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On Graphs Whose Local Subgraphs Are Strongly Regular with Parameters (115, 18, 1, 3)

Corresponding Member of the RAS A. A. Makhnev[a, b]

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We consider undirected graphs without loops or multiple edges. If a and b are vertices in a graph Γ, then $d(a, b)$ denotes the distance between a and b, and $\Gamma_i(a)$ denotes the subgraph of Γ induced by the set of vertices of Γ that are a distance of i away from a. The subgraph $\Gamma_1(a)$ is called the neighborhood of a and is denoted by $[a]$. By $a^\perp$ we denote the subgraph that is the ball of radius 1 centered at a. Let $\mathcal{F}$ be a family of graphs. A graph Γ is said to be a locally $\mathcal{F}$-graph if $[a] \in \mathcal{F}$ for any vertex $a \in \Gamma$.

Γ is called a regular graph of degree k if $[a]$ contains precisely k vertices for any vertex a in Γ. A graph Γ is said to be an edge-regular graph with parameters (v, k, λ) if Γ is a regular graph of degree k on v vertices and each of its edges lies in λ triangles. Γ is called an amply regular graph with parameters (v, k, λ, μ) if Γ is an edge-regular graph with the corresponding parameters and the subgraph $[a] \cap [b]$ contains μ vertices in the case $d(a, b) = 2$. An amply regular graph of diameter 2 is called a strongly regular graph.

A graph Γ of diameter d is said to be antipodal if the relation of coincidence or being a distance of d apart on its vertex set is an equivalence relation. An antipodal quotient Γ' is a graph whose vertices are the antipodal classes of Γ and two classes are adjacent if a vertex of one class is adjacent to a vertex of the other class. An antipodal graph Γ is called an r-covering (of its antipodal quotient) if each of its antipodal classes contains precisely r vertices.

Let $K_{m_1, \ldots, m_n}$ denote a complete n-partite graph with parts of orders $m_1, m_2, \ldots, m_n$. If $m_1 = m_2 = \ldots m_n = m$, then this graph is denoted by $K_{n \times m}$.

If vertices u and w are separated by a distance of i in Γ, then $b_i(u, w)$ ($c_i(u, w)$) denotes the number of vertices in the intersection of $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph of diameter d is called a distance-regular graph with an intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of vertices u and w separated by the distance i. Let $a_i = k - b_i - c_i$.

Various classes of distance-regular graphs whose local subgraphs are isomorphic to a given strongly regular graph were investigated in [1]. An issue of special interest is locally Δ-graphs, where Δ is a strongly regular graph with $\lambda = 1$. The known strongly regular graph with $\lambda = 1$ is the point graph of the generalized quadrangle $GQ(2, t)$ with $t = 1, 2, 4$ or a graph with parameters (81, 20, 1, 6), (243, 22, 1, 2), or (729, 112, 1, 20).

Thus far, the distance-regular graphs whose local subgraphs are isomorphic to a given strongly regular graph with $\lambda = 1$ and $v \le 81$ have been classified (see [2, 3]).

Proposition. *Let Γ be a distance-regular graph whose local subgraphs are isomorphic to a given strongly regular graph Δ with $\lambda = 1$ and $v \le 81$. Then one of the following assertions holds:*

(1) *Δ is a (3×3)-lattice and Γ is the complement of a (4×4)-lattice or the Johnson graph $J(6, 3)$.*

(2) *Δ is the point graph of the generalized quadrangle $GQ(2, 2)$ and Γ is the complement of the triangular graph $T(8)$, a strongly regular graph with parameters (36, 15, 6, 6), or a Taylor graph with the intersection array {15, 8, 1; 1, 8, 15}.*

(3) *Δ is the point graph of the generalized quadrangle $GQ(2, 4)$ and Γ is a strongly regular graph with parameters (64, 27, 10, 12) or a Taylor graph with the intersection array {27, 16, 1; 1, 16, 27}.*

(4) *Δ is a strongly regular graph with parameters (81, 20, 1, 6) and Γ is a distance-regular graph with the intersection array {81, 60, 1; 1, 20, 81}.*

In this work, we classify the distance-regular graphs whose local subgraphs are isomorphic to a strongly regular graph with parameters (115, 18, 1, 3).

Theorem. *Let Γ be a distance-regular graph whose local subgraphs are strongly regular with parameters (115, 18, 1, 3). Then one of the following assertions holds:*

[a] *Krasovskii Institute of Mathematics and Mechanics, Ural Branch, Russian Academy of Sciences, ul. S. Kovalevskoi 16, Yekaterinburg, 620219 Russia*
[b] *Ural Federal University, ul. Mira 19, Yekaterinburg, 620002 Russia*
e-mail: makhnev@imm.uran.ru

(1) Γ *is a strongly regular graph with parameters* (576, 115, 18, 24), (484, 115, 18, 30), *or* (392, 115, 18, 40).

(2) *The diameter of* Γ *is* 3, *and* Γ *has the intersection array* {115, 96, 8; 1, 8, 92} *and the spectrum* $115^1, 23^{217}, 3^{713}, -9^{805}$.

(3) *The diameter of* Γ *is* 4 *and* Γ *has the intersection array* $\left\{115, 96, \frac{40(r-1)}{r}, 1; 1, \frac{40}{r}, 96, 115\right\}$, *where* $r \in \{2, 4, 5\}$.

The graph from assertion (3) in the theorem is an $AT4(3, 5, r)$ graph (see [6]). For such a graph, the second neighborhood of a vertex is a distance-regular graph with the intersection array $\left\{75, 64, \frac{24(r-1)}{r}, 1; 1, \frac{24}{r}, 64, 75\right\}$.

Below are some auxiliary results.

Lemma 1 ([4, Lemma 3.1]). *Let* Γ *be a strongly regular graph with parameters* (v, k, λ, μ). *Then either* $k = 2\mu$ *and* $\lambda = \mu - 1$ (*so-called half case*) *or the nonprincipal eigenvalues* $n - m$ and $-m$ *of* Γ *are integers, where* $n^2 = (\lambda - \mu)^2 + 4(k - \mu)$, $n - \lambda + \mu = 2m$, *and the multiplicity* $n - m$ *is equal to* $\frac{k(m-1)(k+m)}{\mu n}$. *Furthermore, if m is an integer larger than* 1, *then* $m - 1$ *divides* $k - \lambda - 1$ *and*

$$\mu = \lambda + 2 + (m-1) - \frac{k-\lambda-1}{m-1},$$

$$n = m - 1 + \frac{k-\lambda-1}{m-1}.$$

Lemma 2. *Let* Γ *be a strongly regular graph with parameters* (v, k, λ, μ), Δ *be an induced subgraph on N vertices with M edges and vertex degrees* $d_1, \ldots, d_N$. *Then*

$$(v-N)-(kN-2M) + \left(\lambda M + \mu\left(\binom{N}{2} - M\right) - \sum_{i=1}^{N}\binom{d_i}{2}\right) = x_0 + \sum_{i=3}^{N}\binom{i-1}{2}x_i,$$

where x_i *is the number of vertices in* $\Gamma - \Delta$ *that are adjacent to precisely i vertices in* Δ.

Lemma 3. *Let* Γ *be a strongly regular graph with parameters* (115, 18, 1, 3) *and eigenvalues* 3 *and* -5, Δ *be a regular subgraph of* Γ *of degree* 3 *on n vertices*, X_i *be the set of vertices from* $\Gamma - \Delta$ *adjacent to precisely i vertices in* Δ, *and* $x_i = |X_i|$. *Then the following assertions hold:*

(1) $\sum x_i = 115 - n$, $\sum ix_i = 15n$, $\sum\binom{i}{2}x_i = \frac{3n^2-15n}{2}$, *and* $x_0 + \sum\binom{i-1}{2}x_i = 115 + \frac{3n^2-47n}{2}$.

(2) $n \le 40$ *and we have* $nx_0 \le \frac{(115-n)((115-x_0)\cdot 4^2)}{19^2}$.

(3) *If* $n = |X_0|$, *then* $n \le 20$.

Proof. By Lemma 2, we have $\sum x_i = 115 - n$, $\sum ix_i = 15n$, $\sum\binom{i}{2}x_i = \frac{3n}{2} + 3\left(\frac{n(n-1)}{2} - \frac{3n}{2}\right) - 3n = \frac{3n^2-15n}{2}$. Therefore, $x_0 + \sum\binom{i}{2}x_i = 115 + \frac{(3n^2-47n)}{2}$.

We have $-5 \le 3 - \frac{15n}{115-n} \le 3$. Therefore, $23n \le 8 \cdot 115$ and $n \le 40$. Moreover, if $n = 40$, each vertex from $\Gamma - \Delta$ id adjacent to precisely $\frac{15\cdot 40}{115-40} = 8$ vertices from Δ.

Since there are no edges between Δ and X_0, by Proposition 4.6.1 from [5], we have $nx_0 \le \frac{(v-n)(v-x_0)(\theta_2-\theta_1)^2}{(2k-\theta_2-\theta_1)^2}$, where $\theta_2 = -5$ and $\theta_1 = 3$ are nonprincipal eigenvalues of Γ. It follows that $nx_0 \le \frac{(115-n)((115-x_0)\cdot 4^2)}{19^2}$.

If $n = x_0$, we have $19n \le 4(115 - n)$ and $n \le 20$.

Lemma 4. *Let* Γ *be a distance-regular graph of diameter* $d \ge 3$ *whose local subgraphs are strongly regular with parameters* (115, 18, 1, 3), *and let* $0 = k > \theta_1 > \ldots > \theta_d$ *be the eigenvalues of* Γ. *Then* $\theta_1 \le 23$ *and* $\theta_d \ge -25$.

Proof. By Terwilliger's theorem [1, Theorem 4.4.3], we have $-5 \ge b^- = -1 - \frac{b_1}{\theta_1+1}$ and $3 \le b^+ = -1 - \frac{b_1}{\theta_d+1}$. Therefore, $\theta_1 \le 23$ and $\theta_d \ge -25$.

In what follows, let Γ be a distance-regular graph of diameter d whose local subgraphs are strongly regular with parameters (115, 18, 1, 3). We fix a vertex u in Γ and set $k_i = |\Gamma_i(u)|$.

Lemma 5. *The following assertions hold:*

(1) *If the diameter of* Γ *is* 2, *then* Γ *has the parameters* (576, 115, 18, 24), (484, 115, 18, 30), *or* (392, 115, 18, 40).

(2) *If the diameter of* Γ *is larger than* 2, *then* $\mu \in \{6, 8, 10, 12, 16, 20, 24, 30, 32, 40\}$.

(3) *If the diameter of* Γ *is larger than* 3, *then* $\mu \in \{6, 8, 10, 12, 16, 20\}$.

(4) *If the diameter of* Γ *is larger than* 4, *then* $\mu \in \{6, 8, 10, 12\}$.

Proof. By assumption, $k = 115$ and $\lambda = 18$. If the diameter of Γ is 2, then, by Lemma 1, the number $(\lambda - \mu)^2 + 4(k - \mu)$ is the square of a positive integer n. Therefore, $(\mu - 20)^2 + 384 = n^2$ and $(\mu, n) \in \{(10, 22), (16, 20), (24, 20), (30, 22), (40, 28)\}$. It follows that Γ has the eigenvalues 15, –7; 11, –9; 7, –13; 5, –17; or 3, –25. In the first and second cases, the multiplicities of the eigenvalues are not integer. Therefore, Γ has the parameters (576, 115, 18, 24), (484, 115, 18, 30), or (392, 115, 18, 40).

Let the diameter of Γ be larger than 2. By Lemma 3, we have $\mu \le 40$. Since μ is an even divisor of $115 \cdot 96$, we have $\mu \in \{6, 8, 10, 12, 16, 20, 24, 30, 32, 40\}$.

Let the diameter of Γ be larger than 3 and u, w, x, y, z be a geodesic 4-path in Γ. Then there are no edges in the graph $[x]$ between $[u] \cap [x]$ and $[x] \cap [z]$ and, by Lemma 3, we have $\mu \le 20$. From this, $\mu \in \{6, 8, 10, 12, 16, 20\}$.

Let the diameter of Γ be larger than 4. Then $\frac{3\mu}{2} \le c_3 \le b_2$ and $\mu \ne 20$. If $\mu = 16$, then, by Lemma 3, $b_2 \le 24$ and $c_3 \ge 24$. It follows that $d = 5$ and $c_3 = b_2 = 24$. Furthermore, $c_3 - b_3 \ge c_2 - b_2 + a_1 + 2$. Therefore, $b_3 \le 12$, a contradiction to $b_3 \ge c_2$.

Remark 1. Let Δ be a strongly regular graph with parameters (576, 115, 18, 24), (484, 115, 18, 30), or (392, 115, 18, 40), and let Γ be a distance-regular graph of diameter d that is an r-covering of Δ. Then $\mu_\Gamma \ge 6$. If $d = 5$, then Γ has the intersection array $\{115, 96, t(r-1), 24, 1; 1, 24, t, 96, 115\}$, $\{115, 96, t(r-1), 30, 1; 1, 30, t, 96, 115\}$, or $\{115, 96, t(r-1), 40, 1; 1, 40, t, 96, 115\}$. In any case, there are no admissible arrays.

If $d = 4$, then Γ has the intersection array $\left\{115, 96, \frac{24(r-1)}{r}, 1; 1, \frac{24}{r}, 96, 115\right\}$, $\left\{115, 96, \frac{(r-1)30}{r}, 1; 1, \frac{30}{r}, 96, 115\right\}$, or $\left\{115, 96, \frac{(r-1)40}{r}, 1; 1, \frac{40}{r}, 96, 115\right\}$, the new eigenvalues θ_1 and θ_3 of Γ are the roots of the quadratic equation $x^2 - \lambda x - k = 0$, and the multiplicity of $\theta_1 = 23$ is $m_1 = \frac{(r-1)v}{\left(2 + \frac{\lambda\theta_1}{k}\right)}$. Therefore, $v = 392r$ and Γ is an $AT4(3, 5, r)$ graph.

Lemma 6. *The parameter* μ *is at most* 20.

Proof. Let $\mu > 20$. By Lemma 5, the diameter of Γ is 3 and $\mu \in \{24, 30, 32, 40\}$.

If $\mu = 40$, then $k_2 = 23 \cdot 12 = 276$. By Lemma 3, we have $19^2 b_2 \le 30(115 - b_2)$. Therefore, $b_2 \le 8$, $c_3 \ge 40 - b_2 + 20$ and $c_3 \in \{60, 69, 72, 84, 92, 96, 115\}$. From this, Γ has the intersection array $\{115, 96, b_2; 1, 40, c_3\}$. In any case, there are no admissible intersection arrays.

Let $\mu = 32$. Then $k_2 = 23 \cdot 15 = 345$ and, by Lemma 3, we have $19^2 \cdot 2b_2 \le 83(115 - b_2)$. Therefore, $b_2 \le 10$, b_2 is odd, $c_3 \ge 32 - b_2 + 20$, c_3 is odd, $c_3 \in \{45, 69, 75, 105, 115\}$, and Γ has the intersection array $\{115, 96, b_2; 1, 32, c_3\}$. In any case, there are no admissible intersection arrays.

Let $\mu = 30$. Then $k_2 = 16 \cdot 23 = 368$ and, by Lemma 3, we have $19^2 \cdot 3b_2 \le 17(115 - b_2) \cdot 8$. Therefore, $b_2 \le 12$, $c_3 \ge 30 - b_2 + 20$. Hence, Γ has the intersection array $\{115, 96, b_2; 1, 30, c_3\}$. In the cases $b_2 = 12$, $c_3 = 92$ and $b_2 = 5$, $c_3 = 92$, the graph has the integer eigenvalues 23, 1, –25 and 23, 3, –20. In any case, there are no admissible intersection arrays.

Let $\mu = 24$. Then $k_2 = 23 \cdot 20 = 460$ and, by Lemma 3, we have $19^2 \cdot 3b_2 \le 91(115 - b_2) \cdot 2$. Therefore, $b_2 \le 16$, $c_3 \ge 24 - b_2 + 20$. Hence, Γ has the intersection array $\{115, 96, b_2; 1, 24, c_3\}$. In the cases $b_2 = 8$, $c_3 = 92$ and $b_2 = 14$, $c_3 = 92$, the graph has the integer eigenvalues 23, 3, –17 and 23, 1, –21. In any case, there are no admissible intersection arrays.

Lemma 7. *If the diameter of* Γ *is* 3, *then* Γ *has the intersection array* $\{115, 96, 8; 1, 8, 92\}$ *and the spectrum* $115^1, 23^{217}, 3^{713}, -9^{805}$.

Proof. Let the diameter of Γ be equal to 3.

Let $\mu = 20$. Then $k_2 = 24 \cdot 23 = 552$ and, by Lemma 3, we have $19b_2 \le 4(115 - b_2)$. Therefore, $b_2 \le 20$, $c_3 \ge 20 - b_2 + 20$. Hence, Γ has the intersection array $\{115, 96, b_2; 1, 20, c_3\}$. If $b_2 = 10$ and $c_3 = 92$, then the graph has the integer eigenvalues 23, 3, –15. In any case, there are no admissible intersection arrays.

The cases μ = 10, 12, 16 are treated in a similar fashion.

Let $\mu = 8$. Then $k_2 = 60 \cdot 23 = 1380$ and, by Lemma 3, we have $19^2 b_2 \le 107(115 - b_2) \cdot 2$. Therefore, $b_2 \le 41$. There is the unique admissible intersection array $\{115, 96, 8; 1, 8, 92\}$ and Γ has the spectrum $115^1, 23^{217}, 3^{713}, -9^{805}$.

Let $\mu = 6$. Then $k_2 = 80 \cdot 23$ and, in the notation of Lemma 3, we obtain $x_2 = 9$, $x_0 + x_1 = 100$, and $x_1 = 72$. Therefore, $b_2 \le 28$ and Γ has the intersection array $\{115, 96, b_2; 1, 6, c_3\}$. In any case, there are no admissible intersection arrays. The lemma is proved.

Let $d \ge 4$. Fix a geodesic 4-path u, w, x, at, y, z in Γ.

Lemma 8. *If* $\mu = 20$, *then* Γ *has the intersection array* $\{115, 96, 20, 1; 1, 20, 96, 115\}$.

Proof. Let $\mu = 20$. Then $k_2 = 552$. By Lemma 3, we have $b_2 = 20$ and $[x] \cap \Gamma_3(u) = [x] \cap [z]$. Furthermore, $[x] \cap \Gamma_2(u)$ is contained in $\Gamma_2(z)$. Therefore, $\Gamma_4(u) = [z]$ and Γ has the intersection array $\{115, 96, 20, 1; 1, 20, 96, 115\}$.

Lemma 9. *If* $\mu \neq 20$, *then* $\mu \leq 10$.

Proof. Let $\mu = 16$. Then $k_2 = 30 \cdot 23 = 690$ and, by Lemma 4, we have $16 \leq b_2 \leq 24$ and $c_3 - b_3 \geq 16 - b_2 + 20$. If $c_3 \leq 90$, then $\theta_1 > 25$, a contradiction. Therefore, $c_3 \in \{92, 95, 96\}$.

If $c_3 = 95$, then c_4 is divided by 16, $b_2 = 19$, $b_3 = 16$, $k_3 = 6 \cdot 23$, and Γ has the intersection array $\{115, 96, 19, 16; 1, 16, 95, 96\}$, a contradiction to $\theta_4 < -40$.

If $c_3 = 92$, then c_4 is divided by 4, $k_3 = \dfrac{15b_2}{2}$, and $b_2 = 16, 18, 20, 22, 24$. Therefore, $c_4 = 96, 100, 104, 108, 112$. If $c_4 = 96$, then either $b_2 = 16$ and b_3 is divided by 4, $b_2 = 20$ and $b_3 = 16$, or $b_2 = 24$ and b_3 is divided by 8. If $c_4 = 100$, then either $b_2 = 20$ and b_3 is divided by 4 or b_3 is divided by 5. If $c_4 = 104$, we have $b_2 = 16$ and $b_3 = 13$. If $c_4 = 108$, then either $b_2 = 16$ and $b_3 = 9$ or $b_2 = 24$ and b_3 is divided by 6. In any case, $\theta_1 > 31$.

If $c_3 = 96$, then $c_4 = 115$. If the case of the intersection array $\{115, 96, 24, 1; 1, 16, 96, 115\}$, the graph has the spectrum $115^1, 23^{105}, 3^{345}, -5^{483}, -25^{46}$, but $p^4_{44} = \dfrac{1}{2}$. In any case, there are no admissible intersection arrays.

The case $\mu = 12$ is treated in a similar manner.

Lemma 10. *If* $\mu \leq 10$, *then* Γ *has the intersection array* $\{115, 96, 30, 1; 1, 10, 96, 115\}$ *or* $\{115, 96, 32, 1; 1, 8, 96, 115\}$.

Proof. Let $\mu = 10$. Then $k_2 = 48 \cdot 23$ and, by Lemma 4, we have $10 \leq b_2 \leq 36$ and b_2 is divided by 5. If $c_3 \leq 90$, then $\theta_1 > 27$, a contradiction. Therefore, $c_3 \in \{92, 96\}$; $b_2 = 10, 15, 20, 25, 30, 35$; and c_4 is divided by 5.

If $c_3 = 92$, then $k_3 = 12b_2$. Therefore, $c_4 = 100, 105, 110, 115$. If $c_4 = 100$, the number b_2b_3 is divided by 25. If $c_4 = 105$, b_2b_3 is divided by 35. If $c_4 = 110$, b_2b_3 is divided by 55. If $c_4 = 115$, b_2b_3 is divided by 115. In any case, $\theta_1 > 25$.

If $c_3 = 96$, then $k_3 = \dfrac{23b_2}{2}$ and $c_4 = 100, 105, 110, 115$. If $c_4 = 100$, then the number b_2b_3 is divided by 200. If $c_4 = 105$, the number b_2b_3 is divided by 210. If $c_4 = 110$, b_2b_3 is divided by 220. If $c_4 = 115$, b_2b_3 is divided by 10. In this case, Γ has the intersection array $\{115, 96, 30, 1; 1, 10, 96, 115\}$ and the spectrum 115^1, $23^{210}, 3^{345}, -5^{966}, -25^{46}$.

Let $\mu = 8$. Then $k_2 = 60 \cdot 23$ and, by Lemma 4, we have $8 \leq b_2 \leq 41$. If $c_3 \leq 90$, then $\theta_1 > 23$, a contradiction. Therefore, $c_3 \in \{92, 93, 95, 96\}$.

If $c_3 = 95$, then c_4 is divided by 8; $b_2 = 19$; $b_3 = 8, 16$; $k_3 = 12 \cdot 23$; and Γhas the intersection array $\{115, 96, 19, b_3; 1, 12, 95, 96\}$, a contradiction to $\theta_1 > 36$.

If $c_3 = 93$, then c_4 is divided by 8; $b_2 = 31$; $b_3 = 8, 16, 24$; $k_3 = 20 \cdot 23$; and Γ has the intersection array $\{115, 96, 31, b_3; 1, 12, 93, 96\}$, a contradiction to $\theta_1 > 34$.

If $c_3 = 92$, then c_4 is even and $k_3 = 15b_2$, a contradiction to $\theta_1 > 23$.

If $c_3 = 96$, then $k_3 = \dfrac{115b_2}{8}$. In this case, Γ has the intersection array $\{115, 96, 32, 1; 1, 8, 96, 115\}$ and the spectrum $115^1, 23^{280}, 3^{345}, -5^{1288}, -25^{46}$.

Let $\mu = 6$. Then $k_2 = 80 \cdot 23$ and, by Lemma 4, we have $6 \leq b_2 \leq 28$. If $c_3 \leq 90$, then $\theta_1 > 24$, a contradiction. Therefore, $c_3 \in \{92, 95, 96\}$.

If $c_3 = 95$, then c_4 is divided by 6; $b_2 = 19$; $b_3 = 6, 12, 18$; $k_3 = 16 \cdot 23$; and Γ has the intersection array $\{115, 96, 19, b_3; 1, 6, 95, 96\}$, a contradiction to $\theta_1 > 34$.

If $c_3 = 92$, then c_4 is divided by 3 and $k_3 = 20b_2$, a contradiction to $\theta_1 > 24$.

If $c_3 = 96$, then $k_3 = \dfrac{115b_2}{6}$. In this case, there are no admissible intersection arrays.

Lemma 11. *The following assertions hold:*

(1) *If* $\mu = 12$, *then* $d \leq 5$.

(2) *If* $\mu = 6, 8, 10$, *then* $d \leq 6$.

Proof. We have $c_3 - b_3 \geq c_2 - b_2 + 20, \ldots, c_i - b_i \geq c_{i-1} - b_{i-1} + 20$. Summing up the inequalities termwise yields $c_i - b_i \geq c_2 - b_2 + (i - 2) \cdot 20$.

If $\mu = 12$, then, by Lemma 3, we have $b_2 \leq 31$ and $c_3 - b_3 \geq 12 - b_2 + 20$. Therefore, $d \leq 5$.

If $\mu = 6$, then, by Lemma 3, we have $b_2 \leq 28$, $c_4 - b_4 \geq 46 - b_2$, and $d \leq 7$. If $d = 7$, then $c_5 - b_5 \geq 6 - 28 + 60$, a contradiction.

If $\mu = 10$, then, by Lemma 3, we have $b_2 \leq 36$. Therefore, $c_4 - b_4 \geq 50 - b_2$ and $d \leq 7$. If $d = 7$, we obtain $c_5 - b_5 \geq 10 - 36 + 60$. Therefore, $b_5 \leq 2$ and b_5b_6 is not divided by 10, a contradiction.

If $\mu = 8$, then $k_2 = 60 \cdot 23$ and, by Lemma 3, we have $b_2 \leq 41$. Therefore, $c_4 - b_4 \geq 48 - b_2$ and $d \leq 7$. If $d = 7$, then $c_5 - b_5 \geq 8 - 41 + 60 = 37$. Therefore, $b_5 \leq 4$. Since b_5b_6 is divided by 8, we have $b_5 = 4$; $b_6 = 2$; $b_2 = c_5 = 41$; and the numbers b_3, c_4, and c_6 are divided by 8. Moreover, $c_3 - b_3 = -13$ and $c_4 - b_4 = 7$. Now the pair (b_3, c_3) coincides with (40, 27), (32, 19), or (24, 11), a contradiction to the fact that c_3 divides 60.

Lemma 12. *If* $d \in \{5, 6\}$, *then there are no admissible intersection arrays.*

Proof. Elementary computer calculations. The lemma, together with the theorem, is proved.

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