n \times \mathbb{R}^m_+$, we can find a unique saddle point $[x^\sigma_\delta, \lambda^\sigma_\delta]$ of the function $L_\sigma(x,\lambda)$ on the set $M_\delta \times \mathbb{R}^m_+$. We will denote by (P^σ_δ) the problem of finding the point $[x^\sigma_\delta, \lambda^\sigma_\delta]$.

Let us show that problems (20) and (P^σ_δ) are closely related. Thus, the chain of problems

$$(1) \longrightarrow (3) \longrightarrow (20) \longrightarrow (P^\sigma_\delta)$$

will specify one of the possible ways to correct improper CP problems.

Assume that the Lagrange function for problem (20)

$$H_{\xi,\delta}(x,u,u_0) = \|x\|^2 + (u, f(x) - \bar{\xi}_\delta) + u_0(f_0(x) - \delta)$$

has a saddle point $[\widetilde{x}_\delta, \widetilde{u}_\delta, \widetilde{u}_{0\delta}]$ in the domain $\mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^1_+$; i.e., the following equalities hold:

$$\nabla_x H_{\xi,\delta}(\widetilde{x}_\delta, \widetilde{u}_\delta, \widetilde{u}_{0\delta}) = 0, \qquad (\widetilde{u}_\delta, f(\widetilde{x}_\delta) - \bar{\xi}_\delta) = 0, \qquad \widetilde{u}_{0\delta}(f_0(\widetilde{x}_\delta) - \delta) = 0. \tag{22}$$

Let us investigate the connection between problems (20) and (P^σ_δ).

Theorem 5. *Suppose that conditions* (22) *hold for problem* (20) *at the point* $[\widetilde{x}_\delta, \widetilde{u}_\delta, \widetilde{u}_{0\delta}]$; *moreover,* $\widetilde{u}_{0\delta} > 0$ *and the parameter* α *of the function* $L_\sigma(x,\lambda)$ *is equal to* α_0, *where* $\alpha_0 \widetilde{u}_{0\delta} > 1$. *Then, the following estimates are valid*:

$$\|(f(x^\sigma_\delta) - \bar{\xi}_\delta)^+\| \le \beta\, B_0(\delta), \tag{23}$$

$$|f_0(x^\sigma_\delta) - \delta| \le \beta\, B_1(\delta), \tag{24}$$

$$\|x^\sigma_\delta - \widetilde{x}_\delta\| \le \sqrt{\beta}\, B_2(\delta), \tag{25}$$

where $B_0(\delta) = 2\alpha_0\|\widetilde{u}_\delta\|$, $B_1(\delta) = \dfrac{2{\alpha_0}^2\|\widetilde{u}_\delta\|^2}{\alpha_0\widetilde{u}_{0\delta} - 1}$, *and* $B_2(\delta) = \frac{1}{2}\sqrt{\alpha_0}\,\|\widetilde{u}_\delta\|$.

Proof. The definition of the saddle point $[x^\sigma_\delta, \lambda^\sigma_\delta]$ implies the inequality

$$(\nabla_x L_\sigma(x^\sigma_\delta, \lambda^\sigma_\delta),\ x - x^\sigma_\delta) \ge 0 \qquad (\forall x \in M_\delta).$$

Therefore,

$$-2\alpha\,(x^\sigma_\delta,\ \widetilde{x}_\delta - x^\sigma_\delta) \le (\nabla_x L(x^\sigma_\delta, \lambda^\sigma_\delta),\ \widetilde{x}_\delta - x^\sigma_\delta).$$

From this and conditions (22), we obtain

$$2\alpha\,\|\widetilde{x}_\delta - x^\sigma_\delta\|^2 \le \alpha \sum_{i=1}^{m} \widetilde{u}_i^\delta (\nabla f_i(\widetilde{x}_\delta),\ x^\sigma_\delta - \widetilde{x}_\delta) + \alpha\widetilde{u}_{0\delta}(\nabla f_0(\widetilde{x}_\delta),\ x^\sigma_\delta - \widetilde{x}_\delta)$$

$$+ (\nabla_x L(x^\sigma_\delta, \lambda^\sigma_\delta),\ \widetilde{x}_\delta - x^\sigma_\delta) \le \alpha \sum_{i=1}^{m} \widetilde{u}_i^\delta (f_i(x^\sigma_\delta) - \bar{\xi}_\delta - f_i(\widetilde{x}_\delta) + \bar{\xi}_\delta)$$

$$+\alpha\widetilde{u}_{0\delta}(f_0(x^\sigma_\delta)-\delta-f_0(\widetilde{x}_\delta)+\delta)+L(\widetilde{x}_\delta,\lambda^\sigma_\delta)-L(x^\sigma_\delta,\lambda^\sigma_\delta)$$
$$\le(\alpha\widetilde{u}_{0\delta}-1)(f_0(x^\sigma_\delta)-\delta)+\alpha(\widetilde{u}_\delta,f(x^\sigma_\delta)-\bar{\xi}_\delta)+(\lambda^\sigma_\delta,f(\widetilde{x}_\delta)-f(x^\sigma_\delta)) \qquad (26)$$

(here, $\widetilde{u}^\delta_i$ are the components of the vector $\widetilde{u}_\delta,\ \ i=1,\dots,m$).

Further, in view of inequality (21) and the fact that $\lambda^\sigma_\delta=\lambda(x^\sigma_\delta)=\frac{1}{2\beta}f^+(x^\sigma_\delta)$, we estimate

$$(\lambda^\sigma_\delta,\,f(\widetilde{x}_\delta)-f(x^\sigma_\delta))\le(\lambda^\sigma_\delta,\,\bar{\xi}_\delta-f(x^\sigma_\delta))=\frac{1}{2\beta}\big(f^+(x^\sigma_\delta),\,\bar{\xi}_\delta-f(x^\sigma_\delta)\big)$$
$$=\frac{1}{2\beta}\big(\bar{\xi}_\delta,f^+(x^\sigma_\delta)\big)-\frac{1}{2\beta}\big\|f^+(x^\sigma_\delta)\big\|^2=-\frac{1}{2\beta}\big\|f^+(x^\sigma_\delta)-\bar{\xi}_\delta\big\|^2+\frac{1}{2\beta}\big[\big\|\bar{\xi}_\delta\big\|^2-\big(\bar{\xi}_\delta,f^+(x^\sigma_\delta)\big)\big]$$
$$\le-\frac{1}{2\beta}\big\|f^+(x^\sigma_\delta)-\bar{\xi}_\delta\big\|^2\le-\frac{1}{2\beta}\big\|(f(x^\sigma_\delta)-\bar{\xi}_\delta)^+\big\|^2.$$

Using the latter inequality and the conditions of the theorem, we obtain from (26), for $\alpha=\alpha_0$,

$$2\|x^\sigma_\delta-\widetilde{x}_\delta\|^2\le\|\widetilde{u}_\delta\|\,\|(f(x^\sigma_\delta)-\bar{\xi}_\delta)^+\|-\frac{1}{2\alpha_0\beta}\big\|(f(x^\sigma_\delta)-\bar{\xi}_\delta)^+\big\|^2$$
$$=-\Big[\frac{1}{\sqrt{2\alpha_0\beta}}\big\|(f(x^\sigma_\delta)-\bar{\xi}_\delta)^+\big\|-\frac{\sqrt{2\alpha_0\beta}}{2}\big\|\widetilde{u}_\delta\big\|\Big]^2+\frac{\alpha_0\beta}{2}\big\|\widetilde{u}_\delta\big\|^2.$$

This immediately implies estimates (23) and (25).

It also follows from (26) that $(\alpha\widetilde{u}_{0\delta}-1)(\delta-f_0(x^\sigma_\delta))\le\alpha\|\widetilde{u}_\delta\|\,\|(f(x^\sigma_\delta)-\bar{\xi}_\delta)^+\|$. This, in view of (23) and $x^\sigma_\delta\in M_\delta$, leads to estimate (24).

The theorem is proved.

Corollary 2. *Let $\beta\to0$. Then, $f^+(x^\sigma_\delta)\to\bar{\xi}_\delta$, $f_0(x^\sigma_\delta)\to\delta$, and $x^\sigma_\delta\to\widetilde{x}_\delta$.*

Corollary 3. *If (1) is a solvable CP problem, then $\lim\limits_{\delta\to f^*}\lim\limits_{\beta\to0}f_0(x^\sigma_\delta)=f^*$.*

Corollary 4. *If (1) is an improper CP problem of the first kind, then $\lim\limits_{\delta\to\bar{f}}\lim\limits_{\beta\to0}f_0(x^\sigma_\delta)=\bar{f}$, where $\bar{f}$ is the optimal value of problem (14).*

Consider in more detail the case when (1) is a solvable problem but $\delta<f^*$ in (3). Let $f^*>\delta_2>\delta_1$. Evidently, $M_{\delta_2}\supset M_{\delta_1}$, and $X_{\|\bar{\xi}_{\delta_1}\|}\supset X_{\|\bar{\xi}_{\delta_2}\|}$ for the sets $X_{\bar{\xi}_\delta}$. Indeed, since $\bar{\xi}_\delta=f^+(x_\delta)$, where $x_\delta=\arg\min\limits_{x\in M_\delta}\|f^+(x)\|$, we have $\|f^+(x_{\delta_2})\|\le\|f^+(x_{\delta_1})\|$; i.e., $\|\bar{\xi}_{\delta_2}\|\le\|\bar{\xi}_{\delta_1}\|$. Hence, $\lim\limits_{\delta\to f^*}\|\bar{\xi}_\delta\|=0$.

Assume that the set $S=\bigcap_{i=0}^m\{x\colon f_i(x)\le D\}$ is bounded for some $D\in\mathbb{R}^1$. Then, the set $S_1=X_{\|\bar{\xi}_{\delta_1}\|}\cap M_{f^*}$ is also bounded and all the points $\widetilde{x}_\delta$ belong to S_1 for $\delta>\delta_1$. Let $\widetilde{x}$ be a limit point of the sequence $\{\widetilde{x}_\delta\}$ as $\delta\to f^*$. Hence, $\widetilde{x}\in X^*=X\cap M_{f^*}$. According to the definition of the point $\widetilde{x}_\delta$, we have $\|\widetilde{x}_{\delta_k}\|\le\|x\|$ for all $x\in X_{\bar{\xi}_{\delta_k}}\cap M_{\delta_k}$. Therefore, $\|\widetilde{x}\|\le\|x\|$ if $x\in X\cap M_{f^*}$. Consequently, by the uniqueness of the normal solution of problem (1), we obtain $\widetilde{x}=x^*_0$.

Thus, the following statement is valid.

Corollary 5. *Assume that $\delta<f^*$ and the set S is bounded in problem (3). Then,*

$$\lim_{\delta\to f^*}\lim_{\beta\to0}x^\sigma_\delta=x^*_0.$$

Remark. The condition $\widetilde{u}_{0\delta} > 0$ in Theorem 5 is natural. If $\delta < f^*$, then the point $\widetilde{x}_\delta$ lies on the surface $f_0(x) = \delta$ and, by conditions (22), we can assume $\widetilde{u}_{0\delta}$ to be positive. If $\delta \geq f^*$, then the definition of the point $[\widetilde{x}_\delta, \widetilde{u}_\delta, \widetilde{u}_{0\delta}]$ for $\widetilde{u}_{0\delta} = 0$ implies the inequality $\|\widetilde{x}_\delta\| \leq \|x\| \quad (\forall x \in X)$. However, if $\bar{x} = \mathrm{Pr}_X\, 0$ and $\bar{x} \notin X^*$, then $\|\bar{x}\| < \|\widetilde{x}_\delta\|$ for $f_0(\bar{x}) > \delta > f^*$.

5. A CP PROBLEM WITH CONSISTENT SYSTEM OF CONSTRAINTS

Consider situation 1b for problem (1) formulated in Section 4. Suppose that $X \neq \varnothing$ in problem (1) but $\widetilde{f} = \inf\limits_{x \in X} f_0(x)$ is not attained; in particular, $\widetilde{f} = -\infty$ (problem (1) is an improper CP problem of the second kind). Denote by $\{x_k\}$ a sequence of points from X for which $\lim\limits_{k\to\infty} f_0(x_k) = \widetilde{f}$. Evidently, $\|x_k\| \to \infty \quad (k \to \infty)$, and problems (3) and (20) coincide. We set in (3) $\delta = f_k$, where $f_k = f_0(x_k)$. Problem (3) is solvable at a unique point $\bar{x}_k$ for any k, and $f_0(\bar{x}_k) = f_k$ starting with some k.

Assume that Slater's condition holds for problem (1) at a point x^0. Then, there exists a saddle point $[x^*_\alpha, \lambda^*_\alpha]$ of the function $L^\alpha(x, \lambda) = L(x, \lambda) + \alpha\,\|x\|^2$ in the domain $\mathbb{R}^n_+ \times \mathbb{R}^m_+$, $\alpha > 0$. In other words, x^*_α and λ^*_α are solutions of the problem

$$\min\{F_\alpha(x) = f_0(x) + \alpha\,\|x\|^2 \colon x \in X\} \tag{27}$$

and of the problem dual to (27), respectively. Note that problem (27) approximates original statement (1) in Tikhonov's regularization method. It is known (see, for example, [1]) that

$$\min_{\alpha\to 0} f_0(x^*_\alpha) = \min_{\alpha\to 0} F_\alpha(x^*_\alpha) = \widetilde{f}. \tag{28}$$

Using the notation $x^\sigma_{f_k} = x^\sigma_k,\ \lambda^\sigma_{f_k} = \lambda^\sigma_k$, we obtain the following relations:

$$\begin{aligned} f_k + \alpha\,\|\bar{x}_k\|^2 &= f_0(\bar{x}_k) + \alpha\,\|\bar{x}_k\|^2 \geq f_0(\bar{x}_k) + (\lambda^\sigma_k, f(\bar{x}_k)) + \alpha\,\|\bar{x}_k\|^2 - \beta\,\|\bar{\lambda}^\sigma_k\|^2 \\ &= L_\sigma(\bar{x}_k, \lambda^\sigma_k) \geq L_\sigma(x^\sigma_k, \lambda^\sigma_k) \geq L_\sigma(x^\sigma_k, \lambda^*_\alpha) = f_0(x^\sigma_k) + (\lambda^*_\alpha, f(x^\sigma_k)) + \alpha\,\|x^\sigma_k\|^2 - \beta\,\|\lambda^*_\alpha\|^2 \\ &= L^\alpha(x^\sigma_k, \lambda^*_\alpha) - \beta\,\|\lambda^*_\alpha\|^2 \geq L^\alpha(x^*_\alpha, \lambda^*_\alpha) - \beta\,\|\lambda^*_\alpha\|^2 \geq f_0(x^*_\alpha) - \beta\,\|\lambda^*_\alpha\|^2. \end{aligned} \tag{29}$$

Since $[x^*_\alpha, \lambda^*_\alpha]$ is a saddle point of the function $L^\alpha(x, \lambda)$, we have

$$L^\alpha(x^*_\alpha, \lambda^*_\alpha) \leq L^\alpha(x^0, \lambda^*_\alpha).$$

Hence,

$$0 \leq (\lambda^*_\alpha)_i \leq \frac{f_0(x^0) - f_0(x^*_\alpha) + \alpha\|x^0\|^2}{\min\limits_{1\leq i\leq m} |f_i(x^0)|}.$$

Therefore, for $\widetilde{f} > -\infty$, the value $\|\lambda^*_\alpha\|$ in (29) is bounded from above by the constant

$$K_1 = m\biggl(\frac{f_0(x^0) - \widetilde{f} + \alpha_0\|x^0\|^2}{\min\limits_{1\leq i\leq m} |f_i(x^0)|}\biggr)^2$$

for all $0 < \alpha < \alpha_0$. Thus, in view of (28), it follows from (29) that

$$\lim_{k\to\infty}\lim_{\sigma\to 0} L_\sigma(x^\sigma_k, \lambda^\sigma_k) = \widetilde{f}. \tag{30}$$

If $\widetilde{f} = -\infty$, then (30) holds in view of the inequality $f_k + \alpha \|\bar{x_k}\|^2 \geq L_\sigma(x_k^\sigma, \lambda_k^\sigma)$.

In the proof of relation (30), we actually used the close connection between the approach to the optimal correction of improper CP problems based on the regularized Lagrange function and Tikhonov's regularization method. Let us formulate a statement that shows the closeness of Tikhonov's method and the residual method.

Theorem 6. *If $[x_\alpha^*, \lambda_\alpha^*]$ is a saddle point of the function $L^\alpha(x, \lambda)$ in the domain $\mathbb{R}^n \times \mathbb{R}_+^m$, then $\big[x_\alpha^*, \frac{1}{\alpha}\lambda_\alpha^*, \frac{1}{\alpha}\big]$ is a saddle point of the function $H(x, u, u_0)$ in the domain $\mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^1$.*

If $[\bar{x}, \bar{u}, \bar{u}_0]$ is a saddle point of $H(x, u, u_0)$ in the domain $\mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^1$ and $\bar{u}_0 > 0$, then $\big[\bar{x}, \frac{1}{\bar{u}_0}\bar{u}\big]$ is a saddle point of the function $L^\alpha(x, \lambda)$ in the domain $\mathbb{R}^n \times \mathbb{R}_+^m$ for $\alpha = \frac{1}{\bar{u}_0}$.

Proof. If $[x_\alpha^*, \lambda_\alpha^*]$ is a saddle point of the function $L^\alpha(x, \lambda)$, then the relations

$$f_0(x_\alpha^*) + (\lambda, f(x_\alpha^*)) + \alpha \|x_\alpha^*\|^2 \leq f_0(x_\alpha^*) + (\lambda_\alpha^*, f(x_\alpha^*)) + \alpha \|x_\alpha^*\|^2 \leq f_0(x) + (\lambda_\alpha^*, f(x)) + \alpha \|x\|^2$$

hold for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+^n$. However, in this case, the inequalities

$$\frac{1}{\alpha}\big(f_0(x_\alpha^*) - \delta\big) + \Big(\frac{1}{\alpha}\lambda, f(x_\alpha^*)\Big) + \|x_\alpha^*\|^2 \leq \frac{1}{\alpha}(f_0(x_\alpha^*) - \delta) + \Big(\frac{1}{\alpha}\lambda^*, f(x_\alpha^*)\Big) + \|x_\alpha^*\|^2$$

$$\leq \frac{1}{\alpha}\big(f_0(x) - \delta\big) + \Big(\frac{1}{\alpha}\lambda_\alpha^*, f(x)\Big) + \|x\|^2$$

are also valid for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_+^m$, and $\alpha > 0$.

Now, let $[\bar{x}, \bar{u}, \bar{u}_0]$ be a saddle point of the function $H(x, u, u_0)$. The inequality $H(\bar{x}, \bar{u}, \bar{u}_0) \leq H(x, \bar{u}, \bar{u}_0)$ immediately implies

$$\frac{1}{\bar{u}_0}\|\bar{x}\|^2 + \Big(\frac{1}{\bar{u}_0}\bar{u}, f(\bar{x})\Big) + f_0(\bar{x}) \leq \frac{1}{\bar{u}_0}\|x\|^2 + \Big(\frac{1}{\bar{u}_0}\bar{u}, f(x)\Big) + f_0(x)$$

for all $x \in \mathbb{R}^n$; i.e.,

$$L^{\frac{1}{\bar{u}_0}}\Big(\bar{x}, \frac{1}{\bar{u}_0}\bar{u}\Big) \leq L^{\frac{1}{\bar{u}_0}}\Big(x, \frac{1}{\bar{u}_0}\bar{u}\Big) \qquad (\forall x \in \mathbb{R}^n).$$

The inequality $H(\bar{x}, u, u_0) \leq H(\bar{x}, \bar{u}, \bar{u}_0)$ leads to complementary slackness conditions in the form

$$(\bar{u}, f(\bar{x})) = 0, \qquad \bar{u}_0(f_0(\bar{x}) - \delta) = 0.$$

In view of these conditions, the inequality turns into the relation $(u, f(\bar{x})) \leq (\bar{u}, (f(\bar{x})) = 0$. Hence,

$$f_0(\bar{x}) + (u, f(\bar{x})) + \frac{1}{\bar{u}_0}\big\|\bar{x}\big\|^2 \leq f_0(\bar{x}) + \Big(\frac{1}{\bar{u}_0}\bar{u}, f(\bar{x})\Big) + \frac{1}{\bar{u}_0}\big\|\bar{x}\big\|^2;$$

i.e.,

$$L^{\frac{1}{\bar{u}_0}}(\bar{x}, u) \leq L^{\frac{1}{\bar{u}_0}}\Big(\bar{x}, \frac{1}{\bar{u}_0}\bar{u}\Big) \qquad (\forall u \in \mathbb{R}_+^m).$$

The theorem is proved.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 10-01-00273) and by the Presidium of the Ural Branch of the Russian Academy of Sciences (project nos. 12-P-1-1016 and 12-S-1-1017/1).

REFERENCES

1. V. D. Skarin, "Regularized Lagrange function and correction methods for improrer convex programming problems," Proc. Steklov Inst. Math., Suppl. 1, S116–S144 (2002).
2. V. D. Skarin, "On one general approach to the optimal correction of improper problems of convex programming," Trudy Inst. Mat. Mekh. UrO RAN **16** (3), 265–275 (2010).
3. I. I. Eremin, V. D. Mazurov, and N. N. Astaf'ev, *Improper Problems of Linear and Convex Programming* (Nauka, Moscow, 1983) [in Russian].
4. F. P. Vasil'ev, *Methods for Solving Extremal Problems* (Nauka, Moscow, 1981) [in Russian].
5. I. I. Eremin and N. N. Astaf'ev, *Introduction to the Theory of Linear and Convex Programming* (Nauka, Moscow, 1976) [in Russian].
6. E. S. Levitin, "On the correctness of constraints and stability in extremal problems," Vestn. Mosk. Univ., Ser. Mat. Mekh. **1**, 24–34 (1968).
7. E. G. Gol'shtein, *Duality Theory in Mathematical Programming and Its Applications* (Nauka, Moscow, 1971) [in Russian].
8. V. D. Skarin, "On a regularization method for inconsistent convex programming problems," Russ. Math. (Izv. VUZov) **39** (12), 78–85 (1995).

Translated by M. Deikalova