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MATHEMATICS

On Almost Distance-Transitive Graphs

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We consider undirected graphs without loops or multiple edges. Given a vertex a in a graph Γ, let $\Gamma_i(a)$ denote the i-neighborhood of a, i.e., the subgraph induced by Γ on the set of all its vertices that are a distance of i away from a. Let $[a] = \Gamma_1(a)$ and $a^\perp = \{a\} \cup [a]$.

If vertices u and w are separated by a distance of i in Γ, then $b_i(u, w)$ ($c_i(u, w)$) denotes the number of vertices in the intersection of $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph Γ of diameter d is called a distance-regular graph with the intersection array $\{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of the vertices u and w separated by a distance of i in Γ for any $i = 0, 1, ..., d$. Let $a_i = k - b_i - c_i$. Note that, for a distance-regular graph, b_0 is its degree and $c_1 = 1$.

A graph Γ of diameter d is called distance-transitive if the group $G = \mathrm{Aut}(\Gamma)$ is transitive on the vertex set of Γ and, for any vertex $u \in \Gamma$ and for any $i \in \{1, 2, ..., d\}$, the group G_u is transitive on $\Gamma_i(u)$.

Let Γ be a distance-regular graph of diameter d with an automorphism group G. Γ is said to be almost distance-transitive if there exists a number i_0, $1 < i_0 < d$, such that, for any $i \in \{1, 2, ..., d\} - \{i_0\}$, the stabilizer of any vertex a acts transitively on $\Gamma_i(a)$.

Given a subset X of automorphisms of Γ, let $Fix(X)$ denote the set of all vertices of Γ that are fixed under any automorphism from X.

If H is a subgroup of the automorphism group of Γ, then, on the set of H-orbits, the quotient of Γ can be defined by assuming that two orbits are adjacent if one of them has a vertex adjacent in Γ to a vertex of the other orbit.

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The antipodal distance-transitive graphs of diameter 3 were described in [1]. More general is the problem of describing the antipodal almost distance-transitive graphs of diameter 3.

According to [2], an antipodal distance-regular graph Γ of diameter 3 has the intersection array $\{k, \mu(r-1), 1; 1, \mu, k\}$, $v = r(k+1)$ vertices, and the spectrum $k^1, n^f, (-1)^k, (-m)^g$, where n and $-m$ are the roots of the equation $x^2 - (\lambda - \mu)x - k = 0$, $f = \dfrac{m(r-1)(k+1)}{n+m}$, and $g = \dfrac{n(r-1)(k+1)}{n+m}$.

If $\mu \neq \lambda$, then the eigenvalues of the graph are integer and the parameters of the graph can be expressed in terms of r, n, and m: $k = nm$, $\mu = \dfrac{(m-1)(n+1)}{r}$, and $\lambda = \mu + n - m$. Since the multiplicities of the eigenvalues are integer-valued, we have the following divisibility condition: $n + m$ divides $(r-1)m(m^2-1)$. If $r > 2$, the Krein condition $q_{33}^3 \geq 0$ implies $m \leq n^2$. If $m = n^2$, then the neighborhood of any vertex is a strongly regular graph with eigenvalues $a_1 = (n-1)\left(\dfrac{(n+1)^2}{r} - n\right)$, $n - \dfrac{n+1}{r}$, $n - \dfrac{(n+1)^2}{r}$. If $\lambda = 0$, then $r = 2$ or $r - 2 \geq \sqrt{\mu}$.

If $\mu = \lambda$, then Γ has the intersection array $\{r\mu+1, \mu(r-1), 1; 1, \mu, r\mu+1\}$, $v = r(r\mu+2)$, and the spectrum k^1, $\sqrt{k}^f$, $(-1)^k$, $(-\sqrt{k})^f$, where $f = \binom{r}{2}\mu + r - 1 = \dfrac{v-k-1}{2}$. In what follows, the number $\mu(r-1)$ is even.

This paper continues the study of edge-symmetric antipodal distance-regular graphs of diameter 3 by following the program proposed in [3] as based on a classification of twice transitive permutation groups. More specifically, we classify the antipodal almost distance-transitive graphs of diameter 3.

Let Γ be an edge-symmetric distance-regular graph with the intersection array $\{k, (r-1)\mu, 1; 1, \mu, k\}$, $G = \mathrm{Aut}(\Gamma)$, Σ be the set of antipodal classes of Γ, and $F \in \Sigma$. By Proposition 1 in [3], the group G acts twice transitively on Σ. If the stabilizer G_a of a vertex a acts transitively on $\Gamma_3(a)$, then the global stabilizer $G_{\{F\}}$ acts twice transitively on F.

Proposition 1 [1, Theorem 2.9]. *Let G^X be a twice transitive permutation group of degree n, $a \in X$, $H = G_a$, and T be the socle of the group G. Then either*

(1) *G is an almost simple group and one of the following possibilities occurs for (T, n):*

(i) *alternating* (A_n, n), $n \geq 5$;

(ii) *linear* $\left(L_m(q), \dfrac{q^m-1}{q-1}\right)$, $m \geq 2$ *and* $(m, q) \notin \{(2, 2), (2,3)\}$;

(iii) *symplectic* $(\mathrm{Sp}_{2m}(2), 2^{2m-1} \pm 2^{m-1})$, $m \geq 3$;

(iv) *unitary* $(U_3(q), q^3 + 1)$, $q \geq 3$;

(v) *Ree* $({}^2G_2(q), q^3 + 1)$, $q = 3^{2a+1} \geq 27$;

(vi) *Suzuki* $(Sz(q), q^2 + 1)$, $q = 2^{2a+1} \geq 8$;

(vii) *Mathieu* (M_n, n), $n \in \{11, 12, 22, 23, 24\}$;

(viii) *sporadic* $(L_2(11), 11)$, $(M_{11}, 12)$, $(A_7, 15)$, $(L_2(8), 28)$, $(HiS, 176)$, $(Co_3, 276)$ *or*

(2) *$G = TH$, T is an elementary Abelian group of order $n = p^e$, and one of the following possibilities occurs:*

(i) *linear $e = cd$, $d \geq 2$ and $SL_d(p^c) \lhd H \leq \Gamma L_d(p^c)$;*

(ii) *symplectic $e = ct$, t is even, $t \geq 4$, and $\mathrm{Sp}_t(p^c) \lhd H \leq Z_{p^c-1} \circ \Gamma Sp_t(p^c)$;*

(iii) *G_2-type $e = 6c$, $p = 2$, and $G_2(2^c)' \lhd H \leq Z_{2^c-1} \circ \mathrm{Aut}(G_2(2^c))$;*

(iv) *one-dimensional $H \leq \Gamma L_1(p^e)$;*

(v) *exceptional $p^e \in \{9^2, 11^2, 19^2, 29^2, 59^2\}$ and $SL_2(5) \lhd H$, $p^e = 2^4$, and $H \in \{A_6, S_6, A_7\}$ or $p^e = 3^6$ and $H = SL_2(13)$;*

(vi) *extraspecial $p^e \in \{5^2, 7^2, 11^2, 23^2\}$ and $SL_2(3) \lhd H$ or $p^e = 3^4$, $R = D_8 \circ Q_8 \lhd H$, $H/R \leq S_5$, and 5 divides $|H|$.*

Theorem. *Let Γ be an almost distance-transitive graph with the intersection array $\{k, (r-1)\mu, 1; 1, \mu, k\}$, $G = \mathrm{Aut}(\Gamma)$, Σ be the set of antipodal classes of Γ, K be the kernel of a group action of G on Σ, $\bar{G} = G/K$, $\bar{T}$ be the socle of the group $\bar{G}$, $F \in \Sigma$, $a \in F$, and $H = G_{\{F\}}$. Then Γ is the distance-transitive graph from the claim of the theorem in* [1] *or one of the following assertions holds:*

(1) *G contains a normal subgroup N with the property $|N\colon N \cap G_a| = k + 1$, Γ has the intersection array $\{15, 10, 1; 1, 2, 15\}$, $K = 1$, N is an elementary Abelian group of order 16, $H = A_6$, S_6, or $H = GL_2(4).Z_2$.*

(2) *$\bar{T}$ is an elementary Abelian group of order p^e and either*

(i) *Γ has the intersection array $\{8, 6, 1; 1, 3, 8\}$, $|K| = r - 3$, $|H\colon C_H(K)| = 2$, T is an extraspecial group of order 3^3, H contains a subgroup of index 2 that is isomorphic to $Z_3 \times SL_2(3)$ or is the direct product of K and a subgroup of Z_8;*

(ii) *Γ has the intersection array $\{80, 54, 1; 1, 27, 80\}$, $|K| = r = 2$, $|H\colon C_H(K)| = 2$, T is an extraspecial group of order 3^5, $R = D_8 \circ Q_8 \lhd H_a$, $H_a/R = S_5$, or $|H_a : R| \leq 2$ and $R \leq SL_2(5) \circ Z_8$; or*

(iii) *Γ has the intersection array $\{728, 486, 1; 1, 243, 728\}$, $|K| = r - 3$, $|H : C_H(K)| = 2$, $SL_2(13) \lhd H_a$, and T is an extraspecial group of order 3^7.*

The following result is used to prove the theorem and is of interest in itself.

Proposition 2. *Let Γ be an almost distance-transitive graph with the intersection array $\{k, (r-1)\mu, 1; 1, \mu, k\}$, $G = \mathrm{Aut}(\Gamma)$, Σ be the set of antipodal classes of Γ, K be the kernel of a group action of G on Σ, $F \in \Sigma$, and $\bar{T}$ be the socle of the group $\bar{G} = C/K$. Then one of the following assertions holds:*

(1) *T does not contain normal (in G) subgroups N with the property $|N : N \cap G_a| = k + 1$.*

(2) *Γ has the intersection array $\{15, 10, 1; 1, 2, 15\}$.*

The proof of the theorem relies on the results of [3], which were obtained using Higman's method for handling automorphisms of a distance-regular graph.

The following result is well known.

Lemma 1. *Let G be a finite group acting transitively on X and Y. If $|X|$ and $|Y|$ are relatively prime numbers, then, for $x \in X$, the group G_x acts transitively on Y.*

Proof. Let $x \in X$ and $y \in Y$. Then, for a group action of G on $X \cup Y$, we have $|G_y : G_{x,y}| = |G_y|/|G_{x,y}| = |G|/(|Y||G_{x,y}|) = (|G_x||X|)/(|Y||G_{x,y}|) = |G_x : G_{x,y}||X|/|Y|$. Therefore, since $|X|$ and $|Y|$ are relatively prime, $|Y|$ divides $|G_x : G_{x,y}|$, i.e., divides the order of the G_x-orbit on Y that contains y. Thus, G_x acts transitively on Y.

Lemma 2 [1, Theorem 2.5]. *Let Γ be an distance-regular nonbipartite graph with the intersection array k, $\mu(r-1)$, 1; 1, μ, $k\}$, K be an Abelian subgroup of* $\mathrm{Aut}(\Gamma)$ *that is transitive on each antipodal class, and p be a prime divisor of r. Then p divides $k + 1$.*

Lemma 3. *Suppose that r divides p^c, $|\bar{T}| = p^e$, and H contains a normal subgroup* $\mathrm{Sp}(d, p^c)'$ *or* $G_2(2^c)$, *where e is equal to cd $(d \geq 4)$ or $6c$ $(c \geq 2)$, respectively. If H acts twice transitively on F, then Γ is a distance-transitive graph.*

Proof. If $r = p^c$, then Γ is the graph from [1, Example 3.6] and Γ is a classical distance-transitive graph. If $r < p^c$, then Γ is the quotient of a classical graph and, by [1, Lemma 2.8], Γ is a distance-transitive graph.

Throughout the rest of this paper, let Γ be an almost distance-transitive graph with the intersection array $\{\{k, (r-1)\mu, 1; 1, \mu, k\}$, where $r > 2$; $G = \mathrm{Aut}(\Gamma)$; Σ be the set of antipodal classes of Γ; K be the kernel of a group action of G on Σ, $F \in \Sigma$, $a \in F$, $H = G_F$, $\bar{T}$ be the socle of the group $\bar{G} = G/K$; C be the kernel of a group action of H on F; $g \in G$; and $\Omega = Fix(g)$.

If $\mu = 1$, then, by Proposition 4 in [5], we have $K = 2^e$, $L_2(k) \lhd G$, and Γ has the intersection array $\{2^e, 2^e - 1, 1; 1, 1, 2^e\}$. The almost distance transitivity condition implies that $2^e - 1$ divides e. Therefore, $e = 1$ and Γ is a hexagon.

In what follows, we assume that $\mu > 1$.

Lemma 4 [4, Lemma 1]. *The following assertions hold:*

(1) *If Ω is an empty graph and $|g| = p$ is a prime number, then either*

(i) *p does not divide r, $\alpha_3(g) = 0$, $\alpha_1(g) + \alpha_2(g) = v$, and* $\chi_1(g) = \dfrac{\alpha_1(g) - (k+1)}{m+n}$ *or*

(ii) *p divides r, $\alpha_3(g) = tr$, and* $\chi_1(g) = \dfrac{(1-m)t + \alpha_1(g) - (k+1)}{m+n}$; *if $\alpha_3(g) = v$, then* $\chi_1(g) = \dfrac{-m(k+1)}{m+n}$.

(2) *If Ω contains $t > 0$ antipodal classes, then $\alpha_3(g) = 0$, $\alpha_1(g) + \alpha_2(g) = r(k + 1 - t)$, $\chi_2(g) = \dfrac{\alpha_0(g)}{r} - 1$, and*

$$\chi_1(g) = \frac{(m(r-1)+1)\alpha_0(g)}{r(m+n)} + \frac{\alpha_1(g) - (k+1)}{m+n}.$$

(3) *If $\alpha_0(g) = yr$ and $\alpha_0(g) + \alpha_2(g) = v$, then* $\chi_1(g) = \dfrac{(m(r-1)+1)y - (k+1)}{m+n}$.

Lemma 5. *If $r - 1$ and k are relatively prime numbers, then, for $b \in [a]$, the group $H_a \cap H_b$ acts transitively on $F - \{a\}$.*

Proof. By assumption, H_a acts transitively on $F - \{a\}$ and $[a]$. Now the lemma follows from Lemma 1.

Let us prove Proposition 2. Assume that T contains a normal (in G) subgroup N with the property $|N: N \cap G_a| = k + 1$. Then any N-orbit contains a unique element from each antipodal class. For vertex $a \in \Gamma$, the group G_a acts transitively on $[a]$. Therefore, every N-orbit consists of vertices separated by a distance of 2 from one another and there are t orbits ($t < r$) in which a is adjacent to precisely α vertices; specifically, $k = t\alpha$.

A vertex from $[a]$ is adjacent to $\alpha - 1$ vertices in $a^N - \{a\}$, and there are precisely $k(\alpha - 1)$ edges between $[a]$ and $a^N - \{a\}$. Therefore, a vertex from $a^N - \{a\}$ is adjacent on average to $\alpha - 1$ vertices from $[a]$. From this, $\alpha - 1 = \mu$ and $k = t(\mu + 1)$. Since $b_1 = (r - 1)\mu$, we have $\lambda = k - b_1 - 1 = t + \mu - (r - t)\mu - 1$ and $\lambda - \mu = t - 1 - (r - t)\mu$; specifically $t - 1 \geq (r - t - 1)\mu$.

The almost distance transitivity of Γ implies that $t = r - 1$. By [4, Proposition 2], the claim of the proposition holds if $t = r - 1$ and the affine case occurs for an action of G on Σ.

Let $t = r - 1$ and the semisimple case occur for an action of G on Σ. A vertex from $[a]$ is adjacent to $\alpha - 1$ vertices from $a^N - \{a\}$, and there are precisely $k(\alpha - 1)$ edges between $[a]$ and $a^N - \{a\}$. Therefore, a vertex from $a^N - \{a\}$ is adjacent on average to $\alpha - 1$ vertices from $[a]$. From this, $\alpha - 1 = \mu$ and $k = (r - 1)(\mu + 1)$. Since $b_1 = (r - 1)\mu$, we have $\lambda = k - b_1 - 1 = r - 2$.

The set of N_a-orbits on $[a]$ forms an imprimitivity system of the group G_a on $[a]$. For a vertex $b \in F - \{a\}$ such that $|b^N \cap [a]| > 0$, the set $[a] \cap b^N$ is the union of some number of N_a-orbits. Therefore, the order of an N_a-orbit on $[a]$ divides $\mu + 1$.

Furthermore, N_a pointwise fixes F and $N_a = N_F \leq C$.

The group G contains a subgroup NG_a that is twice transitive on a^N (an action of NG_a on a^N is exact, since an element from NG_a that pointwise fixes a^N pointwise fixes each antipodal class). Let S denote the socle of NG_a. We have $S \leq N$. Let $(S, k + 1) \neq (L_2(8), 28)$. Then S is a twice transitive permutation group on a^N and $S_a \leq N_a$. Consequently, N_a is transitive on $a^N - \{a\}$. Since N_a pointwise fixes F, N_a is transitive on $b^N - \{b\}$. Therefore, the vertex a is adjacent to $\mu + 1 = k$ vertices from $b^N - \{b\}$, which yields $r = 2$, a contradiction. Now let $(S, k + 1) = (L_2(8), 28)$. Computer calculations in GAP show that there is no graph with such an intersection array. Proposition 2 is proved.

Throughout the rest of this paper, we assume that the semisimple case occurs for an action of G on Σ.

Lemma 6. *If $K \neq 1$, then $r = 3$, $T = SU_3(q)$, $T_F = T_a \times K$ is the extension of a subgroup of order q^3 by a cyclic group of order $q^2 - 1$, $q + 1$ is comparable to ± 3 modulo 9, and Γ is a distance-transitive graph with the intersection array*

$$\left\{q^3, \frac{2(q^2-1)(q+1)}{3}; 1; 1, \frac{(q^2-1)(q+1)}{3}, q^3\right\}.$$

Proof. Let g be a nonidentity element of K. Then $\alpha_3(g) = v$ and, by Lemma 4, we have $\chi_1(g) = -\dfrac{m(k+1)}{m+n}$. Since K is a normal subgroup of a twice transitive permutation group H^F, K is transitive on F and $|K| = r$. From this, K is an elementary Abelian group of order r.

Let C be the kernel of the conjugation action of G on K. If $C = K$, then G/K acts transitively on nonidentity elements of K. Moreover, G/K has a twice transitive representation of degree $k + 1 > r$ on Σ, a contradiction to [1, Proposition 2.14]. Therefore, C contains T.

Assume that $T' \neq T$. Then $T = K \times N$ and, by Proposition 2, the group $N = T'$ acts transitively on the vertex set of Γ. Let us show that an action of N_F^F is isomorphic to K^F. Since K^F is a regular Abelian subgroup of S_F, it coincides with the centralizer in S_F. However, N_F^F is transitive on F and commutes with K. Therefore, an action of N_F^F is isomorphic to K^F. It follows from [1, Proposition 2.13] that r is an odd prime number and either $N = L_d(q)$, $k + 1 = \frac{q^d - 1}{q - 1}$, and r divides $(d, q - 1)$ or $N = U_3(q)$, $k + 1 = q^3 + 1$, and r divides $\frac{q + 1}{(3, q + 1)}$. In the former case, $d \geq 3$, since $r > 2$ and, for $E \in \Sigma - \{F\}$, the subgroup $G_a \cap G_E$ is transitive on E; a contradiction to the fact that a is adjacent to a single vertex of E. In the latter case, N is transitive on the vertex set of the graph and twice transitive on Σ, while N_a is a single subgroup of index r in N_F (if $r = 3$, then 9 divides $q + 1$). Furthermore, N_a is transitive on $[a]$ and on $F - \{a\}$. Thus, Γ is the graph from Example 3.4 or 3.5 in [1]. This is a contradiction to the fact that, by [1, Theorem 4.3], the socle of the automorphism group of Γ is the group $U_3(q)$.

Thus, K is contained in the Schur multiplier of T/K. Since any prime divisor of r divides $k + 1$, it follows from [1, Lemma 2.11] that $(T/K, k + 1, r)$ is one of the following triplets: $(A_6, 6, 3)$, $(A_7, 15, 3)$, $(L_3(4), 21, 3)$, $(M_{22}, 22, 4)$, $(U_3(q), q^3 + 1, 3)$ (3 divides $q + 1$), $\left(L_m(q), \frac{(q^m - 1)}{q - 1}, r\right)$ (r is a prime number dividing $q - 1$).

In the cases $(A_6, 6, 3)$ and $(A_7, 15, 3)$, we obtain the unique array $\{14, 12, 1; 1, 6, 14\}$ with integer eigenvalues $n = 2$ and $m = -7$, but the multiplicity of 2 is equal to $\frac{15 \cdot 2 \cdot 7}{9}$, a contradiction. In the case $(L_3(4), 21, 3)$, the only admissible array is $\{20, 18, 1; 1, 9, 20\}$ with $n = 1$ and $m = -7$, but, by [2, Theorem 1.2.3], we have $k \geq 2\lambda + 3 - \mu$, a contradiction. In the case $(M_{22}, 22, 4)$, there are no admissible arrays. The case of $L_m(q)$ with $m \geq 3$ leads to a contradiction as before. In the case $(L_2(q), q + 1, r)$, we have $q = 9$, $r = 3$, and there are no admissible arrays. Finally, in the case $(U_3(q), q^3 + 1, 3)$, we have $T = SU_3(q)$, and $T_F = T_a \times K$ is the extension of a subgroup of order q^3 by a cyclic group of order $q^2 - 1$. Therefore, $q + 1$ is congruent to ± 3 modulo 9. Thus, Γ is the distance-transitive graph from Example 3.5 in [1].

Lemma 7. *If $K = 1$, then one of the following assertions holds:*

(1) $T = L_2(q)$, $q = p^e$, r is an odd prime divisor of $q - 1$, T_F is the extension of a group of order q by a group of order $(q - 1)e$, and Γ is a distance-transitive graph with the intersection array $\left\{q, \frac{(r-1)(q-1)}{r}, 1; 1, \frac{q-1}{r}, q\right\}$.

(2) $T = U_3(q)$, r is an odd prime divisor of $q^2 - 1$, T_F is the extension of a group of order q^3 by a group of order $\frac{q^2 - 1}{(3, q + 1)}$, and Γ is a distance-transitive graph with the intersection array $\left\{q^3, \frac{(r-1)(q^2-1)(q+1)}{r}, 1; 1, \frac{(q^2-1)(q+1)}{r}, q^3\right\}$.

Proof. Let $K = 1$. Then T is a simple non-Abelian group, G acts twice transitively on Σ, and $G_F = H$ induces a twice transitive group of degree r on F. By [1, Proposition 2.12], one of eight possibilities can occur for the quadruplet (T, n, r, L) $(n = k + 1)$.

Let $T = L_d(q)$ and $k + 1 = \frac{(q^d - 1)}{q - 1}$. If $r = q^{d-1}$ and $d \geq 3$ (first possibility), then $k - (r - 1)\mu = \lambda + 1$ and $(q^{d-1} - 1)\left(\frac{q}{q - 1} - \mu\right) = \lambda + 1$, and $\mu = 1$, a contradiction.

Assume that r is an odd prime divisor of $q - 1$ (second possibility). If $d = 2$, then Γ is the distance-transitive graph from Example 3.4 in [1]. If $d \geq 3$, then, for $E \in \Sigma - \{F\}$, the group $H_a \cap G_E$ is transitive on E, which contradicts the fact that a is adjacent to a single vertex in E.

Let $T = L_3(3)$, $k + 1 = 13$, and $r = 3, 4$ (fourth possibility). Then there are no admissible arrays.

Let $T = L_3(5)$, $k + 1 = 31$, and $r = 5$ (fifth possibility). Then there is the only array $\{30, 24, 1; 1, 6, 30\}$ with integer eigenvalues $n = 5$ and $m = -6$, but the multiplicity of 5 is equal to $\frac{31 \cdot 4 \cdot 6}{11}$, a contradiction.

Let $T = L_3(8)$, $k + 1 = 73$, and $r = 28$ (sixth possibility). Then there are no admissible arrays.

Let $T = L_5(2)$, $k + 1 = 31$, and $r = 8$ (seventh possibility). Then there is the only array $\{30, 14, 1; 1, 2, 30\}$ with integer eigenvalues $n = 15$ and $m = -2$, but the multiplicity of 15 is $\frac{31 \cdot 7 \cdot 2}{17}$, a contradiction.

Let $T = L_7(2)$, $k + 1 = 127$, and $r = 63$ (eighth possibility). Then there are no admissible arrays.

Let $T = U_3(q)$, $k + 1 = q^3 + 1$, and r be an odd prime divisor of $q^2 - 1$ (third possibility). If $r \geq 5$ or $r = 3$ and r divides $q - 1$, then Γ is the distance-transitive graph from Example 3.4 or 3.5 in [1] (see also [6, Proposition 5.1]). Suppose that $r = 3$ and r divides $q + 1$. Then T is an index 3 subgroup of $PGU_3(q)$, G_a is an index 3 subgroup of G_F, and $G_a \cap T$ is an index 3 subgroup of $G_F \cap T$. It follows that $G \cap PGU_3(q) = U_3(q)$ and $q + 1$ is divided by 9. Applying the conjugation by an element from $P\Gamma U_3(q)$, we can assume that $G \leq P\Sigma U_3(q)$. Now Γ is the distance-transitive graph from Example 3.5 in [1]. Lemma 7, together with the theorem, is proved.

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