The structure of non-nilpotent CTI-groups

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Abstract. A subgroup $H$ of a group $G$ is called a TI-subgroup if $H \cap H^g \notin \{1, H\}$, for all $g \in G$, and a group is called a CTI-group if all of its cyclic subgroups are TI-subgroups. In this paper, we determine the structure of non-nilpotent CTI-groups. Also we will show that if $G$ is a nilpotent CTI-group, then $G$ is either a Hamiltonian group or a non-abelian $p$-group.

1 Introduction and preliminaries

Throughout the following, $G$ always denotes a finite group.

Let $H$ be a subgroup of $G$. If for every $g \in G$ we have $H \cap H^g \notin \{1, H\}$, then $H$ is called a TI-subgroup. Now if every subgroup of $G$ is a TI-subgroup, then $G$ is called a TI-group, and $G$ is an ATI-group if all of its abelian subgroups are TI-subgroups. In [13], G. Walls classified the TI-groups. S. Li and X. Guo in [6] classified the ATI-groups of prime power order; also these authors with P. Flavell in [4] determined the structure of ATI-groups.

A subgroup $H$ of $G$ is called a QTI-subgroup if for every $1 \neq x \in H$, we have

$$C_G(x) \leq N_G(H).$$

A group $G$ is called a QTI-group if all of its subgroups are QTI-subgroups; correspondingly, $G$ is an AQTI-group if all its abelian subgroups are QTI-subgroups. It can be shown that any TI-subgroup is a QTI-subgroup, but the converse is not true. In [8], G. Qian and F. Tang classify AQTI-groups and prove that if $G$ is a $p$-group, then the properties of being TI, ATI and AQTI are equivalent in $G$.

Groups all of whose cyclic subgroups are TI-subgroups are called CTI-groups. Clearly, any ATI-group is a CTI-group; however, the converse is not true. In particular, the center of any non-nilpotent ATI-group is trivial, but this does not hold.
for CTI-groups. In this paper, we classify the CTI-groups with non-trivial center. Also we prove that these groups are necessarily solvable with elementary abelian center. Next, we determine the structure of solvable CTI-groups with trivial center, and show that the centralizers of their minimal normal subgroups are equal to the Fitting subgroup of the group. Also we prove that a CTI-group is solvable if and only if it has a solvable minimal normal subgroup. Finally we classify non-solvable CTI-groups.

Our notation is standard and can be found in [2] and [11]. Throughout this paper, $F(G)$ is the Fitting subgroup of $G$, $Z(G)$ is the center of $G$; also $Q_8$ and $S_4$ are the quaternion group of order 8, and the symmetric group of degree 4, respectively.

The following easy lemmas will be useful.

**Lemma 1.1.** Let $G$ be a CTI-group and $H$ be a subgroup of $G$. Then:

(i) $H$ is a CTI-group.

(ii) If $H$ is cyclic and $\text{Core}_G(H) \neq 1$, then $H \trianglelefteq G$.

**Lemma 1.2.** Let $G$ be a CTI-group and assume that $x, y \in G$ have coprime orders. If $[x, y] = 1$ and $\langle x \rangle \trianglelefteq G$, then $\langle y \rangle \trianglelefteq G$.

**Proof.** As $\langle x \rangle \leq \langle xy \rangle$, we have

\[ \text{Core}_G(\langle xy \rangle) \neq 1 \]

and so $\langle xy \rangle \trianglelefteq G$. Now since $\langle y \rangle$ is a characteristic subgroup of $\langle xy \rangle$, we have $\langle y \rangle \trianglelefteq G$. \qed

As an immediate corollary, we get:

**Corollary 1.3.** Let $G$ be a CTI-group with non-trivial center.

(i) Assume that the order of $1 \neq g \in G$ is coprime to the order of an element of $Z(G)$. Then $\langle g \rangle \trianglelefteq G$.

(ii) If two distinct primes $p$ and $q$ divide the order of $Z(G)$, then $G$ is a Hamiltonian group.

**Proof.** (i) This is trivial.

(ii) Let $x \in G$ be of prime order $r$. Then, we have $(r, p) = 1$ or $(r, q) = 1$. Therefore by (i), $\langle x \rangle \trianglelefteq G$, consequently any cyclic subgroup of $G$ and so any subgroup of $G$ is normal in $G$ (by Lemma 1.1 (ii)). \qed

The preceding corollary implies that a finite non-Hamiltonian nilpotent CTI-group is necessarily a non-abelian $p$-group.
2 CTI-groups with non-trivial center

In this section, we suppose that $G$ is a non-nilpotent CTI-group with non-trivial center.

**Theorem 2.1.** Let $G$ be a non-nilpotent CTI-group with non-trivial center. Then $Z(G)$ is an elementary abelian $p$-subgroup, where $p$ is the smallest prime divisor of $|G|$. In particular, any $p'$-subgroup of $G$ is normal.

**Proof.** Since $G$ is not a Hamiltonian group, it follows that $Z(G)$ is a $p$-subgroup (by Corollary 1.3 (ii)). Also Corollary 1.3 (i) implies that any $p'$-subgroup of $G$ is normal. Now it suffices to prove that every element of $Z(G)$ is of order $p$. Let $x \in Z(G)$ satisfy $x^{p^i} = 1$, where $i > 1$. Also assume that $\langle y \rangle \not\leq G$ is of order $p$. As $\langle x^p \rangle \leq \langle yx \rangle$, we have $\langle yx \rangle \leq G$. Therefore $\langle yx \rangle$ acts trivially on any $p'$-element $t$ of $G$, and this implies that $[t, y] = [t, xy] = 1$. Now since $\langle t \rangle \leq \langle yt \rangle$, it follows that $\langle yt \rangle \leq G$. Thus we conclude that $\langle y \rangle \leq G$ which contradicts our assumption.

Now let $q$ be the smallest prime divisor of $|G|$ and $q \neq p$. Let $y \in G$ be of order $q$. Then by Lemma 1.2, $\langle y \rangle \leq G$. Consequently, $y \in Z(G)$. Hence we get a contradiction and the proof is complete.

**Remark 2.2.** The preceding theorem states that a Hall $p'$-subgroup of any non-nilpotent CTI-group $G$ with non-trivial center is Hamiltonian and normal, so we can write $G = HP$, where $P \leq \text{Sy}_p(G)$ and $H$ is an abelian $p'$-subgroup, because $|H|$ is odd, since $p$ is the smallest prime divisor of $|G|$. Also we immediately see that any non-normal cyclic subgroup is necessarily a $p$-subgroup.

We continue to assume that $p$ is the smallest prime divisor of $|G|$.

**Proposition 2.3.** Let $G$ be a non-nilpotent CTI-group with non-trivial center. Then for every non-normal cyclic subgroup $K$ of $G$, $\mathcal{C}_G(K)$ is a $p$-subgroup. In particular, $\mathcal{C}_H(P) = 1$ and accordingly $H \leq G'$.

**Proof.** Let $K = \langle x \rangle$ and $y \in \mathcal{C}_G(x)$ be a $p'$-element. By Theorem 2.1, we have $\langle y \rangle \leq G$. Lemma 1.2 implies that $\langle x \rangle \leq G$ which contradicts our assumption. Therefore $\mathcal{C}_G(x)$ is a $p$-group and so we will have $\mathcal{C}_H(P) \leq \mathcal{C}_H(x) = 1$. Now the fundamental theorem of coprime actions implies that $H = [H, P]$ and hence $H \leq G'$.

**Theorem 2.4.** Let $G$ be a non-nilpotent CTI-group with non-trivial center and $p$ be the smallest prime divisor of $|G|$. If $G$ has no subgroups isomorphic to a dihedral group of 2-power order, then any cyclic $p'$-subgroup of order greater than $p$ is non-normal.
Proof. Let $\langle x \rangle \unlhd G$ be of order $p$ and let $y \in G$ satisfy $1 \neq y^p \in Z(G)$. If $p = 2$ and $(xy)^2 = 1$, then $y^x = y^{-1}$ and $\langle x, y \rangle$ is a dihedral group of 2-power order, which is contradiction. Thus

$$(xy)^p = y^p[y, x]^{\frac{p(p-1)}{2}},$$

since $[y, x] \in Z(G)$. Therefore $(xy)^p$ is a central element of $G$ and so $\langle xy \rangle \leq G$. Consequently, for any $p'$-element $t$, we have $[t, x] = [t, yx] = 1$ or $t \in C_G(x)$ and this is in contradiction to Proposition 2.3.

It follows from Theorem 2.4 that if a finite non-nilpotent CTI-group has no subgroups isomorphic to a dihedral group of 2-power order, then no power of any non-trivial element of its $p$-subgroups can be central.

We can now prove our main structural theorem:

**Theorem 2.5.** Let $G$ be a non-nilpotent CTI-group with non-trivial center and let $p$ divide $|Z(G)|$. Then $G$ possesses an abelian $p$-subgroup $K$ such that

$$P \cong K \rtimes \mathbb{Z}_{p^i}$$

and every subgroup of $K$ is normal in $G$. Also,

(i) if $p$ is odd or $P$ is an abelian subgroup, then

$$K = Z(G) \quad \text{and} \quad P = Z(G) \times \mathbb{Z}_{p^i},$$

also in this case $G' \cap Z(G) = 1$,

(ii) if $p = 2$ and $P$ is a non-abelian subgroup, then $i = 1$ and $P$ has a subgroup isomorphic to a dihedral group of 2-power order, moreover $G' \cap Z(G) \neq 1$,

(iii) $G' \cap Z(G) \neq 1$ if and only if $G$ possesses a subgroup isomorphic to a dihedral group of 2-power order.

Proof. Let $h \in H$ with $|h| = q \neq p$. Then $\langle h \rangle \unlhd G$ and $P$ acts on $\langle h \rangle$ by conjugation, so there exists a homomorphism $\varphi : P \longrightarrow \text{Aut}(\langle h \rangle)$.

Set $K := \ker \varphi$ and let $P/K = \langle x \rangle$. Then $P = \langle x, K \rangle$. Clearly $\langle x \rangle \not\unlhd G$, otherwise the action of $x$ on $h$ would be trivial. If for some $i$, $x^i \in K$ then we get $\langle x \rangle \unlhd G$ and this is a contradiction. Thus $\langle x \rangle \cap K = 1$ and $P = K \rtimes \langle x \rangle$. As every element of $K$ commutes with $h$, by applying Lemma 1.2, we conclude that every subgroup of $K$ is normal in $G$ and therefore $K$ is a Hamiltonian group. Also it is clear that $Z(G) = \Omega_1(K)$.

(i) Let $p$ be odd or $P$ be an abelian group. Then $G$ has no subgroup isomorphic to a dihedral groups of 2-power order. Thus Theorem 2.4 implies that any element of $K$ is of order $p$ and so $K = Z(G)$. Hence $P = Z(G) \times \mathbb{Z}_{p^i}$ and $G' = H$. Thus $G' \cap Z(G) = 1$. 


(ii) First, we note that for any \( y \in K \) and \( 1 \neq t \in \langle x \rangle \) we have \( \langle yt \rangle \not\subseteq G \); otherwise \( [h, t] = [h, yt] = 1 \) and \( t \in K \cap \langle x \rangle \), which is clearly a contradiction.

Let \( y \in \mathcal{C}_K(x) \). If \( |y| \neq 2 \), then \((yt)^2 = y^2\), whence \( t \in \langle x \rangle \) is a element of order 2. Therefore \( \langle yt \rangle \leq G \), a contradiction. Consequently, \( Z(G) = \mathcal{C}_K(x) \).

Since \( P \) is non-abelian, we have \( Z(G) \neq K \). Therefore, on assuming that \( y \in K \) is of order 4 we see that \( [y, x^2] = 1 \) (since the action of \( \langle x \rangle \) on \( \langle y \rangle \) is at most of order 2). Now, if \( |xy| = l \neq 2 \) then \( y^2 \in \langle yx^{\frac{1}{2}} \rangle \) and so \( \langle yx^{\frac{1}{2}} \rangle \leq G \). This is a contradiction; consequently, \( x^2 = 1 \).

Now let \( y \in K \) be an arbitrary element. Since \( y^x \in \langle y \rangle \), we have \( (yx)^2 \in K \).
So, if \( |yx| > 2 \), then we get \( \langle yx \rangle \leq G \), a contradiction. Thus we have \( |yx| = 2 \) and \( y^x = y^{-1} \), in other words, \( x \) inverts any element of \( K \). Hence \( \langle y, x \rangle \) is a dihedral group of 2-power order. So, \( Z((y, x)) \leq G' \cap Z(G) \).

If \( K \) were a non-abelian group, then \( Q_8 \leq K \), because \( K \) is a Hamiltonian group. Therefore \( K \) would contain two elements \( y \) and \( z \) of order 4 such that \( |yz| = 4 \) and \( y^2 = z^2 \). But in this case we would have
\[
(yz)^{-1} = (yz)^x = y^x z^x = y^{-1} z^{-1} = (zy)^{-1}.
\]
Thus \( [z, y] = 1 \) and so
\[
(zy)^2 = z^2 y^2 = z^4 = 1,
\]
a contradiction. Hence, \( K \) must be an abelian group.

(iii) First, let \( G' \cap Z(G) \neq 1 \). Then \( P \) is non-abelian. Therefore \( K \neq Z(G) \), and so by (ii), \( G \) has a subgroup isomorphic to \( D_{2l} \) for some \( l \).

Conversely, assume that \( P \) has a subgroup isomorphic to \( D_{2l} \). In this case, by (ii), \( K \) has an element \( y \) of order \( 2l^{-1} \), so \( y^{2l-2} \in Z(G) \) and also \( y^{2l-2} \in D_{2l} \). Hence, \( G' \cap Z(G) \neq 1 \).

\[ \square \]

**Corollary 2.6.** Let \( G \) be a non-nilpotent CTI-group such that \( Z(G) \neq 1 \). Also suppose that \( p \) divides \( |Z(G)| \) and let \( H \) be a Hall \( p' \)-subgroup of \( G \). Then \( H \) is abelian and normal, and moreover \( G = HP \) is solvable. Also,

(i) if \( Z(G) \cap G' = 1 \), then \( G \cong K \rtimes (H \rtimes \mathbb{Z}_{p^i}) \), where \( p \) is the smallest divisor of \( |G| \), \( K = Z(G) \), \( P = Z(G) \rtimes \mathbb{Z}_{p^i} \) and \( H = G' \),

(ii) if \( Z(G) \cap G' \neq 1 \), then \( p = 2 \) and \( P = K \rtimes \mathbb{Z}_2 \), where \( K \) is an abelian normal subgroup of \( G \); also \( Z(G) = \Omega_1(K) \), \( G' = H \cong \mathbb{Z}^1(K) \) and \( \mathbb{Z}_2 \) inverts any element of \( HK \),

(iii) the Fitting subgroup \( F(G) = HK \) is abelian.

**Lemma 2.7.** Let \( G \) be a non-nilpotent CTI-group with non-trivial center and let \( \langle x \rangle \not\subseteq G \). Then for any \( y \in Z(G) \), \( \langle x, y \rangle \not\subseteq G \). So the center of any non-nilpotent ATI-group is trivial.
Proof. Assume that \( \langle x, y \rangle \leq G \). Since any \( p' \)-subgroup is normal, it follows that \( x \) is a \( p \)-element. Therefore \( \langle x, y \rangle \leq G \) is a \( p \)-subgroup of \( G \), and so \( x \) acts trivially on any \( p' \)-element of \( G \). Now, by Lemma 1.2, \( \langle x \rangle \leq G \).

Since in every ATI-group, for any \( y \in Z(G) \) and \( g \in G \) we have \( \langle y, g \rangle \leq G \), and any ATI-group is a CTI-group, we get \( \langle g \rangle \leq G \) for every \( g \in G \). Hence, \( G \) is Hamiltonian; a contradiction. \( \square \)

3 Solvable CTI-groups with trivial center

In this section, we show that a CTI-group \( G \) is solvable if and only if it has a solvable minimal normal subgroup. Also assuming that \( G \) is a solvable group with trivial center we show that if \( V \) is a minimal normal subgroup of \( G \), then \( V \) is also solvable. We remark that if a CTI-group \( G \) has a solvable minimal normal subgroup, then, by Corollary 2.6, every minimal normal subgroup of \( G \) is also solvable.

Let \( x \in \mathcal{C}_G(V) \). Then we have \( V \leq \mathcal{C}_G(x) \). Now if \( \mathcal{C}_G(x) \) is Hamiltonian, then \( V \leq Z(\mathcal{C}_G(x)) \) and so \( \mathcal{C}_G(x) \leq \mathcal{C}_G(V) \). If \( \mathcal{C}_G(x) \) is non-nilpotent and \( x \) is a \( p \)-element, then again \( V \leq Z(\mathcal{C}_G(x)) \) (by Corollary 2.6), and so \( \mathcal{C}_G(x) \leq \mathcal{C}_G(V) \).

For the sake of simplicity in the following theorems we set \( C_V = \mathcal{C}_G(V), F = F(G) \) and \( C_x = \mathcal{C}_G(x) \), for any \( x \in G \).

Theorem 3.1. Let \( G \) be a finite CTI-group with trivial center and \( V \) be a minimal normal subgroup of \( G \). If \( V \) is solvable, then \( F = C_V \).

Proof. By the above discussion, it suffices to show that \( C_V \) is nilpotent. Suppose by way of contradiction that \( C_V \) is not nilpotent. Since \( Z(C_V) \neq 1 \), we conclude that \( C_V \cong F \times \mathbb{Z}_{p^i} \) where \( F \) is abelian. We claim that \( C_x \leq C_V \) for any \( x \in C_V \). Therefore \( G \) will be a Frobenius group with kernel \( C_V \), and this is a contradiction, because \( C_V \) is not nilpotent.

Consider first the case \( x \in Z(C_V) \). Then \( C_V \leq C_x \). Therefore, \( C_x \) is also non-nilpotent and so \( V \leq Z(C_x) \). Thus, \( C_V = C_x \). Now assume that \( x \notin Z(C_V) \). In this case, if \( \langle x \rangle \leq C_V \), then \( x \in F(C_V) = F \) and so \( F \leq C_x \). Also either \( x \) is a \( p' \)-element or \( p = 2 \) and \( |x| = 2^l \neq 2 \), so in either case, \( C_x \) is nilpotent by The-
orem 2.1 and since it is not a $p$-group, it is a Hamiltonian group and $V \leq Z(C_x)$. Hence $F = C_x \leq C_V$.

Let $\langle x \rangle \nsubseteq C_V$. If $|x| > p$, then $C_x$ is necessarily nilpotent. Therefore by choosing $y \in V \cap Z(C_x) \neq 1$, $C_y$ will be non-nilpotent because $C_V \leq C_y$. Thus we get $C_x \leq C_y = C_V$. Now if $|x| = p$, then either $C_x$ is nilpotent and so we have $V \cap Z(C_x) \neq 1$, or $C_x$ is non-nilpotent and hence $V \leq Z(C_x)$. So in either case, $C_x \leq C_V$. Thus $C_V$ is nilpotent and so $F = C_V$. □

Notice that the Fitting subgroup of a CTI-group is not necessarily abelian. For example, using the Small Group library of GAP, we see that the group Small\text{-}Group(9477,4035), is a CTI-group with abelian center and non-abelian Fitting subgroup. The structure of this group is as follows:

$$ G \cong ((\mathbb{Z}_3 \times \mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_{13}, $$

and its Fitting subgroup is $F(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_3$.

If the order of $F(G)$ is divisible by more than one prime, then $F(G)$ is abelian.

**Proposition 3.2.** Let $G$ be a finite CTI-group with trivial center and also let its minimal normal subgroup be solvable. If $|F|$ has more than one prime divisor, then $G = FH$ is a Frobenius group with abelian kernel $F$ and complement $H$.

**Proof.** By Corollary 1.3 (ii), $F$ is a Hamiltonian group. Therefore $F' \leq Z(G) = 1$ and so $F$ is an abelian group.

Assume that $q$ is a prime divisor of $|F|$ and $Q \in \text{Syl}_q(G)$. As $F \cap Q \leq Q$, we have $F \cap Z(Q) \neq 1$. Consequently, on assuming $x \in F \cap Z(Q)$, $C_x$ contains both $F$ and $Q$. Next, we show that $F$ is a Hall subgroup of $G$. First we assume that $C_x$ is nilpotent. Since $Q \leq C_x$, $Q$ commutes with a minimal normal subgroup $V$ of order coprime to $q$. Thus, $Q \leq C_V = F$.

Now, let $C_x$ be non-nilpotent. By Lemma 2.1, $C_x$ contains a minimal normal subgroup $V$ of $q$-power order. Also, since $V$ is elementary abelian, it follows that $V \leq Z(C_x)$, therefore $Q \leq C_x \leq C_V = F$. Thus, $F$ is a Hall subgroup of $G$. Consequently, $G = FH$.

Finally, to complete the proof it will suffice to show that for every $x \in F$, $C_x \leq F$. Let $q$ be a prime divisor of $|C_x|$ such that $q \nmid |F|$. Also let $y \in C_x$ be of order $q$. If $C_x$ is nilpotent, then $y \in C_G(F) = F$ and this is a contradiction. Now, let $C_x$ be non-nilpotent. Then since $x$ and $y$ have coprime orders, Corollary 2.6 (iii) implies that $y \in F(C_x)$ and $F(C_x)$ is abelian. So again $y \in C_G(F) = F$, because $F \leq F(C_x)$, which gives the final contradiction. Hence, $C_x = F$ completing the proof. □

In the following theorems, we suppose that $F$ is a $p$-group.
Lemma 3.3. Let $G$ be a CTI-group with trivial center and $K \leq G$. Also assume that a minimal normal subgroup of $G$ is solvable and $F$ is a $p$-group. Then:

(i) for any $x \in F$, $C_x$ is a $p$-group,

(ii) if $P \in \text{Syl}_p(G)$ is maximal in $K$ and $P \not\trianglelefteq G$, then $K$ is a non-nilpotent group with trivial center. Also, $F(K)$ is a $p$-subgroup of $K$ and $P \not\trianglelefteq K$.

Proof. (i) Let $V$ be a minimal normal subgroup of $G$ and $x \in F$. Suppose that $C_x$ is not a $p$-group. Since any $p'$-subgroup of $C_x$ is normal, whether $C_x$ is or is not nilpotent, we see that $F = C_V$ contains a $p'$-element (because $V \subseteq C_x$) and this is a contradiction. Hence for any $x \in F$, we observe that $C_x$ is a $p$-group.

(ii) Suppose $K \leq G$ contains $P$ as a maximal subgroup. Then $V \leq F \leq F(K)$. Now since for every $x \in F$, the subgroup $C_x$ is a $p$-group, so is $F(K)$. Therefore, $F(K) = \text{Core}_G(P) = F$. Thus $K$ is non-nilpotent and also $Z(K) = 1$ (otherwise, since $Z(K) \leq F$, for any $x \in Z(K)$, $K \leq C_x$ would be a $p$-group). □

Theorem 3.4. Let $G$ be a finite solvable CTI-group with trivial center. Assume further that $F$ is a $p$-group. Then either $G$ is isomorphic to $S_4$, or $F$ is a Sylow $p$-subgroup of $G$ and $G$ is a Frobenius group with kernel $F$.

Proof. Let $P$ be a Sylow $p$-subgroup of $G$. If $P$ is normal in $G$, then $F = P$ is the Frobenius kernel and the desired conclusion follows. So let $P \not\trianglelefteq G$. We shall show $G \cong S_4$.

Assume now that $P$ is a maximal subgroup of $K \leq G$. By the preceding lemma, we have $Z(K) = 1$ and $P \not\trianglelefteq K$. Now, if the conclusion is established for $K$ namely, $K \cong S_4$, then $F \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, we get $S_3 \cong K/F \leq G/F \hookrightarrow S_3$, therefore $K = G$. Hence without loss of generality we may assume that $P$ is maximal in $G$.

Let $Q$ be a Sylow $q$-subgroup of $G$, whence $q \neq p$. Then $QF$ is a Frobenius group with kernel $F$. Therefore $Q$ is either cyclic or generalized quaternion. As $P$ is a maximal subgroup of $G$, we have $G = PQ$, furthermore, $QF/F$ is a unique minimal normal subgroup of $G/F$, because $F = \text{Core}_G(P)$. Hence we will have $Q \cong \mathbb{Z}_q$ and so $q \neq 2$ (otherwise, $P \leq G$). Also, $P/F \hookrightarrow \text{Aut}(Q)$. Thus $P/F$ is cyclic and $p|q - 1$.

Now, set $N = \mathcal{N}_G(Q)$. Then by the Frattini argument, we have $G = NF$, because $QF \leq G$. If $F \cap N \neq 1$, then since $[F \cap N, Q] = 1$, we will have $Q \leq C_x$, for any $x \in N \cap F$ and this is a contradiction, since $C_x$ is a $p$-group. Thus, we obtain $F \cap N = 1$ and so $Q \not\subseteq N$. Let $P_1$ be a Sylow $p$-subgroup of $N$. Then $P_1$ is cyclic and $N = QP_1$ is a CTI-group. As $FZ(N) \leq G$, we have

$Z(N) \leq F \cap N = 1$

so $\text{Core}_N(P_1) = 1$, therefore $|P_1| \mid q - 1$. 


Assume that $V$ is a minimal normal subgroup of $G$ and also $a$ and $x$ are generators of $P_1$ and $Q$, respectively.

**Step 1.** $C_F(a) \cap (C_F(a))^x = 1$ and so $Z(P) \cap Z(P^x) = 1$.

Assume that $f \in C_F(a) \cap (C_F(a))^x$. Then there exists an element $f_1 \in C_F(a)$ such that $f = f_1^x$. Therefore $f_1^x = (f_1^x)^{x^a} = f_1^{x^a}$ and so $f_1 \in C_F([x, a]) = 1$, because $[x, a] \in Q$.

**Step 2.** $p = 2$ and $|(VP_1)| = |P_1| = 2$.

Let $|P_1| = p^m$ and $z \in Z(P) \cap V$ be of order $p$. We set $z_i = z^{x^i}$, for any $i \geq 0$. Then $C = \{z_i \mid 0 \leq i < q\}$ is the set of conjugates of $z$ by $Q$. The set $C$ is also invariant under conjugation by $P_1$ and if for some $l \neq 0$ and $i > 1$, $z_i^{a^l} = z_i$, then $z^{x^i} = z^{a^{-l}x^ia^l}$. Thus

$$a^{-l}x^ia^lx^{-i} \in C_Q(z) = 1,$$

so $a^{-l}x^ia^l = x^l$ then $a^l \in \text{Core}_N(P_1) = 1$, which is a contradiction. Consequently, only the element $z = z_0$ of $C$ is invariant under the action of $P_1$. Therefore, we have

$$C = \{z\} \cup \bigcup_{l=1}^k \text{Orbit}_{P_1}(z_{i_l}).$$

Now, let $u = \prod_{l=0}^{q-1} z_i$. Since $u^x = u$, we have $u \in C_F(x) = 1$. Thus

$$1 = \prod_{i=0}^{q-1} z_i = z \prod_{l=1}^k \prod_{t \in \text{Orbit}_{P_1}(z_{i_l})} t. \quad (*)$$

If $\exp(VP_1) = p^m$, then

$$1 = (a^{-1}z_i)^{p^m} = \prod_{l=1}^{p^m} z_i^{a^l} = \prod_{t \in \text{Orbit}_{P_1}(z_i)} t.$$

By $(*)$, $z = 1$ and this is a contradiction. Thus there exists a $z_i \in C$ such that $a^{-1}z_i$ is of order $p^{m+1}$. Since $a^{-1}z_i \notin V$, it follows that $v = (a^{-1}z_i)^{p^m}$ belongs to the center of $VP_1$, therefore $\langle a^{-1}z_i \rangle \leq VP_1$ ($VP_1$ is a CTI-group). Also we will have

$$VP_1/\langle v \rangle \cong V/\langle v \rangle \times \langle a^{-1}z_i \rangle/\langle v \rangle.$$

Thus $[VP_1, VP_1] = \langle v \rangle \leq Z(VP_1)$ and so

$$(az_i)^p = a^p z_i^p [z_i, a]^{p(p-1)/2}.$$
Step 3. \( V \cong \mathbb{Z}_2 \times \mathbb{Z}_2, q = 3 \) and \( VN \cong S_4 \).

We set \( Z = Z(VP_1) \). Then \( Z \cap Z^x = 1 \) by step 1. Since \( C \subseteq Z(F(G)) \), we have \( \langle C \rangle \leq G \) therefore \( V = \langle C \rangle \). Since for any \( i > 1, [z_1, a] = [z_i, a] \), it follows that \( z_1z_i^{-1} \in Z \); consequently, \( V/Z \cong \langle z_1 \rangle \) and so \( Z^x \cong \mathbb{Z}_2 \) and \( V \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Hence \( q = 3 \) and \( VN \cong S_4 \).

Step 4. \( F(G) \) is the unique minimal normal subgroup of \( G \) and thus \( G \cong S_4 \).

Let \( z_1 \) and \( z_2 \) be two distinct central elements of order 2. Then for \( v_1 = z_1^x \) and \( v_2 = z_2^x \), the subgroups \( V_1 = \langle z_1, v_1 \rangle \) and \( V_2 = \langle z_2, v_2 \rangle \) will be two distinct minimal normal subgroup of \( G \). Thus \( v_1^q = z_1v_1 \) and \( v_2^q = z_2v_2 \), and also

\[
(a v_1)^{v_2} = v_2 a v_1 v_2 = a v_1 z_2.
\]

Since \( P \) is a CTI-group and \( (a v_1)^2 = (a v_1 z_2)^2 = z_1 \), we will have

\[
av_1 z_2 = (a v_1)^3 = a v_1 z_1
\]

and so \( z_1 = z_2 \), a contradiction. Thus \( Z(P) \) is cyclic and therefore \( G \) possesses a unique minimal normal subgroup \( \langle z, v \rangle \), where \( z \in Z(P) \) and \( v \in V \).

As \( (va)^2 = z \), we have \( \langle va \rangle \leq P \) and so \( [F, \langle va \rangle] \leq F \cap \langle va \rangle = \langle z \rangle \). Since for every \( f \in F \), \( [f, v] = 1 \), we will have \( [F, a] \leq \langle z \rangle \) and so \( F^2 \leq C_F(a) \); consequently, \( C_F(a) \leq F \) and \( F/C_F(a) \) is elementary abelian.

Finally assume that \( f_1, f_2 \notin C_F(a) \). Then we have \( f_2^{-1} f_1 \notin C_F(a) \), because \( [f_1, a] = [f_2, a] \). Therefore, \( F/C_F(a) \) is cyclic and so it is isomorphic to \( \mathbb{Z}_2 \). By step 1, we have \( |C_F(a)| = |C_F(a)^x| = 2 \), consequently, \( F = V \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) and the desired conclusion follows.

\[\square\]

Theorem 3.5. Let \( G = KH \) be a finite Frobenius CTI-group with kernel \( K \) and complement \( H \). Then,

(i) if \(|H|\) is odd, then \( H \) is cyclic,

(ii) if \(|H|\) is even, then \( K \) is abelian and either \( H \) is cyclic or \( H \cong Q_8 \times \mathbb{Z}_n \), where \( n \) is odd.

In either case \( G \) is solvable.

Proof. (i) Since \( H \) is a solvable group and cannot be Frobenius group by [10, Theorem 12.6.11], it follows that \( Z(H) \neq 1 \) by Theorem 3.4 and 3.2. Now by Corollary 2.6, \( H \) is a nilpotent. Therefore \( H \) is cyclic by [2, Theorem 10.3.1 (iv)].

(ii) By [2, Theorem 10.3.1 (iii), (iv)], \( K \) is abelian and \( Z(H) \neq 1 \) again by Corollary 2.6, \( H \) is nilpotent. We can easily see that the only generalized quaternion CTI-group is \( Q_8 \). Therefore either \( H \) is a cyclic group or \( H \cong Q_8 \times \mathbb{Z}_n \), where \( n \) is odd.

\[\square\]

Theorem 3.6. A CTI-group \( G \) is solvable if and only if it has a solvable minimal normal subgroup.
Proof. If $Z(G) \neq 1$ or $F(G)$ is not a $p$-group, then by Proposition 3.2 and Corollary 2.6, $G$ is solvable. So we assume that $Z(G) = 1$ and $F(G)$ is a $p$-group.

Let $G$ be a minimal counterexample for the theorem. Let $P \in \text{Syl}_p(G)$. By Theorems 3.5 and 3.4, $P \not\trianglelefteq G$. Suppose that a proper subgroup $K$ of $G$ contains $P$ as a maximal subgroup. Therefore we have $P \not\trianglelefteq K$, $F(K) = F(G)$ and $Z(K) = 1$ (by Lemma 3.3), also by the choice of $G$, $K$ is solvable and so $K \cong S_4$. Hence $P \cong D_8$ and $F(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore $G/F(G)$ is solvable which is a contradiction. And so $P$ is a maximal subgroup of $G$. By a well-known theorem of Thompson [2, Theorem 10.3.2], $p = 2$ and by [9, Theorem II], $G/F$ has a unique minimal normal subgroup $K/F$ such that $G/K$ is a 2-group. Hence $K$ is not solvable. Again by the minimality of $G$, we have $K = G$. Now by [5, Theorem 2.13] every involution of $G/F$ inverts an element of odd order in $G/F$, so $G/F$ contains a non-nilpotent dihedral subgroup. Consider the inverse image $R$ of this dihedral subgroup in $G$. Obviously $Z(R) = 1$ and $R$ is solvable with non-normal Sylow 2-subgroup. By using Theorem 3.4, $R \cong S_4$ and $F$ is a four group and this is also a contradiction. 

\section{Non-solvable CTI-groups}

In this section we classify non-solvable CTI-groups. Let $V$ be a minimal normal subgroup of a non-solvable CTI-group $G$. By Theorem 3.6, $V$ cannot be solvable, since the centralizer of any element (in particular any subgroup) of $G$ is solvable, and so $\mathcal{C}_G(V) = 1$. Therefore, $V$ must be simple. Also we have

$$V \leq G \leftrightarrow \text{Aut}(V) \quad \text{and} \quad G/V \leftrightarrow \text{Out}(V).$$

Lemma 4.1. Let $G$ be a non-solvable CTI-group with minimal normal subgroup $V$ and $P \in \text{Syl}_2(V)$. If $N = \mathcal{N}_G(P)$ is non-nilpotent, then $Z(N) = 1$.

Proof. If $Z(N) \neq 1$, then by Corollary 2.6 either $P \leq Z(N)$ or $\mathcal{C}_G(P)$ has index 2 in $N$. In the latter case, we have $\mathcal{N}_V(P) = \mathcal{C}_V(P)$. In either case, we get $P \leq Z(\mathcal{N}_V(P))$ and so $P$ has a normal $p$-complement in $V$, a contradiction. 

Theorem 4.2. Let $G$ be a finite non-solvable CTI-group. Then $G \cong \text{PSL}(2, q)$ or $G \cong \text{PGL}(2, q)$, where $q > 3$ is a prime power.

Proof. Let $G$ be a finite non-abelian simple CTI-group. Since every $p$-local subgroup of $G$ is solvable, then $G$ is an N-group. Now by a theorem of Thompson ([2, Theorem, p. 474]), only the groups $\text{PSL}(2, q)$ and $\text{Sz}(q)$ which do not contain $\text{SL}(2, 3)$ can be CTI (because $\text{SL}(2, 3)$ is not a CTI-group). Let $G \cong \text{Sz}(q)$ and $P \in \text{Syl}_2(G)$. Then by [1, Lemma 1 and Proposition 3] we have $\Omega_1(P) = Z(P)$
and $|P| = |Z(P)|^2$. Since $P$ is a non-abelian CTI-group, $P$ must be a non-abelian Hamiltonian group of order 16. This is a contradiction.

Now we consider the non-simple case: then $G$ is isomorphic to a subgroup of $H = \text{Aut}(\text{PSL}(2, q)) = \text{PGL}(2, q) \rtimes \langle x \rangle$, where $q = p^f$ and $x$ has order $f$. Let $g \in G \setminus \text{PGL}(2, q)$ be a power of $x$. Then $f \neq 1$ also $\text{PSL}(2, p) \leq \mathcal{C}_G(g)$, because $\mathcal{C}_H(x) = \text{PGL}(2, p) \rtimes \langle x \rangle$. Since $\mathcal{C}_G(g)$ is non-Hamiltonian and solvable, it follows that $|g| = 2$ (by Corollary 2.6), and $p = 2$, because a Sylow 3-subgroup of $\text{PSL}(2, 3)$ is non-normal. Now let $S \in \mathfrak{S}_{\ell_2}(G)$ and $P \in \mathfrak{S}_{\ell_2}(\text{PGL}(2, q))$ such that $P \leq S$. Then $S = P\langle g \rangle$. Suppose $N = \mathcal{N}_G(P)$; by Lemma 4.1, $Z(N) = 1$. If $S \leq N$, then $N = S\langle y \rangle$, where $|y| = q - 1$ (by [2, Lemma 15.1.1]). Hence $[g, y] = 1$ and $N$ cannot be a Frobenius group; now by Theorem 3.4, $N \cong S_4$ and $f = 2$. Therefore, $G \cong \text{Aut}(\text{PSL}(2, 4))$ which is isomorphic to $\text{PGL}(2, 5)$.

In the other case, since $G$ is a pre-image of a subgroup of

$$\text{Out}(\text{PSL}(2, q)) = \langle \bar{y} \rangle \times \langle \bar{x} \rangle,$$

then either $G$ is isomorphic to $\text{PGL}(2, q)$, where $q > 3$ is a prime power or $p$ is odd, $f$ is even and $G \cong \langle \text{PSL}(2, q), yx^{f/2} \rangle$. In the latter case $G$ is isomorphic to a non-solvable maximal subgroup of $\text{PGL}(2, q) \rtimes \langle x^{f/2} \rangle$. Now by [3, Lemma 6.6.3], $G$ is isomorphic to $\text{PGL}^*(2, q)$ which has semidihedral Sylow 2-subgroup. This case cannot occur because a semidihedral group is not CTI. \hfill $\Box$

The inverses of Corollary 2.6 and Theorem 3.4 are simple: we just prove the inverse of the non-solvable case. Before proving the inverse theorem, we consider the simple fact that if a non-normal subgroup $\langle x \rangle$ of $G$ is normal in a non-normal maximal subgroup $M$, then $\langle x \rangle \cap \langle x \rangle^g \leq G$, where $g \in G \setminus M$.

**Theorem 4.3.** Let $G$ be isomorphic to $K$, where $\text{PSL}(2, q) \leq K \leq \text{PGL}(2, q)$, $q > 3$ is a power of prime $p$. Then $G$ is a CTI-group.

**Proof.** We can simply check by GAP that $\text{PSL}(2, p)$ is CTI for $p = 5, 7, 9, 11$. Let $x$ be an element of $G$. If $p \mid |x|$, then $x$ must be a $p$-element, because by [2, Lemma 15.1.1] Sylow $p$-subgroups of $G$ are elementary abelian and TI; therefore $|x| = p$. If $|x| \mid (q^2 - 1)$ and $x$ is not a 2-element, then $|x| \mid 2^nm$, where $m$ is odd; hence $x = yz$, where $|z| > 1$ is odd. In this case $z$ belongs to the maximal subgroup $D_{2(q-1)}$ or $D_{2(q+1)}$ by [7, Theorem 2.1 and Theorem 2.2]; since $\langle z \rangle$ is normal in these groups, it follows that $\mathcal{N}_G(x) = \mathcal{N}_G(z)$ is a non-normal maximal subgroup of $G$. Therefore, $\langle x \rangle$ is normal in a non-normal maximal subgroup of $G$, and so is TI. Now, let $x$ be a 2-element and $|x| > 2$; then $p$ is an odd prime and again $\langle x \rangle$ belongs to the dihedral group. Since $\langle x \rangle$ is normal in this group, it follows that $\mathcal{N}_G(x)$ is maximal in $G$. Hence $\langle x \rangle$ is a TI-group. Therefore, $G$ is a CTI-group. \hfill $\Box$
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