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On a Reconstruction Algorithm for the Trajectory and Control in a Delay System

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Abstract—We discuss a problem of the dynamic reconstruction of unmeasured coordinates of the phase vector and unknown controls in nonlinear vector equations with delay. A regularizing algorithm is proposed for the reconstruction of both controls and unmeasured coordinates simultaneously with the processes. The algorithm is stable with respect to information noises and computational errors.
Keywords: dynamic reconstruction, method of auxiliary models.

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1. INTRODUCTION AND PROBLEM STATEMENT

We consider a control system of the form

$$\dot{x}(t) = f_1(t, x_t(s), y_t(s)) + f_2(t, x_t(s), y_t(s))u(t), \tag{1.1}$$

$$\dot{y}(t) = \psi_1(t, x_t(s), y_t(s)) + \psi_2(t, y_t(s))x(t) \tag{1.2}$$

with initial conditions

$$x_{t_0}(s) = x_0(s) \in C([-\tau_m^x, 0]; \mathbb{R}^{n_1}), \qquad y_{t_0}(s) = y_0(s) \in C([-\tau_n^y, 0]; \mathbb{R}^{n_2}). \tag{1.3}$$

Here, t is time from a given interval $T = [t_0, \vartheta]$ ($t_0 < \vartheta < +\infty$); x and y are n_1- and n_2-dimensional vectors (which we assume to be columns), respectively, that characterize the state of the system; $u(t)$ is an r-dimensional vector of control; and $x_t(s)$ and $y_t(s)$ are the functions $x_t(s) = x(t+s)$ for $s \in [-\tau_m^x, 0]$ and $y_t(s) = y(t+s)$ for $s \in [-\tau_n^y, 0]$. The structure of the vector functions f_1 and ψ_1 and of the matrix functions f_2 and ψ_2 is clarified below.

Initial state (1.3) is assumed to be a Lipschitz function. In what follows, the symbol P denotes a fixed compact set in $\mathbb{R}^r$ (the control resources); $x_0(s)$ and $y_0(s)$ are known fixed functions. Any (Lebesgue) measurable function $u(\cdot)$ from the set $P(\cdot) = \{u(\cdot) \in L_2(T; \mathbb{R}^r)\colon u(t) \in P \text{ for a.a. } t \in T\}$ is called a control, and the solution $z(\cdot) = \{x(\cdot), y(\cdot)\}$ (in the sense of Carathéodory) of system of equations (1.1), (1.2) with initial condition (1.3) is called a motion of the system generated by the control $u(\cdot)$ (and starting from the initial state $\{x_0(s), y_0(s)\}$).

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Let $u(\cdot) \in P(\cdot)$ be a control realized on the time interval T, and let $z(\cdot) = \{x(\cdot), y(\cdot)\}$ be the motion generated by it. Assume that a part of the current state vector $\{x(\tau_i), y(\tau_i)\}$, namely, the vector $y(\tau_i)$, is measured during the process at sufficiently frequent times τ_i from T. The measurement results $\xi^h(\tau_i)$ are inaccurate; they satisfy the inequality

$$|\xi^h(\tau_i) - y(\tau_i)| \le h, \tag{1.4}$$

where h is small. Here, the symbol $|\cdot|$ denotes the Euclidean norm. The problem consists in constructing an algorithm for the reconstruction of the unmeasured component $x(\cdot)$ of the state vector and the control $u(\cdot)$ in real time from the current measurements $\xi^h(\tau_i)$. Since it is impossible to reconstruct their exact values (because $y(\cdot)$ is measured with error), we actually require that the algorithm form (in real time) some approximations $v^h(\cdot)$ and $u^h(\cdot)$ that are close to $x(\cdot)$ and $u(\cdot)$. More exactly, the standard deviation of $v^h(\cdot)$ from $x(\cdot)$

$$|v^h(\cdot) - x(\cdot)|^2_{L_2(T)} = \int\limits_{t_0}^{\vartheta} |v^h(t) - x(t)|^2\,dt \tag{1.5}$$

and the deviation of $v^h(\cdot)$ from $u(\cdot)$

$$|u(\cdot) - u^h(\cdot)|^2_{L_2(T)} = \int\limits_{t_0}^{\vartheta} |u(t) - u^h(t)|^2\,dt \tag{1.6}$$

must be arbitrarily small for sufficiently small measurement error h.

This problem belongs to the class of inverse problems of control system dynamics (the input is reconstructed from measurements of the output). A posteriori formulations of inverse problems were studied by many authors [1–5]. In [6], a method of dynamic (positional) reconstruction of the input in a finite-dimensional control-affine dynamic system was proposed. The method is based on the ideas from the theory of positional control [7, 8] and on the smoothing functional method and the residual method known in the theory of ill-posed problems [1]. For systems described by ordinary differential equations, this method was developed in [6, 15, 16]. The case of measuring all the coordinates of the state vector was considered in [6], and the case of measurements of type (1.4) was studied in [15, 16] under some special constraints on the dynamics of the system. In [9–14], the method was developed for various classes of delay systems.

Let elements of the vector function $f_1(\cdot)$ and of the matrix function $f_2(\cdot)$ have the form

$$g(t, x_t(s), y_t(s)) = g\big(t, x(t), x(t-\tau_1^x), \ldots, x(t-\tau_m^x), y(t), y(t-\tau_1^y), \ldots, y(t-\tau_n^y)\big),$$
$$0 < \tau_1^x < \tau_2^x < \ldots < \tau_m^x < +\infty, \quad 0 < \tau_1^y < \tau_2^y < \ldots < \tau_n^y < +\infty,$$
$$g(\cdot) = f_{1i}(\cdot) \;\text{ for }\; i \in [1:n_1], \qquad g(\cdot) = f_{2ij}(\cdot) \;\text{ for }\; i \in [1:n_1],\; j \in [1:r],$$

and satisfy the Lipschitz condition

$$\Big|g(t_1, x_0^{(1)}, x_1^{(1)}, \ldots, x_m^{(1)}, y_0^{(1)}, y_1^{(1)}, \ldots, y_n^{(1)}) - g(t_2, x_0^{(2)}, x_1^{(2)}, \ldots, x_m^{(2)}, y_0^{(2)}, y_1^{(2)}, \ldots, y_n^{(2)})\Big|$$
$$\le c_1\Big(|t_2 - t_1| + \sum_{i=0}^{m} |x_i^{(1)} - x_i^{(2)}| + \sum_{j=0}^{n} |y_j^{(1)} - y_j^{(2)}|\Big). \tag{1.7}$$

Let elements of the matrix function $\psi_2(\cdot)$ and of the vector-function $\psi_1(\cdot)$ have similar properties:

$$g_2(\cdot) = \psi_{2ij}(\cdot) \ \text{ for } \ i \in [1:n_2],\ j \in [1:n_1]; \qquad g_2(t, y_t(s)) = g_2(t, y(t), y(t-\tau_1^y), \ldots, y(t-\tau_n^y));$$

$$g_1(\cdot) = \psi_{1i}(\cdot) \ \text{ for } \ i \in [1:n_2],$$

$$g_1(t, x_t(s), y_t(s)) = g_1(t, x(t-\tau_1^x), \ldots, x(t-\tau_m^x), y(t), y(t-\tau_1^y), \ldots, y(t-\tau_n^y)).$$

Here, the elements $g_1(\cdot)$ and $g_2(\cdot)$ satisfy the conditions

$$\Big|g_2(t_1, y_0^{(1)}, y_1^{(1)}, \ldots, y_n^{(1)}) - g_2(t_2, y_0^{(1)}, y_1^{(2)}, \ldots, y_n^{(2)})\Big| \le C_1\Big(|t_2 - t_1| + \sum_{j=0}^{n} |y_j^{(1)} - y_j^{(2)}|\Big), \tag{1.8}$$

$$\Big|g_1\big(t_1, x_1^{(1)}, \ldots, x_m^{(1)}, y_0^{(1)}, y_1^{(1)}, \ldots, y_n^{(1)}\big) - g_1\big(t_2, x_1^{(2)}, \ldots, x_m^{(2)}, y_0^{(2)}, y_1^{(2)}, \ldots, y_n^{(2)}\big)\Big|$$

$$\le d_1\Big(|t_2 - t_1| + \sum_{i=1}^{m} |x_i^{(1)} - x_i^{(2)}| + \sum_{j=0}^{n} |y_j^{(1)} - y_j^{(2)}|\Big). \tag{1.9}$$

In (1.7)–(1.9) and below, the symbol $|\cdot|$ denotes the Euclidean norm, the corresponding matrix norm, and the absolute value of a number. Under the above conditions, for every initial state (1.3) and control $u(\cdot) \in P(\cdot)$, there exists a unique solution of system (1.1), (1.2). In what follows, we assume $\tau_m^x = \tau_n^y = \tau$ for simplicity.

Let us describe the scheme of the algorithm that solves the problem under consideration. Denote by $\xi^h(\cdot)$ a function $\xi^h(t)$, $t \in [t_0 - \tau, \vartheta]$, such that $\xi^h(t) = y_0(t - t_0)$ for $t \in [t_0 - \tau, t_0)$ and $\xi^h(t) = \xi^h(\tau_i)$ for $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : q-1]$, where $\tau_i = \tau_{h,i}$, $q = q_h$, and $\xi^h(\tau_i)$ satisfies (1.4). For given $h \in (0,1)$, fix the points

$$\tau_i = \tau_{h,i}, \quad i = 0, 1, \ldots, \quad q = q_h, \quad t_0 = \tau_0 < \tau_1 < \ldots < \tau_q = \vartheta, \tag{1.10}$$

of the time interval T. For simplicity, assume that $\tau_i - \tau_{i-1} = \delta = \delta(h)$. Thus, for every h, the uniform grid $\Delta_h = \{\tau_{h,i}\}_{i=0}^{q_h}$ with step $\delta = \delta(h)$ is chosen on T (see (1.10)). Next, we introduce a control system of the form

$$\dot{w}^h(t) = \rho(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s), u_i^h, w_{\tau_i}^h(s)), \tag{1.11}$$

$$\tau_i < t \le \tau_{i+1}, \quad i = 0, 1, \ldots, m-1, \quad w^h(t_0 + s) = w_0^h(s);$$

this system is called a model. Here, $w^h(t)$ is the finite-dimensional state vector of the model at time t; $w_0^h(s) = \{w_{0x}^h(s), w_{0y}^h(s)\} \in L_\infty([-\tau, 0]; \mathbb{R}^{n_1+n_2})$ is the initial state of the model; $w_{\tau_i}^h(s) = w^h(\tau_i + s) \in \mathbb{R}^{n_1+n_2}$ for $s \in [-\tau, 0]$; and $v_i^h \in \mathbb{R}^{n_1}$ and $u_i^h \in \mathbb{R}^r$ are the finite-dimensional feedback controls [7,8] in the model generated at time τ_i:

$$v_i^h = V(\tau_i, w_{\tau_i}^h(s), \xi_{\tau_i}^h(s)), \tag{1.12}$$

$$u_i^h = U(\tau_i, w_{\tau_i}^h(s), \xi_{\tau_i}^h(s), v_{\tau_i}^h(s)), \tag{1.13}$$

where $\tau_i = \tau_{h,i} \in \Delta_h$, $v_t^h(t_0 + s) = x(t_0 + s)$ for $s \in [-\tau, 0]$, and $v_t^h(s) = v^h(t + s)$ for $s \in [-\tau, 0]$, $t \ge t_0$. The functions $V(\cdot)$ and $U(\cdot)$ are called strategies. The model operates in "real time," i.e., synchronously with system (1.1), (1.2). In the process of its operation, piecewise constant controls $v^h(\cdot)$ and $u^h(\cdot)$ are formed:

$$v^h(t) = v_i^h, \quad u^h(t) = u_i^h, \quad \tau_i < t \le \tau_{i+1}, \quad i = 0, 1, \ldots, q-1.$$

They are the required approximations of the functions $x(\cdot)$ and $u(\cdot)$.

2. RECONSTRUCTION OF UNKNOWN COORDINATES

Let us describe the algorithm for the dynamic reconstruction of the unknown coordinate $x(\cdot)$. We will specify the rules for choosing the strategy V (1.12) and model (1.11). This will allow us to construct the function $v^h(\cdot)$, which approximates $x(\cdot)$ (see (1.5)).

Fix the value of measurement error $h \in (0,1)$ and the family of partitions Δ_h (1.10) of the interval T. Denote by $Z(T)$ the bundle of solutions of system (1.1), (1.2) with initial condition (1.3); i.e., $Z(T) = \{z(\cdot) = z(\cdot; t_0, z_0(s), u(\cdot))\colon u(\cdot) \in P(\cdot)\}$. Consider a model described by the equation

$$\dot{w}^{(1)}(t) = F_1(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s), w^{(1)}(\tau_i)), \tag{2.1}$$

$$F_1(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s), w^{(1)}(\tau_i)) = \psi_1(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s))$$
$$+ \psi_2(\tau_i, \xi^h_{\tau_i}(s))v^h_i + 2(\xi^h(\tau_i) - w^{(1)}(\tau_i)), \quad w^{(1)} \in \mathbb{R}^{n_2}, \quad t \in [\tau_i, \tau_{i+1}),$$

with initial condition $w^{(1)}_{t_0}(s) = y(t_0 + s)$, $s \in [-\tau, 0)$, $w^{(1)}(t_0) = \xi^h(t_0)$. The solution of this equation $w^{(1)}(\cdot) = w^{(1)}(\cdot; t_0, w^{(1)}_{t_0}(s), v^h(\cdot))$ is understood in the sense of Carathéodory.

Assume that $\Delta^{(j)} = [t_j, t_{j+1}]$, $t_j = t_0 + \tau_1^x j$, l is the integer part of the number τ/τ_1^x, $j_* = \max\{j\colon t_j < \vartheta\}$, and $g_j(h) = h^{(1/3)^j}$ for $j \in [1 : j_*]$. Below, we assume for simplicity that the partitions Δ_h are chosen so that $t_j \in \Delta_h$. The strategy V (1.12) for $\tau_i \in [t_j, t_{j+1}) \cap T$ is specified as follows:

$$V(\tau_i, w^{(1)}_{\tau_i}(s), \xi^h_{\tau_i}(s)) = V_j(\tau_i, w^{(1)}_{\tau_i}(s), \xi^h_{\tau_i}(s))$$
$$= \arg\min\Big\{2(l_i, \psi_2(\tau_i, \xi^h_{\tau_i}(s))v) + \alpha_j|v|^2\colon v \in S(A)\Big\}. \tag{2.2}$$

Here, α_j is a parameter, $j \in [0 : j_*]$, $S(A) \subset \mathbb{R}^{n_1}$ is a ball of radius $A = \sup\{|x(\cdot)|_{C(T;\mathbb{R}^{n_1})}\colon z(\cdot) = \{x(\cdot), y(\cdot)\} \in Z(T)\} < +\infty$ centered at zero, and $l_i = w^{(1)}(\tau_i) - \xi^h(\tau_i)$.

Assume that the following condition is satisfied.

Condition 1. Assume that $n_1 \le n_2$ and there exists $c_* > 0$ such that the matrix $\psi_2(t, y_t(s))$ has a minor of n_1th order with the following property: the $n_1 \times n_1$-matrix $\bar{\psi}_2(t) = \bar{\psi}_2(t, y_t(s))$ corresponding to this minor satisfies the condition $|\bar{\psi}_2(t)x| \ge c_*|x|$ for all $t \in T$ and $x \in \mathbb{R}^{n_1}$.

Choose the parameter α_j as follows:

$$\alpha_0 = Ch^{2/3}, \quad \alpha_j = Cg_j^{2/3}(h), \quad j \ge 1, \quad C = \text{const} > 0. \tag{2.3}$$

Theorem 1. *Let $\delta = \delta(h) \le h$. Then,*

$$|v^h(\cdot) - x(\cdot)|^2_{L_2(\Delta^{(j-1)};\mathbb{R}^{n_1})} \le c_j g_j(h), \quad j \in [1 : j_*].$$

Before starting to prove the theorem, we give auxiliary statements. Theorem 1 will follow from Lemma 5. Consider the two systems

$$\dot{p}(t) = f_1(t) + f_2(t)u_1(t), \qquad \dot{q}(t) = F_1(t) + F_2(t)u_2(t), \qquad t \in T,$$

where $p(t), q(t) \in \mathbb{R}^n$, $f_1(\cdot), F_1(\cdot) \in L_2(T;\mathbb{R}^n)$, $f_2(\cdot), F_2(\cdot) \in L_2(T;\mathbb{R}^{n\times r})$, $u_1(\cdot), u_2(\cdot) \in L_2(T;\mathbb{R}^r)$, and $|u_p(\cdot)|_{L_\infty(T;\mathbb{R}^r)} \le K$ for $p = 1, 2$.

Introduce the notation: $\Delta_*^{(j)} = [t_j^*, t_{j+1}^*] \cap T$ and $t_j^* = t_0 + \tau_* j$ for $j \in [0 : j_0]$, $\Delta^{(-1)} = [t_0 - \tau_*, t_0]$, $\tau_* = \text{const} \in (0, \vartheta - t_0)$, and $j_0 = \max\{j\colon t_j^* \le \vartheta\}$. Assume that $r \le n$ and there exists $c > 0$ such that the matrix $f_2(t)$ has a minor of rth order with the following property: the $r \times r$-matrix $\bar{f}_2(t)$ corresponding to this minor is such that $|\bar{f}_2(t)u| \geq c|u|$ for all $t \in T$ and $u \in \mathbb{R}^r$.

Lemma 1. *Suppose that $t \to (\bar{f}_2(t))^{-1}u_1(t)$ is a function of bounded variation on T,*

$$|f_1(\cdot) - F_1(\cdot)|^2_{L_2(\Delta_*^{(j)};\mathbb{R}^n)} \leq a_1^{(j)}, \tag{2.4}$$

$$|f_2(\cdot) - F_2(\cdot)|^2_{L_2(\Delta_*^{(j)};\mathbb{R}^{n\times r})} \leq a_2^{(j)}, \tag{2.5}$$

$$|p(t_j^*) - q(t_j^*)|^2 \leq a_4^{(j)}, \tag{2.6}$$

$$|p(t) - q(t)|^2 + \tilde{\alpha}_j \int\limits_{t_j^*}^{t} \{|u_2(\nu)|^2 - |u_1(\nu)|^2\}\, d\nu \leq a_3^{(j)}, \tag{2.7}$$

where $t \in [t_j^*, t_{j+1}^*]$ and $\tilde{\alpha}_j = \text{const} \in (0, +\infty)$. Then,

$$\mu^{(j)} \equiv |u_1(\cdot) - u_2(\cdot)|^2_{L_2(\Delta_*^{(j)};\mathbb{R}^r)} \leq K_j \Big\{ \sum_{l=1}^{4} (a_l^{(j)})^{1/2} + \tilde{\alpha}_j^{1/2} \Big\} + a_3^{(j)}/\tilde{\alpha}_j.$$

Proof. Let $t \in \Delta_*^{(j)}$. Then, by (2.4)–(2.7), we have the estimate

$$\Big| \int\limits_{t_j^*}^{t} \bar{f}_2(\nu)\{u_1(\nu) - u_2(\nu)\}\, d\nu \Big| \leq \Big| \int\limits_{t_j^*}^{t} f_2(\nu)\{u_1(\nu) - u_2(\nu)\}\, d\nu \Big|$$

$$= \Big| \int\limits_{t_j^*}^{t} \{\dot{p}(\nu) - f_1(\nu) - f_2(\nu)u_2(\nu)\}\, d\nu \Big| = \Big| \int\limits_{t_j^*}^{t} \{\dot{p}(\nu) - \dot{q}(\nu) + F_1(\nu) - f_1(\nu) + (F_2(\nu) - f_2(\nu))u_2(\nu)\}\, d\nu \Big|$$

$$\leq (a_3^{(j)} + 2\tau_* K^2 \tilde{\alpha}_j)^{1/2} + (a_4^{(j)})^{1/2} + \tau_*^{1/2}\big\{(a_1^{(j)})^{1/2} + K(a_2^{(j)})^{1/2}\big\}. \tag{2.8}$$

Using relation (2.7), we derive the inequality

$$\mu^{(j)} = |u_1(\cdot)|^2_{L_2(\Delta_*^{(j)};\mathbb{R}^r)} - 2(u_1(\cdot), u_2(\cdot))_{L_2(\Delta_*^{(j)};\mathbb{R}^r)}$$

$$+ |u_2(\cdot)|^2_{L_2(\Delta_*^{(j)};\mathbb{R}^r)} \leq 2|u_1(\cdot)|^2_{L_2(\Delta_*^{(j)};\mathbb{R}^r)} - 2(u_1(\cdot), u_2(\cdot))_{L_2(\Delta_*^{(j)};\mathbb{R}^r)} + a_3^{(j)}/\tilde{\alpha}_j.$$

Hence,

$$\mu^{(j)} \leq 2 \int\limits_{t_j^*}^{t_{j+1}^*} (u_1(\nu) - u_2(\nu), u_1(\nu))_{\mathbb{R}^r}\, d\nu + a_3^{(j)}/\tilde{\alpha}_j$$

$$= 2 \int\limits_{t_j^*}^{t_{j+1}^*} (\bar{f}_2(\nu)(u_1(\nu) - u_2(\nu)), \bar{f}_2^{-1}(\nu)u_1(\nu))_{\mathbb{R}^r}\, d\nu + a_3^{(j)}/\tilde{\alpha}_j. \tag{2.9}$$

Therefore, in view of (2.8), (2.9), and the results of [17], we obtain

$$\mu^{(j)} \le \Big\{(a_3^{(j)} + 2\tau_* K^2\tilde{\alpha}_j)^{1/2} + (a_4^{(j)})^{1/2} + \tau_*^{1/2}((a_1^{(j)})^{1/2} + (a_2^{(j)})^{1/2})\Big\}$$

$$\times\bigg(\sup_{t\in\Delta_*^{(j)}} |\bar{f}_2^{-1}(t)u_1(t)| + \operatorname{var}_{\Delta_*^{(j)}}(\bar{f}_2^{-1}(\cdot)u_1(\cdot))\bigg) + a_3^{(j)}/\tilde{\alpha}_j \le K_j\bigg\{\sum_{l=1}^{4}(a_l^{(j)})^{1/2} + \tilde{\alpha}_j^{1/2}\bigg\} + a_3^{(j)}/\tilde{\alpha}_j.$$

The lemma is proved.

Lemma 2. *The bundle of solutions $Z(T)$ of system* (1.1), (1.2) *is bounded in the space* $W^{1,\infty}(T;\mathbb{R}^{n_1+n_2}) = \{z(\cdot)\in L_2(T;\mathbb{R}^{n_1+n_2})\colon \dot{z}(\cdot)\in L_\infty(T;\mathbb{R}^{n_1+n_2})\}$.

Lemma 3. *The bundle of solutions of system* (2.1) *is bounded in the space* $W^{1,\infty}(T;\mathbb{R}^{n_2})$.

The validity of Lemmas 2 and 3 is easily verified by using conditions (1.3), (1.7)–(1.9). Define

$$\lambda_j(t, x(\cdot), y(\cdot), w^{(1)}(\cdot), v^h(\cdot)) = \varepsilon(t) + \alpha_j\int_{t_j}^{t}\{|v^h(\nu)|^2 - |x(\nu)|^2\}\,d\nu,$$

$$\varepsilon(t) = |y(t) - w^{(1)}(t)|^2, \quad j\in[0:j_*], \quad t\in T.$$

Lemma 4. *Strategy* (2.2) *provides the inequality*

$$\lambda_j(t, x(\cdot), y(\cdot), w^{(1)}(\cdot), v^h(\cdot)) \le b_j, \quad t\in\Delta^{(j)}\cap T, \quad j\in[0:j_*],$$

where

$$b_j = |y(t_j) - w^{(1)}(t_j)|^2 + c_j^{(1)}(h+\delta) + c_j^{(2)}\sum_{k=j-l}^{j}\nu^{(k)}, \qquad \nu^{(j)} = |v^h(\cdot) - x(\cdot)|^2_{L_2(\Delta^{(j-1)};\mathbb{R}^{n_1})},$$

$v^h(t) = x(t)$ *for* $t\in[t_0-\tau, t_0]$, $v^h(t) = x_0(-\tau)$ *for* $t\in[t_0-\tau-\tau_1^x, t_0-\tau)$, *and the constants* $c_j^{(1)}$ *and* $c_j^{(2)}$ *can be written explicitly.*

Proof. Let us estimate the value

$$\varepsilon_j(t) = \varepsilon(t) + \alpha_j\int_{t_j}^{t}\big\{|v^h(\nu)|^2 - |x(\nu)|^2\big\}\,d\nu, \quad t\in\Delta^{(j)}\cap T.$$

Fix $\tau_i\in\Delta^{(j)}$. Then, for $t\in\Delta^{(j)}\cap\delta_i = [\tau_i, \tau_{i+1}]$, we have

$$\varepsilon_j(t) \le \varepsilon_j(\tau_i) + \sum_{j=1}^{4}\Lambda_{ji}(t), \tag{2.10}$$

where

$$\Lambda_{1i}(t) = 2\bigg(s_i, \int_{\tau_i}^{t}\Big\{\psi_1(\nu, x_\nu(s), y_\nu(s)) - \psi_1(\tau_i, v_\nu^h(s), \xi_{\tau_i}^h(s))\Big\}\,d\nu\bigg), \quad s_i = y(\tau_i) - w^{(1)}(\tau_i),$$

$$\Lambda_{2i}(t) = 2\bigg(s_i, \int_{\tau_i}^{t}\Big\{\psi_2(\nu, y_\nu(s))x(\nu) - \psi_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h\Big\}\,d\nu\bigg) + \alpha_j\int_{\tau_i}^{t}\{|v^h(\nu)|^2 - |x(\nu)|^2\}\,d\tau,$$

$$\Lambda_{3i}(t) = -2(t-\tau_i)(s_i, \xi^h(\tau_i) - w^{(1)}(\tau_i)), \qquad \Lambda_{4i}(t) = (t-\tau_i)\int\limits_{\tau_i}^{t} |\dot{w}^{(1)}(\tau) - \dot{y}(\tau)|^2\, d\tau.$$

By Lemmas 2 and 3,

$$\Lambda_{4i}(t) \le K_*^{(j)}(t-\tau_i)^2, \quad t \in \delta_i. \tag{2.11}$$

Note that $v^h(\tau_i + s) = v^h(t+s)$ for $s \ge t_0 - \tau_i$ and $t \in [\tau_i, \tau_{i+1}]$; in addition,

$$|\xi^h(\tau_i + s) - y(t+s)| \le K_*(h + t - \tau_i) \quad \text{for} \quad \tau_i + s \ge t_0 - \tau. \tag{2.12}$$

Therefore, using Lemma 2, the Lipschitz property of the functions $x_0(s)$ and $y_0(s)$, and inequalities (1.9) and (2.12), we obtain the following relation for $t \in \delta_i$:

$$\int\limits_{\tau_i}^{t} \big|\psi_1(\nu, x_\nu(s), y_\nu(s)) - \psi_1(\tau_i, v_\nu^h(s), \xi_{\tau_i}^h(s))\big|\, d\nu$$

$$\le K_*^{(j)} \int\limits_{\tau_i}^{t} \Big\{(\nu - \tau_j) + \sum_{k=1}^{m} \big|x(\nu - \tau_k^x) - v^h(\nu - \tau_k^x)\big| + \sum_{k=0}^{n} \big|y(\nu - \tau_k^y) - \xi^h(\tau_i - \tau_k^y)\big|\Big\}\, d\nu$$

$$\le K_0^{(j)} \Big\{(t-\tau_i)^2 + \int\limits_{\tau_i}^{t} \sum_{k=0}^{n} \big|y(\nu - \tau_k^y) - \xi^h(\tau_i - \tau_k^y)\big|\, d\nu + \int\limits_{\tau_i}^{t} \Big(\sum_{k=1}^{m} \big|x(\nu - \tau_k^x) - v^h(\nu - \tau_k^x)\big|\Big)\, d\nu\Big\}$$

$$\le K_1^{(j)}(t-\tau_i)(h + t - \tau_i) + K_2^{(j)}(t-\tau_i)^{1/2} \sum_{k=1}^{m} \Bigg(\int\limits_{\tau_i - \tau_k^x}^{t-\tau_k^x} |x(\nu) - v^h(\nu)|^2\, d\nu\Bigg)^{1/2},$$

where $\tau_0^y = 0$. Thus, for $t \in \delta_i$, we have the estimate

$$\Lambda_{1i}(t) \le 2(t-\tau_i)|y(\tau_i) - w^{(1)}(\tau_i)|^2 + K_3^{(j)} \Bigg\{(t-\tau_i)(h + t - \tau_i)^2 + \sum_{k=1}^{m} \int\limits_{\tau_i - \tau_k^x}^{t-\tau_k^x} |x(\nu) - v^h(\nu)|^2\, d\nu\Bigg\}. \tag{2.13}$$

Further, in view of (1.4), we conclude that

$$\Lambda_{3i}(t) \le -2(t-\tau_i)|y(\tau_i) - w^{(1)}(\tau_i)|^2 + K_4^{(j)} h (t-\tau_i), \quad t \in \delta_i. \tag{2.14}$$

Note that (1.4), (1.8), and (2.12) imply the inequalities

$$\big|\psi_2(\nu, y_\nu(s))x(\nu) - \psi_2(\tau_i, \xi_{\tau_i}^h(s))x(\nu)\big| \le \big|\psi_2(\nu, y_\nu(s)) - \psi_2(\tau_i, \xi_{\tau_i}^h(s))\big||x(\nu)| \le K_0(h + \nu - \tau_i)$$

for $\nu \in [\tau_i, \tau_{i+1}]$. Thus,

$$\Lambda_{2i}(t) \le K_5^{(j)}(t-\tau_i)(h+t-\tau_i) + \int\limits_{\tau_i}^{t} \Big\{2(l_i, \psi_2(\tau_i, \xi_{\tau_i}^h(s))\big\{v_i^h - x(\nu)\big\} + \alpha_j \big\{|v_i^h|^2 - |x(\nu)|^2\big\}\Big\}\, d\nu.$$

By the choice of the control v_i^h and strategy $V(\tau_i, w_{\tau_i}(s), \xi_{\tau_i}^h(s))$ (see (1.12) and (2.2)), we get

$$\Lambda_{2i}(t) \le K_5^{(j)}(t-\tau_i)(h+t-\tau_i). \tag{2.15}$$

Combining (2.10)–(2.15), we obtain for $t \in \Delta^{(j)} \cap \delta_i$

$$\varepsilon_j(t) \le \varepsilon_j(\tau_i) + K_6^{(j)}\delta(h+\delta) + K_3^{(j)} \sum_{k=1}^{m} \int\limits_{\tau_i-\tau_k^x}^{t-\tau_k^x} |x(\nu) - v^h(\nu)|^2\, d\nu.$$

Thus, for $t \in \Delta^{(j)} = [t_j, t_{j+1}]$,

$$\varepsilon_j(t) \le \varepsilon_j(t_j) + K_7^{(j)}(h+\delta) + K_8^{(j)} \sum_{k=1}^{m} \int\limits_{t_j-\tau_k^x}^{t-\tau_k^x} |x(\nu) - v^h(\nu)|^2\, d\nu.$$

Hence,

$$\varepsilon_j(t) \le \varepsilon_j(t_j) + K_7^{(j)}(h+\delta) + K_9^{(j)} \int\limits_{t_j-\tau}^{t_{j+1}-\tau_1^x} |x(\nu) - v^h(\nu)|^2\, d\nu.$$

Note that $\tau = l\tau_1^x + \gamma$ and $\gamma \ge 0$. Therefore, $t_{j+1} - \tau_1^x = t_j$ and $t_{j-l-1} \le t_j - \tau \le t_{j-l}$. Thus, for $t \in \Delta^{(j)}$, we have

$$\varepsilon_j(t) \le \varepsilon_j(t_j) + K_7^{(j)}(h+\delta) + K_9^{(j)} \int\limits_{t_{j-l-1}}^{t_j} |x(\nu) - v^h(\nu)|^2\, d\nu = \varepsilon_j(t_j) + K_7^{(j)}(h+\delta) + K_9^{(j)} \sum_{k=j-l}^{j} \nu^{(k)}.$$

Here, the constants $K_k^{(j)}$, $k \in [0:9]$, can be specified explicitly. Thus, we can assume $c_j^{(1)} = K_7^{(j)}$, $c_j^{(2)} = K_9^{(j)}$. The lemma is proved.

Lemma 5. *Suppose that $\delta \le h$ and the values α_j are specified according to* (2.3). *Then,*

$$\nu^{(j)} \le c_j g_j(h), \tag{2.16}$$

$$b_j \le c_j^{(0)} g_j(h). \tag{2.17}$$

Proof. For simplicity, let $t_{j_*+1} = \vartheta$. By Lemma 4, for $t \in \Delta^{(j)}$, we have

$$|y(t) - w^{(1)}(t)| = \varepsilon^{1/2}(t) \le \Big(\lambda_j(t, x(\cdot), y(\cdot), w^{(1)}(\cdot), v^h(\cdot)) + \alpha_j \int\limits_{t_j}^{t} \{|v^h(\nu)|^2 + |x(\nu)|^2\}\, d\nu\Big)^{1/2} \le \big(b_j + \alpha_j \rho_A\big)^{1/2}, \tag{2.18}$$

where $\rho_A = 2\tau_* d^2(A)$, $d(A) = \sup\{|u|\colon u \in S(A)\}$. Since $t_j \in \Delta_h$, it follows that, for each $j \in [0:j_*]$, there exists $i = i_j(h)$ such that $t_j = \tau_{i_j(h)}$. Define $\varrho_j \equiv |f_1(\cdot) - F_1(\cdot)|^2_{L_2(\Delta^{(j)};\mathbb{R}^{n_2})}$. Then, in view of Lemma 2, (1.9), and (2.12),

$$\varrho_j \le d_j^{(1)} \sum_{i=i_j(h)}^{i=i_{j+1}(h)-1} \int\limits_{\tau_i}^{\tau_{i+1}} \Big\{\delta^2 + h^2 + \gamma^h(\nu) + \gamma_i^h(\nu) + |\xi^h(\tau_i) - w^{(1)}(\tau_i)|^2\Big\}\, d\nu,$$

where

$$\gamma^h(\nu) = \sum_{k=1}^{m} \left|x(\nu - \tau_k^x) - v^h(\nu - \tau_k^x)\right|^2, \quad \gamma_i^h(\nu) = \sum_{k=0}^{n} \left|y(\nu - \tau_k^y) - \xi^h(\tau_i - \tau_k^y)\right|^2.$$

Note that the following inequalities hold:

$$\int_{t_j}^{t_{j+1}} \gamma^h(\nu)\,d\nu \le d_j^{(2)} \int_{t_j-\tau}^{t_j} |x(\nu) - v^h(\nu)|^2\,d\nu \le d_j^{(2)} \int_{t_{j-l-1}}^{t_j} |x(\nu) - v^h(\nu)|^2\,d\nu = d_j^{(2)} \sum_{k=j-l}^{j} \nu^{(k)}, \quad (2.19)$$

$$\int_{t_j}^{t_{j+1}} \gamma_i^h(\nu)\,d\nu \le d_j^{(3)}(h^2 + \delta^2). \quad (2.20)$$

In addition,

$$\nu^{(k)} = 0 \quad \text{for} \quad k \in [-l : 0]. \quad (2.21)$$

Then, in view of (2.18)–(2.20), we obtain the estimates

$$\varrho_j \le d_j^{(5)}\left\{h^2 + \delta^2 + \sum_{k=j-l}^{j} \nu^{(k)} + b_j + \alpha_j\right\}, \quad j \in [0 : j_*]. \quad (2.22)$$

It is easy to see that

$$|f_2(\cdot) - F_2(\cdot)|^2_{L_2(\Delta^{(j)};\mathbb{R}^{n_2\times n_1})} \le d_j^{(5)}(h^2 + \delta^2), \quad j \in [0 : j_*]. \quad (2.23)$$

Here, the constants $d_j^{(1)}$–$d_j^{(5)}$ can be specified explicitly. By Lemma 4, (2.18), and (2.21), for $\delta \le h$, we obtain

$$\lambda_0(t, x(\cdot), y(\cdot), w^{(1)}(\cdot), v^h(\cdot)) \le b_0 \le c_0^* h, \quad t \in \Delta^{(0)}, \quad (2.24)$$

$$|y(t_1) - w^{(1)}(t_1)|^2 \le \rho_A \alpha_0 + c_0^* h \le c_* h^{2/3}. \quad (2.25)$$

Next, in view of (2.21)–(2.24), for $h \in (0, 1)$, we have

$$\varrho_0 \le d_0^{(1)}\{h^2 + \delta^2 + b_0 + h^{2/3}\} \le d_0^* h^{2/3}, \qquad |f_2(\cdot) - F_2(\cdot)|^2_{L_2(\Delta^{(0)};\mathbb{R}^{n_2\times n_1})} \le c_j^{(*)} h^2.$$

By Condition 1, we can use Lemma 1. Write $p = y$, $q = w^{(1)}$, $u_1 = x$, $u_2 = v^h$, $f_1(t) = \psi_1(t, x_t(s), y_t(s))$, $f_2(t) = \psi_2(t, y_t(s))$, $F_1(t) = \psi_1(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s)) + 2(\xi^h(\tau_i) - w^{(1)}(\tau_i))$, and $F_2(t) = \psi_2(\tau_i, \xi^h_{\tau_i}(s))$ for $t \in [\tau_i, \tau_{i+1})$. Then, taking $a_1^{(0)} = d_0^* h^{2/3}$, $a_2^{(0)} = c_j^{(*)} h^2$, $a_3^{(0)} = c_0^* h$, $a_4^{(0)} = c_* h^{2/3}$, and $\tilde{\alpha}_0 = \alpha_0 = c h^{2/3}$, we get

$$\nu^{(1)} = |x(\cdot) - v^h(\cdot)|^2_{L_2(\Delta^{(0)};\mathbb{R}^{n_1})} \le \tilde{c}_1 h^{1/3} = c_1 g_1(h); \quad (2.26)$$

i.e., inequality (2.16) is valid for $j = 1$. Further, in view of (2.25) and (2.26), we derive

$$b_1 = |y(t_1) - w^{(1)}(t_1)|^2 + c_1^{(1)}(h + \delta) + c_1^{(2)} \sum_{k=1-l}^{1} \nu^{(k)} \le \tilde{c}_1^{(0)} h^{1/3} = c_1^{(0)} g_1(h). \quad (2.27)$$

Inequality (2.17) for $j = 1$ is also established. From (2.18), we get the inequalities

$$|y(t_j) - w^{(1)}(t_j)|^2 \le b_{j-1} + \rho_A \alpha_{j-1}, \quad j \in [1 : j_* - 1]. \tag{2.28}$$

Consequently, using (2.28) and the rule for choosing b_j, we obtain

$$b_j \le b_{j-1} + \rho_A \alpha_{j-1} + c_j^{(1)}(h + \delta) + c_j^{(2)} \sum_{k=j-l}^{j} \nu^{(k)} \le b_{j-1} + d_j(h + \alpha_{j-1} + \sum_{k=j-l}^{j} \nu^{(k)}), \tag{2.29}$$

where $d_j = \text{const} \in (0, +\infty)$. For $j \ge 1$, write in Lemma 1 $a_1^{(j)} = d_j^{(4)}\{h^2 + \delta^2 + \sum_{k=j-l}^{j} \nu^{(k)} + a_3^{(j)} + \alpha_j\}$, $a_3^{(j)} = b_j$, $a_2^{(j)} = d_j^{(5)}(h^2 + \delta^2)$, and $a_4^{(j)} = b_{j-1} + \rho_A \alpha_{j-1}$ for $j \in [1 : j_*]$. (In the choice of the values $a_i^{(j)}$, we have used Lemma 4 and inequalities (2.23), (2.24), and (2.28).) Then, from this lemma and (2.29), we obtain

$$\nu^{(j+1)} \le c^{(j)} \Big\{ h^{1/2} + \Big(\sum_{k=j-l}^{j} \nu^{(k)} \Big)^{1/2} + b_{j-1}^{1/2} + \alpha_{j-1}^{1/2} + \alpha_j^{1/2} \Big\} + b_j \alpha_j^{-1}, \quad j \in [1 : j_*]. \tag{2.30}$$

Now, we can prove (2.16) and (2.17) by induction. For $j = 1$, inequalities (2.16) and (2.17) are true (see (2.26), (2.27)). Setting $j = 1$ in (2.30), we find that (2.16) is valid for $j = 2$. This and (2.29) yield inequality (2.17) for $j = 2$. Assume that inequalities (2.16) and (2.17) hold for $j > 2$. By the relations $h \in (0, 1)$, $g_j(h) < g_{j+1}(h)$, and $g_{j+1}(h) = g_j^{1/3}(h)$, we have

$$\Big(\sum_{k=j-l}^{j} \nu^{(k)} \Big)^{1/2} \le l(\nu^{(j)})^{1/2} \le l g_j^{1/3}(h) = l g_{j+1}(h).$$

Hence, in view of (2.30) and the inequalities $b_{j-1} \le c_{j-1}^{(0)} g_{j-1}(h) \le c_{j-1}^{(0)} g_{j+1}(h)$ and $\alpha_j < \alpha_{j+1}$, we have

$$\nu^{(j+1)} \le c_{j+1}\{g_{j+1}(h) + \alpha_j^{1/2}(h)\} + g_j(h)\alpha_j^{-1}(h). \tag{2.31}$$

Using the equality $g_j(h)\alpha_j^{-1}(h) = 1/C\, g_j^{1/3}(h)$ (see (2.3)), we derive (2.16) from (2.31). Inequality (2.17) is derived similarly from (2.29) and (2.16). The lemma is proved.

3. RECONSTRUCTION OF UNKNOWN CONTROLS

Let us describe the algorithm for the dynamic reconstruction of the unknown input $u(\cdot)$. We will specify the rules for choosing the strategy U (1.13) and the model. This will allow us to construct the function $u^h(\cdot)$, which approximates $u(\cdot)$ (see (1.6)).

Fix the value of measurement error $h \in (0, 1)$ and the family of partitions Δ_h (1.10) of the interval T. Consider a model described by the equation

$$\dot{w}^{(0)}(t) = F_0(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s), u_i^h),$$
$$\dot{w}^{(1)}(t) = F_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s), w^{(1)}(\tau_i)), \tag{3.1}$$

where $F_0(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s), u_i^h) = f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + f_2(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s))u_i^h$, $w^{(0)} \in \mathbb{R}^{n_1}$, and $t \in [\tau_i, \tau_{i+1})$, with initial condition $w_{t_0}^{(0)}(s) = x(t_0 + s)$ for $s \in [-\tau, 0]$, $w_{t_0}^{(1)}(s) = y(t_0 + s)$ for $s \in [-\tau, 0)$, and $w^{(1)}(t_0) = \xi^h(t_0)$. The solution $w^h(\cdot) = \{w^{(0)}(\cdot), w^{(1)}(\cdot)\} = w^h(\cdot; t_0, w^h_{t_0}(s), v^h(\cdot), u^h(\cdot))$ of this equation is also understood in the sense of Carathéodory.

Thus, in the equation of the model (see (1.11)),

$$\rho(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s), u^h_i, w^{(1)}(\tau_i)) = \left\{ \begin{array}{c} F_0(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s), u^h_i) \\ F_1(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s), w^{(1)}(\tau_i)) \end{array} \right\}.$$

The strategy U (1.13) is specified as follows:

$$U(\tau_i, w^{(0)}_{\tau_i}(s), \xi^h_{\tau_i}(s), v^h_{\tau_i}(s)) = \arg\min \Big\{ 2(l^{(1)}_i, f_2(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s))u) + \alpha^{(1)}|u|^2 \colon u \in P \Big\}. \tag{3.2}$$

Here, $\alpha^{(1)} = \alpha^{(1)}(h) \colon (0,1) \to R_+$ is some function and $l^{(1)}_i = w^{(0)}(\tau_i) - v^h_i$.

Let $U(y(\cdot))$ denote the set of all controls $u(\cdot) \in P(\cdot)$ corresponding to the output $y(\cdot)$. It is easy to verify that this set is convex, bounded, and closed in $L_2(T; \mathbb{R}^r)$. Therefore, there exists an element

$$u_*(\cdot) = u_*(\cdot; y(\cdot)) = \arg\min \big\{ |u(\cdot)|_{L_2(T;\mathbb{R}^r)} \colon u(\cdot) \in U(y(\cdot)) \big\}.$$

Theorem 2. *Suppose that* $\delta = \delta(h) \le h$, $\alpha^{(1)}(h) \to 0$, *and* $g^{1/2}_{j_*}(h)/\alpha^{(1)}(h) \to 0$ *as* $h \to 0$. *Then,*

$$u^h(\cdot) \to u_*(\cdot) \quad in \quad L_2(T; \mathbb{R}^r) \quad as \quad h \to 0.$$

Let us first prove two auxiliary statements.

Lemma 6. *The bundle of solutions of system* (3.1) *is bounded in the space* $W^{1,\infty}(T; \mathbb{R}^{n_1+n_2})$.

The validity of Lemma 6 is easily verified by using conditions (1.3), (1.4), and (1.7)–(1.9).

Introduce the value

$$\lambda(t) = \lambda(t, x(\cdot), w^{(0)}(\cdot), u_*(\cdot), u^h(\cdot)) = |x(t) - w^{(0)}(t)|^2 + \alpha^{(1)} \int_{t_0}^{t} \{|u^h(\nu)|^2 - |u_*(\nu)|^2\}\, d\nu.$$

Lemma 7. *Strategy* (3.2) *provides the inequality*

$$\lambda(t, x(\cdot), w^{(0)}(\cdot), u_*(\cdot), u^h(\cdot)) < |x(t_0) - w^{(0)}(t_0)|^2 + C^{(1)}(h + \delta) + C^{(2)} \int_{t_0}^{t} |x(\nu) - v^h(\nu)|\, d\nu,$$

where the constants $C^{(1)}$ *and* $C^{(2)}$ *can be specified explicitly.*

Proof. Let us estimate $\lambda(t)$. Fix τ_i. Then, for $t \in \delta_i = [\tau_i, \tau_{i+1}]$,

$$\lambda(t) \le \lambda(\tau_i) + \sum_{j=1}^{3} \Lambda^0_{ji}(t), \tag{3.3}$$

where

$$\Lambda^0_{1i}(t) = 2\left(s^{(1)}_i, \int_{\tau_i}^{t} \{f_1(\nu, x_\nu(s), y_\nu(s)) - f_1(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s))\}\, d\nu \right), \quad s^{(1)}_i = x(\tau_i) - w^{(0)}(\tau_i),$$

$$\Lambda^0_{2i}(t) = 2\left(s_i^{(1)}, \int\limits_{\tau_i}^{t} \{ f_2(\nu, x_\nu(s), y_\nu(s))u_*(\nu) - f_2(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s))u_i^h \}\, d\nu \right)$$

$$+\, \alpha^{(1)} \int\limits_{\tau_i}^{t} \{ |u^h(\nu)|^2 - |u_*(\nu)|^2 \}\, d\nu, \qquad \Lambda^0_{3i}(t) = (t-\tau_i) \int\limits_{\tau_i}^{t} |\dot w^{(0)}(\tau) - \dot x(\tau)|^2\, d\tau.$$

By Lemmas 2 and 6,

$$\Lambda^0_{3i}(t) \le k_0 (t-\tau_i)^2, \quad t \in \delta_i, \tag{3.4}$$

$$|s_i^{(1)}| \le k_1. \tag{3.5}$$

Here and below, the constants k_j, $j = 0, 1, \ldots$, are independent of i and t. Since $x_0(s)$ and $y_0(s)$ are Lipschitz functions, we have, for $t \in \delta_i$, the relations

$$\int\limits_{\tau_i}^{t} \big| f_1(\nu, x_\nu(s), y_\nu(s)) - f_1(\tau_i, v^h_\nu(s), \xi^h_{\tau_i}(s)) \big|\, d\nu$$

$$\le k_2 \int\limits_{\tau_i}^{t} \left\{ (\nu - \tau_j) + \sum_{k=1}^{m} |x(\nu - \tau_k^x) - v^h(\nu - \tau_k^x)| + \sum_{k=0}^{n} |y(\nu - \tau_k^y) - \xi^h(\tau_i - \tau_k^y)| \right\} d\nu$$

$$\le k_3 (t-\tau_i)(h + (t-\tau_i)) + k_4 \int\limits_{\tau_i}^{t} \left(\sum_{k=0}^{m} |x(\nu - \tau_k^x) - v^h(\nu - \tau_k^x)| \right) d\nu, \quad \tau_0^x = 0.$$

Thus, by (3.5), for $t \in \delta_i$, we have the estimate

$$\Lambda^0_{1i}(t) \le k_5 \left\{ (t-\tau_i)(h + (t-\tau_i)) + \int\limits_{\tau_i}^{t} \left(\sum_{k=0}^{m} |x(\nu - \tau_k^x) - v^h(\nu - \tau_k^x)| \right) d\nu \right\}. \tag{3.6}$$

The following estimate is established for $t \in \delta_i$ similarly to (3.6):

$$\int\limits_{\tau_i}^{t} \big| f_2(\eta, x_\eta(s), y_\eta(s))u_*(\eta) - f_2(\tau_i, v^h_{\tau_i}(s), \xi^h_{\tau_i}(s))u_*(\eta^h) \big|\, d\eta$$

$$\le k_6 \left\{ (t-\tau_i)^2 + \int\limits_{\tau_i - \tau}^{t} |y(\tau) - \xi^h(\tau)|\, d\tau + \int\limits_{\tau_i}^{t} \left(\sum_{k=0}^{m} |x(\nu - \tau_k^x) - v^h(\nu - \tau_k^x)| \right) d\nu \right\}. \tag{3.7}$$

In addition, in view of Lemma 2 and inequality (1.4), for $t \in \delta_i$, we obtain

$$\int\limits_{\tau_i - \tau}^{t} |y(\tau) - \xi^h(\tau)|\, d\tau \le k_7 (h + \delta). \tag{3.8}$$

Therefore, by (3.7) and (3.8), we have for $t \in \delta_i$

$$\Lambda_{2i}^0(t) \le k_8(t-\tau_i)(h+\delta) + \int\limits_{\tau_i}^{t} \Big\{ 2(s_i^{(1)}, f_2(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s))\{u_*(\nu) - u_i^h\}$$

$$+ \alpha^{(1)}\{|u_i^h|^2 - |u_*(\nu)|^2\}\Big\}\, d\nu + k_9|s_i^{(1)}| \int\limits_{\tau_i}^{t} \Big(\sum_{k=0}^{m} \big|x(\nu - \tau_k^x) - v^h(\nu - \tau_k^x)\big| \Big)\, d\nu. \qquad (3.9)$$

Thus, using Lemma 2, we establish for $t \in \delta_i$ the estimate

$$\int\limits_{\tau_i}^{t} |l_i^{(1)} - s_i^{(1)}|\, d\nu = \int\limits_{\tau_i}^{t} |x(\tau_i) - v_i^h|\, d\nu \le \int\limits_{\tau_i}^{t} |x(\nu) - v^h(\nu)|\, d\nu + k_{10}(t-\tau_i). \qquad (3.10)$$

Since the control u_i^h and the strategy $U(\tau_i, w_{\tau_i}^h(s), \xi_{\tau_i}^h(s), v_{\tau_i}^h(s))$ were chosen according to rules (1.13) and (3.2), we obtain from (3.9) and (3.10) the following estimate for $t \in \delta_i$:

$$\Lambda_{2i}^0(t) \le k_{11}(t-\tau_i)(h+\delta) + k_{12} \int\limits_{\tau_i}^{t} \Big(\sum_{j=0}^{m} |x(\nu - \tau_j^x) - v^h(\nu - \tau_j^x)| \Big)\, d\nu. \qquad (3.11)$$

Combining estimates (3.3), (3.4), (3.6), and (3.11), we obtain for $t \in \delta_i$

$$\lambda(t) \le \lambda(\tau_i) + k_{13}\delta(h+\delta) + k_{14} \int\limits_{\tau_i}^{t} \Big(\sum_{j=0}^{m} |x(\nu - \tau_j^x) - v^h(\nu - \tau_j^x)| \Big)\, d\nu.$$

Thus, for $t \in T$,

$$\lambda(t) \le \lambda(t_0) + k_{15}(h+\delta) + k_{15} \int\limits_{t_0}^{t} \Big(\sum_{j=1}^{m} |x(\nu - \tau_j^x) - v^h(\nu - \tau_j^x)| \Big)\, d\nu$$

$$\le k_{15}(h+\delta) + k_{16} \int\limits_{t_0}^{t} |x(\nu) - v^h(\nu)|^2\, d\nu, \qquad (3.12)$$

because $v^h(t_0+s) = x(t_0+s)$ for $s \in [-\tau, 0]$. The statement of the lemma follows from (3.12). The lemma is proved.

Proof of Theorem 2. In view of Lemma 7, Theorem 1, and the equality $w^{(0)}(t_0) = x(t_0)$, we have

$$\lambda(t, x(\cdot), w^{(0)}(\cdot), u_*(\cdot), u^h(\cdot)) \le c^{(3)}\big(h + \delta(h) + g_{j_*}^{1/2}(h)\big).$$

Thus,

$$\sup_{t \in T} |x(t) - w^{(0)}(t)|^2 \le c^{(4)}\big(\alpha^{(1)}(h) + h + \delta(h) + g_{j_*}^{1/2}(h)\big),$$

$$\int\limits_{t_0}^{t} |u^h(\nu)|^2\, d\nu \le \int\limits_{t_0}^{t} |u_*(\nu)|^2\, d\nu + \big(h + \delta(h) + g_{j_*}^{1/2}(h)\big)/\alpha^{(1)}(h), \quad t \in T.$$

Further proof follows the standard scheme (see, for example, [6]). The theorem is proved.

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