ON A QUESTION BY EDMOND W. H. LEE*

Introduction

Let \( A_2 = \langle a, b \mid aba = a^2 = a, \ bab = b, \ b^2 = 0 \rangle = \{a, b, ab, ba, 0\} \) be the 5-element idempotent-generated 0-simple semigroup. The semigroup \( A_2 \) as well as the 5-element Brandt semigroup

\( B_2 = \langle c, d \mid cdc = c, \ dcd = d, \ c^2 = d^2 = 0 \rangle = \{c, d, cd, dc, 0\} \)

plays a distinguished role in the theory of semigroups and especially in the theory of semigroup varieties (see, for instance, a discussion in [1, Sections A5, A6, A15]). We denote by \( A_2 \) and \( B_2 \) the varieties generated by respectively \( A_2 \) and \( B_2 \).

It was observed by N. R. Reilly (unpublished) that the variety \( A_2 \) has a unique maximal subvariety which we denote by \( \overline{A}_2 \). Clearly, \( \overline{A}_2 \) can be thought of as the largest subvariety of \( A_2 \) that does not contain the semigroup \( A_2 \). Similarly, the largest subvariety of \( A_2 \) that does not contain the semigroup \( B_2 \) is denoted by \( \overline{B}_2 \) (the existence of such a largest subvariety in each variety of periodic semigroups follows from a general result by E. V. Sukhanov, see [2, Theorem 1]). Recently E. W. H. Lee [3, Question 5.3] has asked whether or not

\[ \overline{A}_2 = \overline{B}_2 \vee B_2 \]  

where the right hand side means the join of \( \overline{B}_2 \) and \( B_2 \) in the lattice of semigroup varieties. In the present note we answer this question in the affirmative.

1. Preliminaries

We adopt the standard terminology and notation of semigroup theory (see [1, 4–6]) and universal algebra (cf. [7]). For the reader’s convenience, we recall a few basic definitions related to words.

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We fix a countably infinite set $\Sigma$ (the *alphabet*) whose elements are referred to as *letters*. As usual, $\Sigma^+$ is the free semigroup over $\Sigma$ and $\Sigma^* = \Sigma^+ \cup \{1\}$ is the free monoid over $\Sigma$. We call elements of $\Sigma^+$ *words* and denote the equality relation on $\Sigma^+$ by $\equiv$. If $u, v$ are words, we say that $u$ *occurs* in $v$ or $u$ is a *factor* in $v$ whenever there exist $v', v'' \in \Sigma^*$ such that $v$ can be decomposed as $v \equiv v'u v''$. For a word $w \in \Sigma^+$ we denote by $\text{alph}(w)$ the set of letters from $\Sigma$ that occur in $w$. If $w \equiv x_1 x_2 \ldots x_n$ where $x_1, x_2, \ldots, x_n$ are letters in $\text{alph}(w)$, then the number $n$ is called the *length* of the word $w$ and is denoted by $|w|$.

Our proof of the equality (1) is based on a graph-theoretical description of the identities holding in $A_2$. This description is well known. It is often attributed to G. Mashevitsky [8] (see, for instance, [3] or [9]) even though the paper [8] does not deal with the identities of $A_2$ at all. Apparently, this mistake originates from an erroneous reference in the survey paper [10]. In fact, the description has been found by A. Trahtman, see his preprint [11].

Given a word $w \in \Sigma^+$, we assign it a directed graph $G(w)$ whose vertex set is $\text{alph}(w)$ and whose edges correspond to factors of length 2 in $w$ as follows: $G(w)$ has a directed edge from $x$ to $y$ ($x, y \in \text{alph}(w)$) if and only if $xy$ appears as a factor in $w$. We will distinguish two (not necessarily different) vertices in $G(w)$: the *initial vertex*, that is the first letter of $w$, and the *final vertex*, that is the last letter of $w$. Then the word $w$ can be thought of as a walk through the graph $G(w)$ that starts at the initial vertex, ends at the final vertex and traverses each edge of $G(w)$ (some of the edges can be traversed more than once). Fig. 1 shows the graph

![Graph](image)

Fig. 1: The graph of the word $x^2yzzzy^2zt^2$ and the corresponding walk $G(w)$ for the word $w \equiv x^2yzzzy^2zt^2$. The ingoing and the outgoing marks show
respectively the initial and the final vertices of the graph. On Fig. 1 each edge of the graph is labelled by the number(s) corresponding to the occurrence(s) of the edge in the walk induced by the word \( w \). We stress that, in contrast to the vertex names and the ingoing/outgoing marks, these labels are not considered as a part of the data making the graph \( G(w) \). Therefore the graph does not determine the word \( w \): for instance, as the reader can easily check, the word \( xy^3zyzx^2zyzt^3 \) has exactly the same graph (but corresponds to a different walk through it, see Fig. 2).

![Fig. 2: Another walk through the graph of Fig. 1](image)

Let \( u, v \in \Sigma^+ \) be words and \( S \) a semigroup. We say that \( S \) satisfies the identity \( u = v \) (or that the identity \( u = v \) holds in \( S \)) if \( u\varphi = v\varphi \) for every homomorphism \( \varphi : \Sigma^+ \to S \).

**Proposition 1** (A. Trahtman, [11]). The semigroup \( A_2 \) satisfies the identity \( u = v \) if and only if the graphs \( G(u) \) and \( G(v) \) are equal.

A system \( \Xi \) of identities is said to be an identity basis for a semigroup variety \( V \) if \( V \) consists precisely of semigroups which satisfy all identities in \( \Xi \). In this situation we also say that \( V \) is defined by \( \Xi \). For the sake of completeness, we list the identity bases for the varieties that play a major role in this note.

**Proposition 2.** (i) The variety \( A_2 \) is defined by the identities

\[
x^2 = x^3, \quad xyx = xyxyx, \quad xyzzx = zzxyx.
\]

(ii) The variety \( B_2 \) is defined by the identities

\[
x^2 = x^3, \quad xyx = xyxyx, \quad x^2y^2 = y^2x^2.
\]
(iii) The variety $\overline{A}_2$ is defined by the identities
\[ x^2 = x^3, \ xyx = xyxyx, \ xyxx = xzxyx, \ x^2y^2x^2 = x^2yx^2. \] (4)

(iv) The variety $\overline{B}_2$ is defined by the identities
\[ x^2 = x^3, \ xyx = xyxyx = xy^2x, \ xyxx = xzxyx. \] (5)

Proof. The identity bases (2) for $A_2$ and (3) for $B_2$ were found by A. Trahtman, see respectively [11] and [12]. The bases (4) for $\overline{A}_2$ and (5) for $\overline{B}_2$ were discovered by E.W.H. Lee [3, Theorems 2.7 and 3.6] via direct manipulations with identities. We outline here an alternative (and calculation-free) way to obtain the basis (5) which is most important for the proof of our main result.

Recall that $\overline{B}_2$ is the largest subvariety of $A_2$ that does not include the semigroup $B_2$. Clearly, $\overline{B}_2$ satisfies the identities (2). By [2, Theorem 1] $\overline{B}_2$ satisfies also the identity
\[(xy)^2(yx)^2(xy)^2 = (xy)^2.\]
Multiplying through on the right by $x$, we obtain the identity
\[xyxy^2xyx^2yxyx = xyxyx.\] (6)
From Proposition 1 we see that the left hand side of (6) is equal in $A_2$ to the word $x^2y^2x^2$ while the right hand side reduces to $xyx$. Thus, $\overline{B}_2$ satisfies the identity
\[x^2y^2x^2 = xyx.\] (7)
Applying (7) to the word $xy^2x$, we obtain
\[xy^2x = x^2(y^2)x^2 = x^2y^2x^2 = xyx.\]
We conclude that the identity
\[xy^2x = xyx\] (8)
holds in $\overline{B}_2$. Since adding the identity (8) to the system (2) gives exactly the identity system (5), the variety $\overline{B}_2$ is contained in the variety defined by the latter system. Conversely, the variety defined by (5) is a subvariety of $A_2$ and does not include $B_2$ (because the identity (8) fails in $B_2$). Therefore, this variety is contained in $\overline{B}_2$. Thus, $\overline{B}_2$ is indeed defined by the identities (5).

A word $w \in \Sigma^+$ of length at least 2 is said to be connected if its directed graph $G(w)$ is strongly connected. In fact, this concept is known in the literature, although under a different name. In [14] G. Mashevsy introduced the following definition: a word $w \in \Sigma^+$ of length at least 2 is said to be covered by cycles if each of its
factor of length 2 occurs in a factor of \( w \) that starts and ends with the same letter. In the language of the graph \( G(w) \), this property means that each edge \( x \to y \) of \( G(w) \) belongs to a directed cycle (namely, to the walk induced by a factor of \( w \) that starts and ends with the same letter and contains \( xy \)). It is one of the basic facts of the theory of directed graphs (cf. [13, Theorem 8.1.5]) that such a graph is strongly connected if and only if each its edge belongs to a directed cycle.

Our next proposition reveals the semigroup meaning of the concept of a connected word. Recall that an element \( s \) of a semigroup \( S \) is regular in \( S \) if there exists \( s' \in S \) such that \( ss's = s \).

**Proposition 3.** A word \( w \in \Sigma^+ \) is connected if and only if for every semigroup \( S \in A_2 \) and for every homomorphism \( \varphi : \Sigma^+ \to S \) the element \( w\varphi \) is regular in \( S \).

**Proof.** First suppose that \( w \) is a connected word, and let \( x \) and \( y \) be respectively the first and the last letters of \( w \) (we do not assume that \( x \neq y \)). Since the graph \( G(w) \) is strongly connected, there is a walk

\[
y \equiv x_0 \to x_1 \to \cdots \to x_n \equiv x.
\]

Let \( u \equiv x_1 \cdots x_{n-1}wx_1 \cdots x_{n-1} \), then the graphs of the words \( w \) and \( wuw \) are equal. Here if \( n = 0 \) (which means that \( y \equiv x \)) or \( n = 1 \), the product \( x_1 \cdots x_{n-1} \) interprets as the empty word. By Proposition 1 \( w\psi = (wuw)\psi = w\psi \cdot w\psi \cdot w\psi \) where \( \psi \) denotes the canonical homomorphism from \( \Sigma^+ \) to \( F_2(A_2) \), the free semigroup of the variety \( A_2 \) over the alphabet \( \Sigma \). Thus, \( w\psi \) is regular in \( F_2(A_2) \) and, since every homomorphism \( \varphi : \Sigma^+ \to S \) factors through \( \psi \), the element \( w\varphi \) is regular in \( S \).

Now suppose that \( w \) is not connected. This means that the graph \( G(w) \) contains a bridge whence the walk induced by \( w \) splits into the part preceding the bridge, the bridge, and the part following the bridge. (The reader may see such a situation on Fig. 1 or 2 where the edge \( z \to t \) forms a bridge.) Accordingly, \( w \) decomposes as \( w \equiv w'w'' \) where the words \( w' \) and \( w'' \) correspond to the parts of the walk respectively before and after the bridge. Clearly, \( \text{alph}(w') \cap \text{alph}(w'') = \emptyset \). Consider the subsemigroup \( A_0 = \{b, ba, ab, 0\} \) of \( A_2 \) and let the homomorphism \( \zeta : \Sigma^+ \to A_0 \) be defined as follows:

\[
x\zeta = \begin{cases} ba & \text{if } x \in \text{alph}(w'), \\ ab & \text{otherwise.} \end{cases}
\]

Then using the defining relations of the semigroup \( A_2 \), one readily calculates that

\[
w\zeta = w'\zeta \cdot w''\zeta = (ba)^{|x|} \cdot (ab)^{|y|} = ba \cdot ab = bab = b.
\]

However, it is easy to check that \( b \) is not regular in \( A_0 \).
Remark. The fact that, under the canonical homomorphism $\Sigma^+ \to F_2(A_2)$, every word covered by cycles maps onto a regular element of $F_2(A_2)$ is a partial case of a similar result claimed by G. Mashevitsky in [14, Lemma 6], see also [15, Lemma 7]. Since then, this result has been used (with a reference to [14]) in several important papers including, for instance, [16] and [17]. However, its proof in [14] contains a fatal flaw (and so does the translation of the proof into English published in [15]). Namely, in [14] Lemma 6 is deduced from Lemma 5 which claims that every word $u$ covered by cycles can be transformed modulo certain identities to a word of the form $z_1u_1z_1 \cdots z_ku_kz_k$ where $z_1, \ldots, z_k$ are letters and $z_i+1 \in \text{alph}(u_i)$ for all $i = 1, \ldots, k - 1$ provided that $k > 1$. In order to justify the latter claim, Mashevitsky uses induction on $|\text{alph}(u)|$ but in the course of the proof he illegitimately applies the induction assumption to a factor of $u$ that generally speaking is not covered by cycles. The word $u \equiv xyzxy$ can be used as a concrete counter example showing that the argument from [14] does not work: here the induction assumption should have been applied to the factor $zy$ which is certainly not covered by cycles.

In fact, a correct proof of the described intermediate claim can be achieved by simple graph-theoretic means, and moreover, the claim can be avoided because we can prove Lemma 6 of [14] by a suitable modification of reasoning applied in the above proof of Proposition 3. Thus, results of [16] and [17] that rely on the lemma are correct.

Proposition 3 also allows us to give a semigroup proof of the following combinatorial property:

**Corollary 4.** If a word $w \in \Sigma^+$ is connected, then for every homomorphism $\gamma : \Sigma^+ \to \Sigma^+$ the word $w\gamma$ is connected.

**Proof.** Take an arbitrary semigroup $S \in A_2$ and an arbitrary homomorphism $\varphi : \Sigma^+ \to S$. Then $(w\gamma)\varphi = w(\gamma\varphi)$ is regular by the “only if” part of Proposition 3 whence the word $w\gamma$ is connected by the “if” part of the proposition.

We will utilize the following partial case of an important lemma due to S. Kublanovskii, see [16, Lemma 3.2]:

**Proposition 5** (S. Kublanovskii, [16]). For any semigroup $S \in A_2$ and distinct regular elements $s, s' \in S$ there exists a completely 0-simple semigroup $K$ and a surjective homomorphism $\varphi : S \to K$ such that $s\varphi \neq s'\varphi$.

2. The main result and its proof

Recall that the join $X \vee Y$ of two semigroup varieties $X$ and $Y$ is the least variety containing both $X$ and $Y$; in other words, $X \vee Y$ is the closure of the
class-theoretic union $X \cup Y$ under the operators of taking homomorphic images, subsemigroups, and direct products. As mentioned in the introduction, our main result is the following

**Theorem.** $\overline{A}_2 = \overline{B}_2 \lor B_2$.

**Proof.** Let $V = \overline{B}_2 \lor B_2$. Since $\overline{A}_2$ is the largest proper subvariety in $A_2$ and both $\overline{B}_2$ and $B_2$ are proper subvarieties in $A_2$, we conclude that $V \subseteq \overline{A}_2$. Arguing by contradiction, assume that this inclusion is strict. Then there exists an identity that holds in the variety $V$ but fails in the variety $\overline{A}_2$. We choose an identity $u = v$ with this property with minimum possible value of $|\text{alph}(u)|$. Observe that necessarily $\text{alph}(u) = \text{alph}(v)$ – otherwise the identity $u = v$ would fail in the two element semilattice and could not be satisfied by $V$.

The core of the proof consists in verifying the following

**Claim.** The words $u$ and $v$ are connected.

**Proof.** Suppose that the word $u$ is not connected. Then arguing as in the proof of Proposition 3 we may decompose $u$ as a product of two non-empty words $u'$ and $u''$ such that $\text{alph}(u') \cap \text{alph}(u'') = \emptyset$. Now we make use of the fact that the subsemigroup $A_0 = \{b, ba, ab, 0\}$ of $A_2$ belongs to the variety $\overline{B}_2$, and therefore, it must satisfy the identity $u = v$. Let the homomorphism $\zeta : \Sigma^+ \to A_0$ be defined as follows:

$$x\zeta = \begin{cases} ba & \text{if } x \in \text{alph}(u'), \\ ab & \text{otherwise.} \end{cases} \quad (9)$$

As in the proof of Proposition 3, one readily obtains that $u\zeta = b$. Hence also $v\zeta = b$. By (9) $v\zeta$ is a product of the idempotents $ab$ and $ba$ in some order. Since $ab \cdot ba = 0$, if such a product is not equal to 0, then no occurrence of $ab$ precedes an occurrence of $ba$. This implies that in the word $v$ no occurrence of a letter from $\text{alph}(u'')$ precedes an occurrence of a letter from $\text{alph}(u')$. Therefore $v$ decomposes as $v = v'v''$ where $\text{alph}(v') = \text{alph}(u')$, $\text{alph}(v'') = \text{alph}(u'')$.

We want to show that both the identities $u' = v'$ and $u'' = v''$ hold in the variety $V$. First consider an arbitrary homomorphism $\varphi : \Sigma^+ \to B_2$. Suppose that $u'\varphi \neq v'\varphi$. Then one of the two elements is not equal to 0; without any loss we may assume that $s = u'\varphi \neq 0$. Let $s'$ be the unique inverse of $s$ in $B_2$. Then $s's$ is a non-zero idempotent and, as one can easily verify, for any $r \in B_2$ the product $rs's$ is equal to either $r$ or 0. In particular,

$$u'\varphi \cdot s's = s \neq v'\varphi \cdot s's \in \{v'\varphi, 0\}. $$

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Now we define a “modification” $\xi$ of the homomorphism $\varphi$ by letting

$$x\xi = \begin{cases} x\varphi & \text{if } x \in \alpha(u'), \\ s' s & \text{otherwise.} \end{cases}$$

Taking into account the equalities $\alpha(u') = \alpha(u')$, $\alpha(v'^n) = \alpha(u'^n)$ and $\alpha(u') \cap \alpha(u'^n) = \emptyset$, we obtain

$$u_\xi = u' \xi \cdot u'' \xi = u' \varphi \cdot s' s \neq v' \varphi \cdot s' s = v' \xi \cdot v'' \xi = v_\xi.$$

This contradicts the assumption that the identity $u = v$ holds in the variety $V$ and hence in the semigroup $B_2$. Thus, we conclude that $u' \varphi = v' \varphi$ under every homomorphism $\varphi : \Sigma^+ \to B_2$, that is, $B_2$ satisfies the identity $u' = v'$. By the left-right symmetry of $B_2$, it also satisfies the identity $u'' = v''$.

Since the identity $u = v$ holds in the variety $B_2$, then by Birkhoff’s Completeness Theorem for Equational Logic (cf. \cite[Theorem 14.19]{7}) there is a deduction of $u = v$ from the identity basis (5) of $B_2$. Recall that such a deduction is a sequence of words

$$u \equiv w_0 \Rightarrow w_1 \Rightarrow \cdots \Rightarrow w_k \equiv v$$

in which $w_i \Rightarrow w_{i+1}$ ($i = 0, 1, \ldots, k - 1$) denotes that either $w_{i+1} \equiv w_i$ or $w_{i+1}$ is obtained from $w_i$ by a single application of an identity $g_i = h_i$ from the system (5), that is, there exist $p_i, q_i \in \Sigma^*$ and an endomorphism $\gamma_i : \Sigma^+ \to \Sigma^*$ such that $w_i \equiv p_i(g_i \gamma_i)q_i$ and $w_{i+1} \equiv p_i(h_i \gamma_i)q_i$. Since for each $i = 0, 1, \ldots, k$ the identity $u = w_i$ holds in $B_2$, and therefore, in the semigroup $A_0$, we can apply to each word $w_i$ the argument from the first paragraph of the proof of our claim. This gives for each $i = 0, 1, \ldots, k$ a (necessarily unique) decomposition

$$w_i \equiv w'_i w''_i$$

with $\alpha(w'_i) = \alpha(u')$, $\alpha(w''_i) = \alpha(u'')$ and $\alpha(w'_i) \cap \alpha(w''_i) = \emptyset$.

Given an index $i < k$, we want to analyze how the application of the identity $g_i = h_i$ to the word $w_i$ interacts with the decomposition (11). Observe that all words involved in the identity system (5) are connected and by Corollary 4 so are their images under endomorphisms of the free semigroup $\Sigma^+$. In particular, the factor $g_i \gamma_i$ of $w_i$ is connected whence it must occur in $w_i$ either before the bridge corresponding to the decomposition (11) or after this bridge. In the first case, we have $w'_i \equiv p_i(g_i \gamma_i)q'_i$ and $q_i \equiv q'_i w''_i$, see the left hand side of Fig. 3. Then

$$w_{i+1} \equiv p_i(h_i \gamma_i)q_i \equiv p_i(h_i \gamma_i)q'_i w''_i.$$

(12)

For every identity $g_i = h_i$ in the system (5) $\alpha(h_i) = \alpha(g_i)$ whence also $\alpha(p_i(h_i \gamma_i)q'_i) = \alpha(p_i(g_i \gamma_i)q'_i) = \alpha(w'_i)$. We see that the right hand side

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of (12) gives a decomposition of the word $w_{i+1}$ into a product of a word over $\text{alph}(w'_i) = \text{alph}(u')$ with a word over $\text{alph}(w''_i) = \text{alph}(u'')$. As mentioned, such a decomposition is unique but, on the other hand, $w_{i+1}$ decomposes as $w_{i+1} \equiv w'_{i+1}w''_{i+1}$ where again $\text{alph}(w'_{i+1}) = \text{alph}(u')$, $\text{alph}(w''_{i+1}) = \text{alph}(u'')$. Thus, we must have $w'_{i+1} \equiv p_i(h_i\gamma_i)q_i$ and $w''_{i+1} \equiv p''_i$. In the second case, when the factor $g_i\gamma_i$ of $w_i$ occurs within $w''_i$, we have $w''_i \equiv p''_i(g_i\gamma_i)q_i$ and $p_i \equiv w'_ip''_i$ (this situation is illustrated by the right hand side of Fig. 3). Repeating the above argument, we then conclude that $w'_{i+1} \equiv w'_i$ and $w''_{i+1} \equiv p''_i(h_i\gamma_i)q_i$.

We see that whenever the deduction step $w_i \Rightarrow w_{i+1}$ is ensured by an application of one of the identities (5), then also $w'_i \Rightarrow w'_{i+1}$ and $w''_i \Rightarrow w''_{i+1}$. Of course, the same conclusion holds true if the deduction step is trivial, that is, $w_{i+1} \equiv w_i$. Thus, the deduction (10) gives rise to the two deductions

$$u' \equiv w'_0 \Rightarrow w'_1 \Rightarrow \cdots \Rightarrow w'_k \equiv v',$$

$$u'' \equiv w''_0 \Rightarrow w''_1 \Rightarrow \cdots \Rightarrow w''_k \equiv v'',$$

that show that both the identities $u' = v'$ and $u'' = v''$ follow from the identity basis (5) of $\overline{B}_2$. Thus, the identities $u' = v'$ and $u'' = v''$ hold in the variety $\overline{B}_2$. As we already have proved that they hold in the variety $B_2$, they hold also in $V = B_2 \vee B_2$.

Since $|\text{alph}(u')|, |\text{alph}(u'')| < |\text{alph}(u)|$, our choice of the identity $u = v$ ensures that both the identities $u' = v'$ and $u'' = v''$ hold in the variety $\overline{A}_2$. However, together they obviously imply the identity $u = v$ that cannot hold in $\overline{A}_2$. This contradiction completes the proof of our claim.

We return to the proof of the main theorem. Consider $F_2(\overline{A}_2)$, the free semigroup of the variety $\overline{A}_2$ over the alphabet $\Sigma$, and let $\chi : \Sigma^+ \rightarrow F_2(\overline{A}_2)$ be the canonical homomorphism. By the above claim and Proposition 3, $u\chi$ and $v\chi$ are distinct regular elements of $F_2(\overline{A}_2)$. We are in a position to apply Kublanovskii’s lemma (Proposition 5) according to which there exists a completely 0-simple semigroup $K$ and a surjective homomorphism $\varphi : F_2(\overline{A}_2) \rightarrow K$ such that $(u\chi)\varphi \neq (v\chi)\varphi$. We will arrive to a final contradiction by analyzing the possible structure.
of the sandwich matrix $P$ of the semigroup $K$ in its presentation as a Rees matrix semigroup (cf. [4, Chapter 3]).

Since the homomorphism $\varphi$ is surjective, the semigroup $K$ belongs to the variety $\mathbb{A}_2$. This means, in particular, that the subgroups of $K$ are trivial whence all entries of the matrix $P$ are 1’s and possibly 0’s. Further, $K$ cannot contain a subsemigroup isomorphic to $A_2$ because $A_2 \notin \mathbb{A}_2$. The sandwich matrix in any presentation of $A_2$ as a Rees matrix semigroup over the trivial group is a $2 \times 2$-matrix with three entries equal to 1 and one entry equal to 0. Therefore none of $2 \times 2$-submatrices of the matrix $P$ can have exactly one entry equal to 0. It is easy to realize that, permuting rows and columns of such a matrix, one can collect all non-zero entries in rectangular blocks placed along the main diagonal. We may thus assume that the matrix $P$ is written in this block-diagonal form. Fig. 4 presents a typical example of such a block-diagonal matrix.

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 
\end{pmatrix}
$$

Fig. 4: A typical block-diagonal matrix

Now let $B_\omega$ be the countable Brandt semigroup over the trivial group and $R_{\omega \times \omega}$ the rectangular band with countably many rows and columns. It is well known (and easy to verify) that $B_\omega$ belongs to the variety $\mathbb{B}_2$ and it is obvious that $R_{\omega \times \omega}$ belongs to the variety $\overline{\mathbb{B}}_2$. In the direct product $B_\omega \times R_{\omega \times \omega}$ take the set $I$ of all pairs of the form $(0, r)$ with $r \in R_{\omega \times \omega}$. Clearly, $I$ is an ideal of $B_\omega \times R_{\omega \times \omega}$. It is well known (see, e.g., [18]) that the Rees quotient $B_\omega \times R_{\omega \times \omega}/I$ is a completely 0-simple semigroup which has a Rees matrix presentation with the sandwich matrix $Q$ being the Kronecker product of the sandwich matrices of $B_\omega$ and $R_{\omega \times \omega}$. The sandwich matrix of $B_\omega$ is the identity $\omega \times \omega$-matrix and the sandwich matrix of $R_{\omega \times \omega}$ is the $\omega \times \omega$-matrix filled by 1’s, whence $Q$ can be thought of as the block-diagonal matrix with countably many blocks of 1’s and with countably many rows and columns in each such block. Since the semigroup $F_\Sigma(\overline{A}_2)$ is countable, the semigroup $K$ is countable or finite, and the matrix $P$ has at most countably many rows and columns. Therefore we can select some rows and columns of the matrix $Q$ such that the submatrix formed by the intersection of these rows and columns coincides with $P$. This proves that the semigroup $K$ embeds into the semigroup $B_\omega \times R_{\omega \times \omega}/I$, and thus, $K$ is a subsemigroup of a homomorphic image of a direct
product of two semigroups from $\mathbb{B}_2 \cup \mathbb{B}_2$. Hence $K \in \mathbb{V} = \mathbb{B}_2 \vee \mathbb{B}_2$. Since the identity $u = v$ has been chosen to hold in $\mathbb{V}$, the images of the words $u$ and $v$ under the homomorphism $\chi \varphi : \Sigma^+ \to K$ must coincide. This contradicts the assumption that $(u \chi) \varphi \neq (v \chi) \varphi$ in $K$. The theorem is proved.

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