Noise-Induced Excitability of
the Complex Liquid Flows

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Abstract

We study an excitability for the stochastically forced system modeling a dynamics of the complex liquid flows. A phenomenon of noise-induced generation of large-amplitude oscillations in a zone of stable equilibria is studied.

Keywords: Excitability, stochastic disturbances, complex liquid flows

Deterministic model

Consider a complex liquid flow bounded by two parallel planes \( z = 0, z = h \). The lower plane \( z = 0 \) is fixed and the unidirectional shear stress \( \Sigma \) is applied to the upper plane \( z = h \). Due to the symmetry, one can limit by one space variable \( z \). The physical state of the system is uniquely determined by the function of the viscous stress \( \sigma(t, z) \). This function in \([0, \infty) \times [0, h] \) satisfies the equation [1]

\[
\rho \frac{\partial}{\partial t} \left( f(\sigma) + G \frac{\partial \sigma}{\partial t} \right) = \frac{\partial^2 \sigma}{\partial z^2}
\]  

(1)

with boundary conditions

\[
\sigma(t, h) = \Sigma, \quad \frac{\partial \sigma}{\partial z} (t, 0) = 0.
\]  

(2)

Here, the function \( f(\sigma) \) reflects the nonlinear N-shaped character of the shear rate as a function of the stress (see Fig.1), \( \rho \) is a medium density, \( G \) is a relaxation parameter.
In this paper, for the distributed model of the flow stream (1)-(2) a three-layer discretization is used. For lines \( z = z_i \) \((z_0 = 0, z_1 = \frac{h}{2}, z_2 = h)\) consider corresponding approximations \( \sigma_i(t) \) for functions \( \sigma(t, z_i) \). Using formulas of numerical differentiation, one get the following approximation of the equation (1) on the line \( z = z_1 \)

\[
\rho \left( \frac{df(\sigma_1)}{d\sigma} \frac{d\sigma_1}{dt} + G \frac{d^2\sigma_1}{dt^2} \right) = 4 \frac{\sigma_0(t) - 2\sigma_1(t) + \sigma_2(t)}{h^2}. \tag{3}
\]

It follows from boundary conditions (2) that

\[
\sigma_2(t) = \Sigma, \quad \sigma_1(t) - \sigma_0(t) = 0. \tag{4}
\]

Equations (3),(4) imply

\[
G \rho \frac{d^2\sigma_1}{dt^2} + \rho f'(\sigma_1) \frac{d\sigma_1}{dt} = 4 \frac{\Sigma - \sigma_1}{h^2}. \tag{5}
\]

A solution \( \sigma_1(t) \equiv \Sigma \) of the equation (5) is a unique equilibrium. The equation (5) for the variables \( x = \sigma_1, \ y = \frac{d\sigma_1}{dt} \) can be rewritten as a system

\[
\dot{x} = y, \quad \dot{y} = -\frac{4}{G\rho h^2} x - \frac{f'(x)}{G} y + \frac{4}{G\rho h^2} \Sigma. \tag{6}
\]

For a study of the possible dynamical regimes of this system we fix parameters \( G = \rho = h = 1, \ f(x) = k(x^3/3 - x^2 + x/2) \). Here, \( f'(x) = k(x^2 - 3x + 2) \). The function \( f(x) \) models a characteristic type of N-shaped curve mentioned above, and the parameter \( k \) reflects a stiffness of this nonlinearity.

The equilibrium \( x = \Sigma, y = 0 \) of the system (6) is stable for \( 0 < \Sigma < 1 \) and \( \Sigma > 2 \). On the interval \( 1 < \Sigma < 2 \), this equilibrium is unstable and a stable limit cycle is observed. The extreme values of the variable \( x \) for attractors of the system (equilibria and cycles) on the interval \( 0 < \Sigma < 3 \) for \( k = 1, 20 \) are plotted. The instability of the equilibrium for \( \Sigma \in (1, 2) \) leads to the appearance of the large-amplitude auto-oscillations of the flow.
In this paper, we focus on the zone $\Sigma < 1$ close to the point $\Sigma_1 = 1$ of Andronov-Hopf bifurcation. In Fig.3, a phase portrait of the deterministic system (6) for $\Sigma = 0.9$ is plotted. Small deviations of the initial state from the equilibrium result in small-amplitude trajectories that correspond to subthreshold responses. If we take initial deviations larger than some threshold, large-amplitude trajectories appear that correspond to suprathreshold response. Around the equilibrium, one can find a set of initial points corresponding to the subthreshold response. A size of this subthreshold domain essentially depends on the parameter $\Sigma$. The closer $\Sigma$ to the bifurcation value, the less a size of this subthreshold domain.

Such non-uniformity of the phase portrait is a underlying reason of the stochastic excitability of the studied system. Noise-induced excitability was studied for various dynamical systems [2-5].

**Stochastic model**

For the analysis of stochastic effects, consider the randomly forced system

$$\dot{x} = y, \quad \dot{y} = -4x - k(x^2 - 3x + 2)y + 4\Sigma + \varepsilon \dot{w}(t),$$

where $w(t)$ is a standard Wiener process, $\varepsilon$ is a noise intensity.

In Figs.4-5, random trajectories and time series of the system (7) for two values of the noise intensity are plotted. For weak noise $\varepsilon = 0.1$, random trajectories leave the stable equilibrium and concentrate in the subthreshold zone.
Here, time series demonstrate small-amplitude stochastic oscillations near the equilibrium (see Fig.5a). As the noise intensity increases, random trajectories transit to the suprathreshold zone. As one can see in Fig.4b for $\varepsilon = 0.5$, the system exhibits stochastic oscillations of large amplitude. An intermittency of small- and large-amplitude oscillations is clearly seen in Fig.5b. So, this system is highly excitable to stochastic disturbances. This model exhibits a noise-induced stochastic cycle even when the deterministic system has a stable equilibrium only.

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References


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