Purely Finitely Additive Measures as Generalized Elements in a Maximin Problem

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Abstract

We study the asymptotic behavior of maximin values of a payoff function, when admissible controls tend to infinity. The payoff function is superposition of a continuous function and a function that is uniform limit of step functions. An extension in the class of finitely additive measures is used.

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It is natural in control problems to use extensions constructions due to the non-existence of optimal solutions [1]. Moreover, extensions are needed for a 'regularization' of practically interesting problems that are associated with asymptotic relaxations of constraints (see [2, 3]). Obviously, game problems require extensions [4]. These problems can be divided into two types: those for which it is possible to well define an extension only on the product of ordinary controls sets of players [4–6], and those for which it is sufficient to construct an extension of an ordinary controls set for each player separately [7–10]. If we deals with a maximin problem of the second type, then we can use the following representation of the maximin value $V_{ext}$ after an extension:

$$V_{ext} = \max_{\nu \in B} \min_{\mu \in A} \tilde{\alpha}(\mu, \nu),$$  \hspace{1cm} (1)
where $A, B$ are some sets of generalized elements (controls) and $\bar{\alpha}$ is the generalized payoff function. In [7–10] the extension in the class of finitely additive measures (FAM) was used. In these papers sets $A$ and $B$ were some compacta in $*$-weak topology. The using of FAM helps to deal with the case of discontinuous control coefficients in the right-hand part of a differential equation. Often in this case purely FAM are essential elements of sets $A, B$ (see [7,10]).

In this paper we consider a maximin problems such that admissible controls of players in some sense tends to infinity. We study asymptotics of values of the problems. We show that in this case sets $A, B$ are subsets of purely FAM.

The present paper extends results of [11].

Let $(X, \rho_X)$ and $(Y, \rho_Y)$ be unbounded metric spaces; fix $x_0 \in X$ and $y_0 \in Y$. By $S_X^\varepsilon$ we denote open $\varepsilon$-neighborhood of $x_0$ w.r.t. $\rho_X$ and by $S_Y^\kappa$ we denote open $\kappa$-neighborhood of $y_0$ w.r.t. $\rho_Y$; here $\varepsilon \in ]0, \infty[; \kappa \in ]0, \infty[$. Let $H_X^\varepsilon \triangleq X \setminus S_X^\varepsilon$, $H_Y^\kappa \triangleq Y \setminus S_Y^\kappa$. By definition, put $\mathcal{H}_X \triangleq \{ H_X^\varepsilon : \varepsilon \in ]0, \infty[ \}$, $\mathcal{H}_Y \triangleq \{ H_Y^\kappa : \kappa \in ]0, \infty[ \}$. Families $\mathcal{H}_X$ and $\mathcal{H}_Y$ are filter bases in $X$ and $Y$ respectively [12]. By $\mathcal{L}_X$ and $\mathcal{L}_Y$ we denote semialgebras of subsets of $X$ and $Y$ respectively such that: $(\forall x_0 \in X \forall \varepsilon \in ]0, \infty[: S_X^\varepsilon \in \mathcal{L}_X) \& (\forall y_0 \in Y \forall \kappa \in ]0, \infty[ : S_Y^\kappa \in \mathcal{L}_Y)$. By $\mathbb{R}^n$ we denote $n$-dimensional arithmetic space. Let $u : X \to \mathbb{R}^k$ and $v : Y \to \mathbb{R}^l$ be uniform limits of $\mathcal{L}_X$-step and $\mathcal{L}_Y$-step functions respectively. Let $\Upsilon, \Upsilon : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$, be a jointly continuous payoff function. Thus,

$$\Upsilon(u(x), v(y)) \in \mathbb{R} \forall \varepsilon \in ]0, \infty[, \forall x \in H_X^\varepsilon \forall \kappa \in ]0, \infty[, \forall y \in H_Y^\kappa.$$ 

Now we can consider the following maximin problem for some $\varepsilon \in ]0, \infty[, \kappa \in ]0, \infty[$. The first player minimizes value of $\Upsilon$ by choosing $x \in H_X^\varepsilon$, the second player maximizes value of $\Upsilon$ by choosing $y \in H_Y^\kappa$. Thus we deal with problems

$$\Upsilon(u(x), v(y)) \to \sup_{y \in H_Y^\kappa} \inf_{x \in H_X^\varepsilon} , \varepsilon \in ]0, \infty[, \kappa \in ]0, \infty[. \tag{2}$$

We will investigate asymptotics of values (2) as $\varepsilon, \kappa \to \infty$. Note that this asymptotics do not depend on $x_0$ and $y_0$.

By definition, put $V(\varepsilon, \delta) \triangleq \sup_{y \in H_Y^\kappa} \inf_{x \in H_X^\varepsilon} \Upsilon(u(x), v(y))$.

Using continuity of $\Upsilon$, boundedness of $v^1(H_Y^\kappa), u^1(H_X^\varepsilon)$, and [13, (2.34)], we obtain that

$$V(\varepsilon, \delta) = \max_{b \in cl(v^1(H_Y^\kappa)), \tau_R^{(l)}} \min_{a \in cl(u^1(H_X^\varepsilon)), \tau_R^{(k)}} \Upsilon(a, b) \in \mathbb{R} \forall \varepsilon, \kappa \in ]0, \infty[,$$

where $\tau_R^{(k)}$ and $\tau_R^{(l)}$ are the topologies of coordinate-wise convergence in $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively. Moreover, the asymptotic maximin as well-defined:

$$V \triangleq \max_{b \in G_2} \min_{a \in G_1} \Upsilon(a, b) \in \mathbb{R},$$
where the following attraction sets (see [2, (3.3.10)])

$$G_1 \triangleq \bigcap_{\varepsilon \in ]0,\infty[} cl(u^1(H^\varepsilon_X), \tau_{\mathbb{R}^k}), G_2 \triangleq \bigcap_{\kappa \in ]0,\infty[} cl(v^1(H^\kappa_Y), \tau_{\mathbb{R}})$$

are compacta. We now ready to state the specific version of [13, theorem 1].

**Theorem 1.** The following approximation property of $V$ holds:

$$\forall \xi \in ]0, \infty[ \, \exists \theta \xi \in ]0, \infty[ : |V - V(\varepsilon, \delta)| < \xi \forall \varepsilon \in ]0, \theta \xi[ \, \forall \kappa \in ]0, \theta \xi[.$$

The extension of the original problem (2) is constructed in the following way: for each point of the sets $X$ and $Y$ we assign the Dirac measure supported at this point (see immersion operator $\Delta$ in [3, p. 1090] and [14, (4.4)]). The closures of resulting sets w.r.t. $*$-weak topology coincide, respectively, with the set of all $\{0, 1\}$-valued FAM on $L_X$ and on $L_Y$ (see [3, p. 1090] and [14, (4.5)]). We now introduce the specific version of sets $A$ and $B$ (see (1)):

$$D_X \triangleq \bigcap_{\varepsilon \in ]0,\infty[} cl(\{\delta^X_x : x \in H^\varepsilon_X\}, \tau_{L_X}), D_Y \triangleq \bigcap_{\kappa \in ]0,\infty[} cl(\{\delta^Y_y : y \in H^\kappa_Y\}, \tau_{L_Y}); (3)$$

where $cl(Z, \tau)$ is the closure of $Z$ w.r.t. topology $\tau$, $\delta^X_x$ is the Dirac measure on $L_X$ supported on $\{x\}$, $\delta^Y_y$ is the Dirac measure on $L_Y$ supported at $y$, and $\tau_{L_X}, \tau_{L_Y}$ are $*$-weak topologies. From [11, proposition 2.2] it follows that sets $D_X, D_Y$ contains only purely FAM. Now we define the generalized payoff function $\tilde{\Upsilon}: D_X \times D_Y \to \mathbb{R}$ by the rule

$$\tilde{\Upsilon} \triangleq \Upsilon \left( \left( \int_{X} u_i \, \mu(dx) \right)_{i \in \overline{1,k}}, \left( \int_{Y} v_j \, \nu(dy) \right)_{j \in \overline{1,l}} \right) \forall \mu \in D_X \forall \nu \in D_Y. (4)$$

Using [13, proposition 5],(3), and (4), we obtain the next statement.

**Theorem 2.** The following representation of asymptotics of values of problems (2) holds: $V = \max_{\nu \in D_Y} \min_{\mu \in D_X} \tilde{\Upsilon}(\mu, \nu)$.

It is shown that if we use the extension in the class of FAM, then it is possible to obtain representation of asymptotics of values of problems (2). Moreover, from Theorem 2 it follows that these asymptotics can be defined in terms of generalized elements. These elements are purely FAM.

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References


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