ISSN 1064-5624, Doklady Mathematics, 2013, Vol. 88, No. 3, pp. 625–629. © Pleiades Publishing, Ltd., 2013.
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MATHEMATICS

On Distance-Regular Graphs without 4-Claws

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Received February 5, 2013

DOI: 10.1134/S1064562413060057

We consider undirected graphs without loops or multiple edges. Given a vertex a in a graph Γ, let $\Gamma_i(a)$ denote the i-neighborhood of a, i.e., the subgraph induced by Γ on the set of all its vertices that are a distance of i away from a. Let $[a] = \Gamma_1(a)$ and $a^\perp = \{a\} \cup [a]$. A graph $\{u; y_1, y_2, \ldots, y_m\}$ is called an m-claw if the degree of u is m and the degrees of the vertices y_1, $y_2, \ldots, y_m$ are equal to 1. The Terwilliger graph is an incomplete connected graph in which the intersection of the neighborhoods of any two vertices separated by a distance of 2 is a μ-clique.

If vertices u and w are separated by a distance of i in Γ, then $b_i(u, w)$ ($c_i(u, w)$) denotes the number of vertices in the intersection of $\Gamma_{i+1}(u)$($\Gamma_{i-1}(u)$) with $\Gamma(w)$. A graph Γ of diameter d is called a distance-regular graph with the intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of the vertices u and w separated by a distance of i in Γ.

An incidence system $\mathbf{S} = (P, B, I)$ with a set of points P, a set of blocks B, and a symmetric incidence relation $I \subseteq (P \times B) \cup (B \times P)$ is called a geometry of rank 2. The flag (antiflag) of a geometry $\mathbf{S}$ defined as a pair $(x, L) \in (P, B)$ such that $x \in L$ ($x \notin L$). Given an antiflag (x, L) of a geometry $\mathbf{S}$, let $\alpha(x, L)$ denote the number of points belonging to L and collinear to x or, equivalently, the number of blocks containing x and intersecting L. A geometry is said to be φ-homogeneous if $\alpha(x, L) = 0$ or φ for any antiflag (x, L). A geometry is said to be strongly φ-homogeneous if $\alpha(x, L) = \varphi$ for any antiflag (x, L).

A geometry of rank 2 is called a partial space of straight lines if each of its elements (an element from $P \cup B$) is incident to at least two elements and any two points are incident to at most one block or, equivalently, if any two blocks are incident to at most one point. In this case, B is said to be a set of straight lines. A partial space of straight lines is called $(0, \alpha)$-geometry if it is α-homogeneous. If each point lies on $t + 1$ lines and each line contains $s + 1$ points, where $s, t \geq 1$, then $\mathbf{S}$ is called a partial space of straight lines of order (s, t). It is easy to see that, if $\mathbf{S}$ is a $(0, \alpha)$-geometry with $\alpha \geq 2$, then there are positive integers s and t such that $\mathbf{S}$ is of order (s, t).

A graph with a vertex set P and with the adjacency relation obtained deleting the equality from the collinearity relation is called the point graph of geometry $\mathbf{S}$. A geometry is said to be connected, regular, etc. if these properties are possessed by its point graph. The block graph of a geometry of rank 2 is defined in a similar manner. Finally, the flag graph of a geometry $\mathbf{S}$ has the vertex set consisting of the flags of the geometry and two distinct flags are adjacent if they share a vertex or a block.

A strongly α-homogeneous partial space of straight lines of order (s, t) is called a partial geometry and is denoted by $pG_\alpha(s, t)$. A partial geometry with $\alpha = 1$ is called a generalized quadrangle and is denoted by $GQ(s, t)$.

An incidence system (X, L), where X is a set of points and L is a set of straight lines, is called an almost $2n$-gon of order (s, t) if each line contains $s + 1$ points; each point lies on $t + 1$ lines (the lines intersect in at most one point); the diameter of the point graph is n; and, for any pair $(a, l) \in (X, L)$, the line l has a single point nearest to a in the point graph. An almost $2n$-gon is called a generalized $2n$-gon if any two points u and w separated by a distance shorter than n lie in a unique geodesic path joining u to w.

A distance-regular graph of diameter d with the smallest eigenvalue θ_d is called geometric if any of its edges lies in the unique $\left(1 - \dfrac{k}{\theta_d}\right)$-clique.

If a distance-regular graph contains a 4-claw, then $k \geq 4a_1 + 10 - 6c_2$ (see Lemma 1). In [1, Theorem 4.3], Bang classified the geometric distance-regular graphs with the smallest eigenvalue -3. It turned out that this class of graphs contains any distance-regular graph with $\max\left\{3, \dfrac{8(a_1 + 3)}{3}\right\} < k < 4a_1 + 10 - 6c_2$. Let Γ be

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a distance-regular geometric graph with the smallest eigenvalue -3. Then Γ is one of the following graphs:

(1) the Heawood graph, the Pappus graph, the Tutte 8-cage, a cube, or the Foster graph;

(2) the generalized 6-gon of order (8, 2) with the intersection array $\{24, 16, 16; 1, 1, 3\}$;

(3) one of two generalized 6-gons of order (2, 2) with the intersection array $\{6, 4, 4; 1, 1, 3\}$;

(4) the generalized 8-gon of order (4, 2) with the intersection array $\{12, 8, 8, 8; 1, 1, 1, 3\}$;

(5) the Johnson graph $J(n, 3)$, $n \geq 6$;

(6) a graph of diameter 3 with the intersection array $\{3\alpha + 3, 2\alpha + 2, \alpha + 2 - \beta; 1, 2, 3\beta\}$, $\alpha \geq \beta \geq 1$;

(7) the halved graph of the Foster graph with the intersection array $\{6, 4, 2, 1; 1, 1, 4, 6\}$;

(8) a graph with $d = h + 2 \geq 4$, and the triplet (c_i, a_i, b_i) is equal to $(1, \alpha, 2\alpha + 2)$ for $1 \leq i \leq h$, to $(2, 2\alpha + \beta - 1, \alpha - \beta + 2)$ for $i = h + 1$, and to $(3\beta, 3\alpha - 3\beta + 3, 0)$ for $i = h + 2$, where $\alpha \geq \beta \geq 2$;

(9) a graph with $d = h + 2 \geq 3$, and the triplet (c_i, a_i, b_i) is equal to $(1, \alpha, 2\alpha + 2)$ for $1 \leq i \leq h$, to $(1, \alpha + 2\beta - 2, 2\alpha - 2\beta + 4)$ for $i = h + 1$, and to $(3\beta, 3\alpha - 3\beta + 3, 0)$ for $i = h + 2$, where $\alpha \geq \beta \geq 2$;

(10) the graph Δ_2 for a distance-biregular graph Δ of degree 3, and the triplet (c_i, a_i, b_i) is equal to $(1, \alpha, 2\alpha + 2)$ for $1 \leq i \leq h$, to $(1, \alpha + 2, 2\alpha)$ for $i = h + 1$, to $(4, 2\alpha - 1, \alpha)$ for $i = h + 2, ..., d - 2$, to $(4, 2\alpha + \beta - 3, \alpha - \beta + 2)$ for $i = d - 1$, and to $(3\beta, 3\alpha - 3\beta + 3, 0)$ for $i = d$, where $\alpha \geq \beta$ and $\beta \in \{2, 3\}$.

All distance-regular graphs of diameter larger than 2 without 3-claws were found in [2, Theorem 1.2]. These are an icosahedral graph, an n-gon for $n \geq 6$, the line graph of a Moore graph (of a strongly regular graph with parameters $(k^2 + 1, k, 0, 1)$, $k = 2, 3, 7, 57, 57$), or the flag graph of a generalized n-gon of order (s, s) for some s.

This paper continues the study of distance-regular graphs without 4-claws.

Theorem 1. *Let Γ be a distance-regular graph of diameter $d \geq 3$ and degree $k > 3$. If $k > \frac{5a_1 + c_2 + 4}{2}$, then the following assertions are equivalent:*

(1) *Γ does not contain 4-claws.*

(2) *Γ is a geometric graph with the smallest eigenvalue -3.*

Proposition. *Let Γ be a distance-regular graph of diameter $d \leq 3$ without 4-claws. If Γ has a set $\mathcal{L}$ of straight lines such that $|L| \geq 3$ for any line $L \in \mathcal{L}$ and $|M| \geq 4$ for some line $M \in \mathcal{L}$, each edge of Γ lies on a single line, and each vertex of Γ lies on precisely three lines, then Γ is a geometric graph with the smallest eigenvalue -3.*

Theorem 2. *Let Γ be a distance-regular graph of diameter $d \geq 5$ without 4-claws. Then Γ is the flag graph of a generalized n-gon of order (s, s) for some s or a geometric graph with $\theta_d = -3$.*

First, we present auxiliary results.

Lemma 1. *Let Γ be an amply regular graph with parameters (v, k, λ, μ). If for some vertex $u \in \Gamma$, the subgraph $[u]$ contains a c-clique, then $\mu - 1 \geq \dfrac{c(\lambda + 1) - k}{\binom{c}{2}}$.*

Proof. The proposition follows from the proof of Lemma 3 in [3].

Lemma 2. *Let Γ be a connected amply regular graph with parameters (v, k, λ, μ), and let s be a maximum number such that, for all $x \in \Gamma$ and any two nonadjacent vertices $y, z \in [x]$ in the graph $[x]$, there is an s-coclique containing y and z. Then the following assertions hold:*

(1) $s \geq \dfrac{k}{\lambda + 1}$.

(2) $\mu - 1 \geq \max\left\{ \dfrac{s'(\lambda + 1) - k}{\binom{s'}{2}} \;\middle|\; 2 \leq s' \leq s \right\}$; *moreover, in the case of equality, Γ is a Terwilliger graph.*

(3) *If $\mu \geq 2$ and the equality holds in (2), then Γ is isomorphic to an icosahedral graph, the Conway–Smith graph, or the Doro graph.*

Proof. This is Proposition 4.1 and Theorem 4.2 from [4].

Lemma 3. *Let Γ be an amply regular Terwilliger graph with parameters (v, k, λ, μ) without 7-claws. If $\mu \geq 2$, then Doro graph is isomorphic to an icosahedral graph, the Conway–Smith graph, or the Doro graph.*

Proof. By Lemma 3.1 in [4], there is a positive integer α such that, for any vertex $u \in \Gamma$, the subgraph $\Delta = [u]$ is a clique α-extension of a strongly regular Terwilliger graph $\bar{\Delta}$ with parameters $\bar{v} = \dfrac{k}{\alpha}$, $k = \dfrac{\lambda - \alpha + 1}{\alpha}$, and $\bar{\mu} = \dfrac{\mu - 1}{\alpha}$, where $\alpha \geq \bar{\lambda} + 1$.

Assume that $\Delta_1 = \bar{\Delta}$ and, for a vertex $u_i \in \Delta_i$, $\Delta_{i+1} = \bar{\Delta}_i(u_i)$. Then, for some i, $\bar{\Delta}_i$ is a graph with $\mu_i = 1$. Since $\bar{\Delta}_i$ does not contain 7-claws, by Lemma 2, $\bar{\Delta}_i$ is a pentagon or a Petersen graph. By Lemma 3.2 in [4], Δ_i is a pentagon or a Petersen graph, $i = 1$, and Γ is isomorphic to an icosahedral graph, the Conway–Smith graph, or the Doro graph.

Lemma 4. *Let Γ be a distance-regular graph of diameter $d \geq 3$ without 4-claws. If for some vertices $w, z \in \Gamma$ with $d(w, z) = 3$, the subgraph $[w] \cap \Gamma_2(z)$ is a clique, then the following assertions hold:*

(1) *$[z] \cap \Gamma_2(w)$ is the union of at least two isolated cliques.*

(2) *If $[z] \cap \Gamma_2(w)$ is the union of two isolated cliques, then $c_3 = 2c_2$ and, for any vertices $x \in [w] \cap \Gamma_2(z)$ and*

$y \in [z] \cap \Gamma_2(w)$, *the subgraphs* $[x] \cap [z]$ *and* $[w] \cap [y]$ *are cliques.*

(3) *If* Γ *does not contain* 4-*claws and* $d \geq 4$, *then* $[z] \cap \Gamma_2(w)$ *is the union of two isolated cliques.*

Proof. By [5, Theorem 5.4.1], we have $c_3 > c_2$. Assume that $\Delta = [w] \cap \Gamma_2(z)$ and $\Sigma = [z] \cap \Gamma_2(w)$. Then, for any vertex $y \in \Sigma$, the subgraph $\Delta \cap [y]$ is a c_2-clique. Note that, for adjacent vertices $y, y' \in \Sigma$, we have $\Delta \cap [y] = \Delta \cap [y']$; otherwise, for $x \in \Delta \cap [y] - [y']$, the subgraph $[x] \cap [y']$ contains y and c_2 vertices from Δ, a contradiction. If Σ contains a geodesic 2-path y_1, y_2, y_3, then $\Delta \cap [y_1] = \Delta \cap [y_3]$ and $[y_1] \cap [y_3]$ contains y_2 and c_2 vertices from Δ, a contradiction. Thus, Σ is the union of isolated cliques. If Σ is a clique, then, for nonadjacent vertices $x \in \Delta$ and $y \in \Sigma$, the subgraph $[x] \cap [y]$ contains $2c_2$ vertices, a contradiction.

Assume that Σ is the union of two isolated cliques. Then, for two nonadjacent vertices $y, y'' \in \Sigma$, the subgraphs $\Delta \cap [y]$ and $\Delta \cap [y'']$ do not intersect; otherwise, for $x \in \Delta \cap [y] \cap [y'']$, the subgraph $[x]$ contains Σ, a contradiction. Now Σ is the union of two isolated cliques of order c_2 and $c_3 = 2c_2$. Furthermore, for any vertices $x \in [w] \cap \Gamma_2(z)$ and $y \in [z] \cap \Gamma_2(w)$, the subgraphs $[x] \cap [z]$ and $[w] \cap [y]$ are cliques.

Suppose that Γ does not contain 4-claws and $d \geq 4$. Then, for a vertex $t \in [z] \cap \Gamma_4(w)$, the subgraph $[z] - t^{\perp}$ does not contain 3-cocliques. Therefore, Σ is the union of two isolated cliques. The lemma is proved.

In Lemmas 5 and 6, it is assumed that Γ is a distance-regular graph of diameter $d \geq 3$ without 4-claws with the smallest eigenvalue θ_d that contains a 3-claw and satisfies the inequality $k \leq \max\left\{3, \dfrac{8(a_1+1)}{3}\right\}$.

Lemma 5. *Let* e^* *be the highest vertex degree in a* μ-*subgraph. Then any* μ-*subgraph does not contain* 3-*cocliques and* $e^* \geq \dfrac{c_2-2}{2}$.

Proof. Let u and w be vertices separated by a distance of 2 in Γ. If $[u] \cap [w]$ contains a 3-coclique, then the union of the neighborhoods of vertices in this coclique contains $[w] - [u]$. Therefore, $[w]$ does not intersect $\Gamma_3(u)$, a contradiction.

If the degree of u in a μ-subgraph $[y_1] \cap [y_2]$ is ν, then $[y_1] \cap [y_2] - u^{\perp}$ is a $(c_2 - 1 - \nu)$-clique. Therefore, $e^* \geq \dfrac{c_2-2}{2}$. The lemma is proved.

Lemma 6. *The following assertions hold:*

(1) $k \geq 3a_1 + 6 - 3c_2$ and $c_2 > 1$.

(2) Γ *is not a Terwilliger graph and* $d(\Gamma) \leq 5$.

Proof. Since Γ has a vertex u such that $[u]$ contains a 3-coclique, by Lemma 1, we have $c_2 - 1 \geq \dfrac{3a_1+3-k}{3}$. Therefore, $k \geq 3a_1 + 6 - 3c_2$. If $c_2 = 1$, then $k \geq 3(a_1 + 1)$, a contradiction to $k \leq \max\left\{3, \dfrac{8(a_1+1)}{3}\right\}$.

If Γ is a Terwilliger graph, then, by Lemma 3, Γ is locally a pentagonal graph or locally a Petersen graph. However, in the former case, Γ does not contain 3-claws, while, in the latter case, it contains a 4-claw, a contradiction. Now, by [5, Corollary 5.2.2], $d(\Gamma) \leq \dfrac{2k}{a_1+2}$. Since $k < \dfrac{8(a_1+1)}{3}$, we conclude that $d(\Gamma) \leq 5$.

Let us prove the proposition. Suppose that Γ has a set $\mathcal{L}$ of lines such that $|L| \geq 3$ for any line $L \in \mathcal{L}$ and $|M| \geq 4$ for some line $M \in \mathcal{L}$, each edge of Γ lies on a single line, and each vertex of Γ lies on precisely three lines.

Let A be the adjacency matrix of Γ, and let B be the incidence matrix of the scheme $(V(\Gamma), \mathcal{L})$ whose rows and columns are indexed by the vertices and lines of Γ, respectively. Then $BB^{\mathrm{T}} = A + 3E$ and the number of lines is less than the number of points. Therefore, BB^{T} is a singular matrix and -3 is an eigenvalue of the matrix A. Since BB^{T} is positive semidefinite, $\theta_d = -3$.

The neighborhood of a vertex in Γ is divided by three lines. Therefore, there is a line L with $|L| - 1 \geq \dfrac{k}{3}$. On the other hand, $|L| \leq 1 - \dfrac{k}{\theta_d}$. Therefore, $|L| = 1 + \dfrac{k}{3}$ for any line. The proposition is proved.

Let us prove Theorem 1. Let Γ be a distance-regular graph of diameter $d \geq 3$ without 4-claws, and let $k > \dfrac{5a_1+c_2+4}{2}$. By [1, Theorem 3.2], we may assume that $k \leq \dfrac{8(a_1+1)}{3}$. Then $a_1 > 1$ and, for any two adjacent vertices u and y, the vertex y is contained in a 3-coclique from $[u]$. Furthermore, $c_2 > 1$; otherwise, k is divided by $a_1 + 1$, a contradiction.

Let $\{u; y_1, y_2, y_3\}$ be a 3-claw; $Y_i = [u] \cap y_i^{\perp}$, M_i be a maximal clique containing $Y_i - (Y_{i+1} \cup Y_{i+2})$, $i \in \{1, 2, 3\}$; and the indices be taken modulo 3. Then the cliques $Y_1 - (Y_2 \cup Y_3)$, $Y_2 - (Y_1 \cup Y_3)$, and $Y_3 - (Y_1 \cup Y_2)$ are pairwise disjoint and $|Y_1 - (Y_2 \cup Y_3)| \geq k - 2(a_1 + 1)$.

Let the line be a maximal clique L with $|L| \geq k - 1 - 2a_1$. Then $|L| > \dfrac{a_1+c_2+2}{2}$; any edge lies on a line; and if $\dfrac{a_1+c_2+2}{2} < |L| \leq 3$, then $c_2 = 2$, $a_1 = 3$, and $k = 10$. However, in this case, Γ is either a locally Petersen graph of diameter 4 or $d = 3$ and Γ has the intersection array $\{10, 6, b_2; 1, 2, c_3\}$. In the former case, Γ contains a 4-claw, while, in the latter case, it has the intersection array $\{10, 6, 4; 1, 2, 5\}$ and the spectrum $10^1, 5^{13}, 0^{26}, -3^{25}$. By [5, Proposition 12.2.2], Γ is a locally Petersen graph. In any case, we have a contradiction.

Assume that two lines L and M share an edge $\{u, y\}$. Then $a_1 = |[u] \cap [y]| \ge 2(k - 3 - 2a_1) - (|L \cap M| - 2)$. Therefore, $c_2 \ge |L \cap M| \ge 2k - 5a_1 - 4$ and $k \le \frac{5a_1 + c_2 + 4}{2}$, a contradiction. Thus, any two lines intersect in at most one point.

Let $\{y_1, y_2, y_3\}$ be a 3-coclique from $[u]$ and M_i be a line containing u and y_i. If $u^\perp$ contains a fourth line L, then each vertex from $L - \{u\}$ belongs to $[y_i] \cap [y_j]$ for two indices $i, j \in \{1, 3, 4\}$ and we can assume that $[u] \cap [y_1]$ contains at least $\frac{2(|L| - 1)}{3}$ vertices from L. Therefore, $c_2 - 1 \ge |(L - \{u\}) \cap [y_1]| \ge \frac{2(|L| - 1)}{3}$, which yields $\frac{a_1 + c_2}{2} < |L| \le \frac{3(c_2 - 1)}{2} + 1$ and $a_1 < 2c_2 - 1$. On the other hand, $[u] \cap [y_1]$ contains more than $\frac{a_1 + c_2 - 4}{2}$ vertices from M_1 and at least $\frac{a_1 + c_2 - 1}{3}$ vertices from L. Therefore, $a_1 \ge \frac{5(a_1 + c_2 - 14)}{6}$ and $a_1 \ge 5c_2 - 14$, $c_2 \le 4$. If $c_2 = 4$, we have $a_1 = 6$ and $18 < k = 18$, a contradiction. If $c_2 = 3$, then $a_1 = 2, 3, 4$, respectively, $7 < k = 8$, $10 < k = 10$, $12 < k = 13$, and the number k_2 is not an integer, a contradiction. If $c_2 = 2$, we have $a_1 = 2$, $k = 8$, $k_2 = 20$, and Γ has the intersection array $\{8, 5, b_2; 1, 2, c_3\}$. In this case, there are no admissible intersection arrays, a contradiction.

Thus, for any vertex u, the subgraph $u^\perp$ contains precisely three lines and Theorem 1 follows from the proposition.

Lemma 7. *If Γ is a distance-regular graph of diameter larger than 4 without 4-claws, then Γ is a geometric graph with eigenvalue $\theta_d = -3$.*

Proof. Let the diameter of Γ be 5. Then $k \ge \frac{5(a_1 + 2)}{2}$. We choose vertices u and w separated by a distance of 2 in Γ such that $[u] \cap [w]$ is not a clique. Let $z \in \Gamma_5(u) \cap \Gamma_3(w)$. Then $\Delta = [w] \cap \Gamma_2(z)$ is a c_3-clique. By Lemma 4, we have $c_3 = 2c_2$.

Assume that $\Sigma = \Gamma_4(u) \cap \Gamma_2(w)$ and $\Delta = [w] \cap \Gamma_3(u)$. For any vertex $z \in \Sigma$, the subgraph $[z] \cap \Delta$ is a c_2-clique. Note that, for adjacent vertices $z, z' \in \Sigma$, we have $[z] \cap \Delta = [z'] \cap \Delta$; otherwise, for $x \in [z] \cap \Delta - [z']$, the subgraph $[x] \cap [z']$ contains z and c_2 vertices from Δ, a contradiction. If Σ contains a geodesic 2-path z_1, z_2, z_3, then $[z_1] \cap \Delta = [z_3] \cap \Delta$ and $[z_1] \cap [z_3]$ contains z_2 and c_2 vertices from Δ, a contradiction. Thus, Σ is the union of t isolated cliques.

Assume that $[y] \cap \Gamma_4(u)$ is not a clique for some vertex $y \in \Delta$. Then the subgraph $[y] \cap \Gamma_2(u)$ is a c_3-clique. By Lemma 4, for any vertex $a \in [y] \cap \Gamma_2(u)$, the subgraph $[u] \cap [a]$ is a clique, a contradiction to the choice of the vertex w.

Since $[y] \cap \Gamma_4(u)$ is a b_3-clique for any vertex $y \in \Delta$, we have, as was shown above, $b_2 = tc_2$ and $p^2_{24} = tb_3$. Furthermore, $b_2 \ge c_3$. Therefore, $t \ge 3$.

Let $t \ge 3$, and let x and x' be two nonadjacent vertices from $[u] \cap [w]$. Then the degree of one of the vertices u and w in the graph $[x] \cap [x']$ is at least $\frac{c_2 - 2}{2}$ and $k \ge (2a_1 + 2) - \frac{c_2 - 2}{2} + 3c_2$. From this, $2a_1 + \frac{5c_2}{2} + 3 \le k \le \frac{5a_1 + c_2 + 2}{2}$ and $a_1 \ge 4c_2 + 4$.

Let $\{u; y_1, y_2, y_3\}$ be a 3-claw; $Y_i = [u] \cap y_i^\perp$; M_i be a maximal clique containing $Y_i - (Y_{i+1} \cup Y_{i+2})$, $i \in \{1, 2, 3\}$; and the indices be taken modulo 3. Then the cliques $Y_1 - (Y_2 \cup Y_3)$, $Y_2 - (Y_1 \cup Y_3)$, and $Y_3 - (Y_1 \cup Y_2)$ are pairwise disjoint. If $[y_2] \cap [y_3]$ is not a clique, then $|M_1| \ge 3c_2 + 1$. If $[y_2] \cap [y_3]$ is a clique, then $|M_1| \ge k + 1 - 2(a_1 + 1) + (c_2 - 1) \ge \frac{7c_2}{2} + 1$.

Let the line be a maximal clique L with $|L| \ge \max\{3c_2 + 1, a_1 - 2c_2 + 4\}$. Then any edge lies on a line and $|L| \ge 4$.

Assume that two lines L_1 and L_2 share an edge $\{u, y\}$. Then $a_1 = |[u]| \cap [y]| \ge (3c_2 - 1) + (a_1 - 2c_2 + 2) - (|L_1 \cap L_2| - 2)$. Therefore, $c_2 \ge |L_1 \cap L_2| \ge c_2 + 3$, a contradiction. Thus, any two lines intersect in at most one point.

If $u^\perp$ contains four lines L, M_1, M_2, and M_3, then each vertex from $L - \{u\}$ is adjacent to at least two vertices from $\{y_1, y_2, y_3\}$ and we can assume that $c_2 - 1 \ge |(L - \{u\}) \cap [y_1]| \ge \frac{2(|L| - 1)}{3}$, which is a contradiction to $|L| \le \frac{3(c_2 - 1)}{2} + 1$. Thus, each vertex of Γ lies on precisely three lines and, by the proposition, Γ is a geometric graph with the smallest eigenvalue -3.

Now it can be assumed that $t = 2$ and $b_2 = c_3 = 2c_2$. We have $c_3 - b_3 \ge c_2 - b_2 + a_1 + 2$. Therefore, $a_1 + b_3 \le 3c_2 - 2$ and $a_1 \le 2c_2 - 2$. As before, $2a_1 + \frac{3c_2}{2} + 3 \le k \le \frac{5a_1 + c_2 + 2}{2}$ and $a_1 \ge 2c_2 + 4$, a contradiction.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (grant no. 11371335), by the Russian Foundation for Basic Research (project no. 12-01-00012 and joint project no. 12-01-91155 with the National Science Fund of China), by the Branch of Mathematics of the Russian Academy of Sciences (project no. 12-T-1-1003), and by the Ural Branch of the Russian Academy of Sciences jointly with the Siberian Branch of the Russian Academy of Sciences (project no. 12-S-1-1018) and with the National Academy of Sciences of Belarus (project no. 12-S-1-1009).

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Translated by I. Ruzanova