UDK 512.532

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ON A QUESTION BY EDMOND W. H. LEE*

Introduction

Let

$$A_2 = \langle a, b \mid aba = a^2 = a,\ bab = b,\ b^2 = 0\rangle = \{a, b, ab, ba, 0\}$$

be the 5-element idempotent-generated 0-simple semigroup. The semigroup A_2 as well as the 5-element Brandt semigroup

$$B_2 = \langle c, d \mid cdc = c,\ dcd = d,\ c^2 = d^2 = 0\rangle = \{c, d, cd, dc, 0\}$$

plays a distinguished role in the theory of semigroups and especially in the theory of semigroup varieties (see, for instance, a discussion in [1, Sections A5, A6, A15]). We denote by $\mathbf{A_2}$ and $\mathbf{B_2}$ the varieties generated by respectively A_2 and B_2. It was observed by N. R. Reilly (unpublished) that the variety $\mathbf{A_2}$ has a unique maximal subvariety which we denote by $\overline{\mathbf{A_2}}$. Clearly, $\overline{\mathbf{A_2}}$ can be thought of as the largest subvariety of $\mathbf{A_2}$ that does not contain the semigroup A_2. Similarly, the largest subvariety of $\mathbf{A_2}$ that does not contain the semigroup B_2 is denoted by $\overline{\mathbf{B_2}}$ (the existence of such a largest subvariety in each variety of periodic semigroups follows from a general result by E. V. Sukhanov, see [2, Theorem 1]). Recently E. W. H. Lee [3, Question 5.3] has asked whether or not

$$\overline{\mathbf{A_2}} = \overline{\mathbf{B_2}} \vee \mathbf{B_2} \qquad (1)$$

where the right hand side means the join of $\overline{\mathbf{B_2}}$ and $\mathbf{B_2}$ in the lattice of semigroup varieties. In the present note we answer this question in the affirmative.

1. Preliminaries

We adopt the standard terminology and notation of semigroup theory (see [1, 4–6]) and universal algebra (cf. [7]). For the reader's convenience, we recall a few basic definitions related to words.

*The author acknowledges support from the Federal Education Agency of Russia, grants 49123 and 04.01.437, the Russian Foundation for Basic Research, grant 05-01-00540, and the Federal Science and Innovation Agency of Russia, grant 2227.2003.01.

We fix a countably infinite set Σ (the *alphabet*) whose elements are referred to as *letters*. As usual, Σ^+ is the free semigroup over Σ and $\Sigma^* = \Sigma^+ \cup \{1\}$ is the free monoid over Σ. We call elements of Σ^+ *words* and denote the equality relation on Σ^+ by $\equiv$. If u, v are words, we say that u *occurs* in v or u is a *factor* in v whenever there exist $v', v'' \in \Sigma^*$ such that v can be decomposed as $v \equiv v'uv''$. For a word $w \in \Sigma^+$ we denote by $\mathrm{alph}(w)$ the set of letters from Σ that occur in w. If $w \equiv x_1x_2\cdots x_n$ where $x_1, x_2, \ldots, x_n$ are letters in $\mathrm{alph}(w)$, then the number n is called the *length* of the word w and is denoted by $|w|$.

Our proof of the equality (1) is based on a graph-theoretical description of the identities holding in A_2. This description is well known. It is often attributed to G. Mashevitsky [8] (see, for instance, [3] or [9]) even though the paper [8] does not deal with the identities of A_2 at all. Apparently, this mistake originates from an erroneous reference in the survey paper [10]. In fact, the description has been found by A. Trahtman, see his preprint [11].

Given a word $w \in \Sigma^+$, we assign it a directed graph $G(w)$ whose vertex set is $\mathrm{alph}(w)$ and whose edges correspond to factors of length 2 in w as follows: $G(w)$ has a directed edge from x to y ($x, y \in \mathrm{alph}(w)$) if and only if xy appears as a factor in w. We will distinguish two (not necessarily different) vertices in $G(w)$: the *initial vertex*, that is the first letter of w, and the *final vertex*, that is the last letter of w. Then the word w can be thought of as a walk through the graph $G(w)$ that starts at the initial vertex, ends at the final vertex and traverses each edge of $G(w)$ (some of the edges can be traversed more than once). Fig. 1 shows the graph

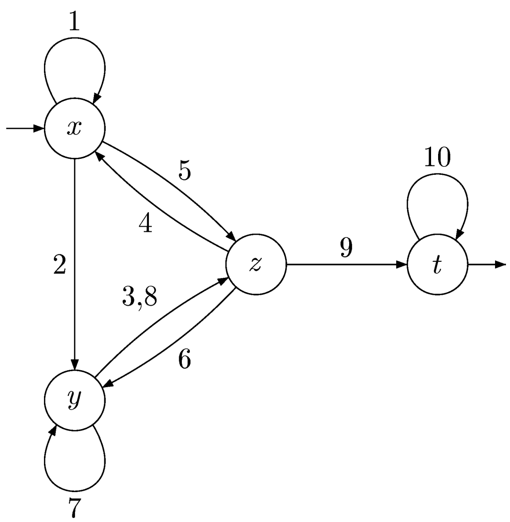

Fig. 1: The graph of the word $x^2yzxzy^2zt^2$ and the corresponding walk

$G(w)$ for the word $w \equiv x^2yzxzy^2zt^2$. The ingoing and the outgoing marks show respectively the initial and the final vertices of the graph. On Fig. 1 each edge of the graph is labelled by the number[s] corresponding to the occurrence[s] of the edge in the walk induced by the word w. We stress that, in contrast to the vertex names and the ingoing/outgoing marks, these labels are not considered as a part of the data making the graph $G(w)$. Therefore the graph does not determine the word w: for instance, as the reader can easily check, the word $xy^3zyzx^2zyzt^3$ has exactly the same graph (but corresponds to a different walk through it, see Fig. 2).

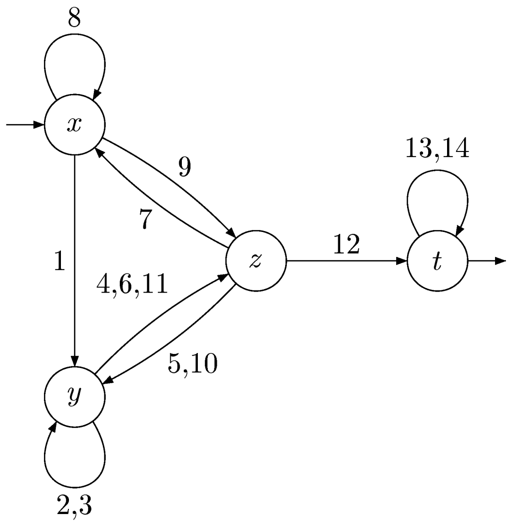

Fig. 2: Another walk through the graph of Fig. 1

Let $u, v \in \Sigma^+$ be words and S a semigroup. We say that S *satisfies the identity* $u = v$ (or that *the identity* $u = v$ *holds in* S) if $u\varphi = v\varphi$ for every homomorphism $\varphi : \Sigma^+ \to S$.

Proposition 1 (A. Trahtman, [11]). *The semigroup* A_2 *satisfies the identity* $u = v$ *if and only if the graphs* $G(u)$ *and* $G(v)$ *are equal.*

A system Ξ of identities is said to be an *identity basis* for a semigroup variety $\mathbf{V}$ if $\mathbf{V}$ consists precisely of semigroups which satisfy all identities in Ξ. In this situation we also say that $\mathbf{V}$ *is defined by* Ξ. For the sake of completeness, we list the identity bases for the varieties that play a major role in this note.

Proposition 2. (i) *The variety* $\mathbf{A_2}$ *is defined by the identities*

$$x^2 = x^3,\ xyx = xyxyx,\ xyxzx = xzxyx. \tag{2}$$

(ii) *The variety* $\mathbf{B_2}$ *is defined by the identities*

$$x^2 = x^3,\ xyx = xyxyx,\ x^2y^2 = y^2x^2. \tag{3}$$

(iii) *The variety* $\overline{\mathbf{A_2}}$ *is defined by the identities*

$$x^2 = x^3,\ xyx = xyxyx,\ xyxzx = xzxyx,\ x^2y^2x^2 = x^2yx^2. \tag{4}$$

(iv) *The variety* $\overline{\mathbf{B_2}}$ *is defined by the identities*

$$x^2 = x^3,\ xyx = xyxyx = xy^2x,\ xyxzx = xzxyx. \tag{5}$$

Proof. The identity bases (2) for $\mathbf{A_2}$ and (3) for $\mathbf{B_2}$ were found by A. Trahtman, see respectively [11] and [12]. The bases (4) for $\overline{\mathbf{A_2}}$ and (5) for $\overline{\mathbf{B_2}}$ were discovered by E. W. H. Lee [3, Theorems 2.7 and 3.6] via direct manipulations with identities. We outline here an alternative (and calculation-free) way to obtain the basis (5) which is most important for the proof of our main result.

Recall that $\overline{\mathbf{B_2}}$ is the largest subvariety of $\mathbf{A_2}$ that does not include the semigroup B_2. Clearly, $\overline{\mathbf{B_2}}$ satisfies the identities (2). By [2, Theorem 1] $\overline{\mathbf{B_2}}$ satisfies also the identity

$$(xy)^2(yx)^2(xy)^2 = (xy)^2.$$

Multiplying through on the right by x, we obtain the identity

$$xyxy^2xyx^2yxyx = xyxyx. \tag{6}$$

From Proposition 1 we see that the left hand side of (6) is equal in A_2 to the word $x^2y^2x^2$ while the right hand side reduces to xyx. Thus, $\overline{\mathbf{B_2}}$ satisfies the identity

$$x^2y^2x^2 = xyx. \tag{7}$$

Applying (7) to the word xy^2x, we obtain

$$xy^2x = x^2(y^2)^2x^2 = x^2y^2x^2 = xyx.$$

We conclude that the identity

$$xy^2x = xyx \tag{8}$$

holds in $\overline{\mathbf{B_2}}$. Since adding the identity (8) to the system (2) gives exactly the identity system (5), the variety $\overline{\mathbf{B_2}}$ is contained in the variety defined by the latter system. Conversely, the variety defined by (5) is a subvariety of $\mathbf{A_2}$ and does not include B_2 (because the identity (8) fails in B_2). Therefore, this variety is contained in $\overline{\mathbf{B_2}}$. Thus, $\overline{\mathbf{B_2}}$ is indeed defined by the identities (5).

A word $w \in \Sigma^+$ of length at least 2 is said to be *connected* if its directed graph $G(w)$ is strongly connected. In fact, this concept is known in the literature, although under a different name. In [14] G. Mashevitsky introduced the following definition: a word $w \in \Sigma^+$ of length at least 2 is said to be *covered by cycles* if each of its factor of length 2 occurs in a factor of w that starts and ends with the same letter. In the language of the graph $G(w)$, this property means that each edge $x \to y$ of $G(w)$ belongs to a directed cycle (namely, to the walk induced by a factor of w that starts and ends with the same letter and contains xy). It is one of the basic facts of the theory of directed graphs (cf. [13, Theorem 8.1.5]) that such a graph is strongly connected if and only if each its edge belongs to a directed cycle.

Our next proposition reveals the semigroup meaning of the concept of a connected word. Recall that an element s of a semigroup S is *regular in* S if there exists $s' \in S$ such that $ss's = s$.

Proposition 3. *A word* $w \in \Sigma^+$ *is connected if and only if for every semigroup* $S \in \mathbf{A_2}$ *and for every homomorphism* $\varphi : \Sigma^+ \to S$ *the element* $w\varphi$ *is regular in* S.

Proof. First suppose that w is a connected word, and let x and y be respectively the first and the last letters of w (we do not assume that $x \not\equiv y$). Since the graph $G(w)$ is strongly connected, there is a walk

$$y \equiv x_0 \to x_1 \to \cdots \to x_n \equiv x.$$

Let $u \equiv x_1 \cdots x_{n-1} w x_1 \cdots x_{n-1}$, then the graphs of the words w and wuw are equal. Here if $n = 0$ (which means that $y \equiv x$) or $n = 1$, the product $x_1 \cdots x_{n-1}$ interprets as the empty word. By Proposition 1 $w\psi = (wuw)\psi = w\psi \cdot u\psi \cdot w\psi$ where ψ denotes the canonical homomorphism from Σ^+ to $F_\Sigma(\mathbf{A_2})$, the free semigroup of the variety $\mathbf{A_2}$ over the alphabet Σ. Thus, $w\psi$ is regular in $F_\Sigma(\mathbf{A_2})$ and, since every homomorphism $\varphi : \Sigma^+ \to S$ factors through ψ, the element $w\varphi$ is regular in S.

Now suppose that w is not connected. This means that the graph $G(w)$ contains a bridge whence the walk induced by w splits into the part preceding the bridge, the bridge, and the part following the bridge. (The reader may see such a situation on Fig. 1 or 2 where the edge $z \to t$ forms a bridge.) Accordingly, w decomposes as $w \equiv w'w''$ where the words w' and w'' correspond to the parts of the walk respectively before and after the bridge. Clearly, $\mathrm{alph}(w') \cap \mathrm{alph}(w'') = \varnothing$. Consider the subsemigroup $A_0 = \{b, ba, ab, 0\}$ of A_2 and let the homomorphism $\zeta : \Sigma^+ \to A_0$ be defined as follows:

$$x\zeta = \begin{cases} ba & \text{if } x \in \mathrm{alph}(w'), \\ ab & \text{otherwise.} \end{cases}$$

Then using the defining relations of the semigroup A_2, one readily calculates that

$$w\zeta = w'\zeta \cdot w''\zeta = (ba)^{|u|} \cdot (ab)^{|v|} = ba \cdot ab = bab = b.$$

However, it is easy to check that b is not regular in A_0.

Remark. The fact that, under the canonical homomorphism $\Sigma^+ \to F_\Sigma(\mathbf{A_2})$, every word covered by cycles maps onto a regular element of $F_\Sigma(\mathbf{A_2})$ is a partial case of a similar result claimed by G. Mashevitsky in [14, Lemma 6], see also [15, Lemma 7]. Since then, this result has been used (with a reference to [14]) in several important papers including, for instance, [16] and [17]. However, its proof in [14] contains a fatal flaw (and so does the translation of the proof into English published in [15]). Namely, in [14] Lemma 6 is deduced from Lemma 5 which claims that every word u covered by cycles can be transformed modulo certain identities to a word of the form $z_1u_1z_1 \cdots z_ku_kz_k$ where $z_1, \ldots, z_k$ are letters and $z_{i+1} \in \mathrm{alph}(u_i)$ for all $i = 1, \ldots, k-1$ provided that $k > 1$. In order to justify the latter claim, Mashevitsky uses induction on $|\,\mathrm{alph}(u)|$ but in the course of the proof he illegitimately applies the induction assumption to a factor of u that generally speaking is not covered by cycles. The word $u \equiv xyxzy$ can be used as a concrete counter example showing that the argument from [14] does not work: here the induction assumption should have been applied to the factor zy which is certainly not covered by cycles.

In fact, a correct proof of the described intermediate claim can be achieved by simple graph-theoretic means, and moreover, the claim can be avoided because we can prove Lemma 6 of [14] by a suitable modification of reasoning applied in the above proof of Proposition 3. Thus, results of [16] and [17] that rely on the lemma are correct.

Proposition 3 also allows us to give a semigroup proof of the following combinatorial property:

Corollary 4. *If a word $w \in \Sigma^+$ is connected, then for every homomorphism $\gamma : \Sigma^+ \to \Sigma^+$ the word $w\gamma$ is connected.*

Proof. Take an arbitrary semigroup $S \in \mathbf{A_2}$ and an arbitrary homomorphism $\varphi : \Sigma^+ \to S$. Then $(w\gamma)\varphi = w(\gamma\varphi)$ is regular by the “only if” part of Proposition 3 whence the word $w\gamma$ is connected by the “if” part of the proposition.

We will utilize the following partial case of an important lemma due to S. Kublanovskii, see [16, Lemma 3.2]:

Proposition 5 (S. Kublanovskii, [16])**.** *For any semigroup $S \in \mathbf{A_2}$ and distinct regular elements $s, s' \in S$ there exists a completely 0-simple semigroup K and a surjective homomorphism $\varphi : S \to K$ such that $s\varphi \neq s'\varphi$.*

2. The main result and its proof

Recall that the *join* $\mathbf{X} \vee \mathbf{Y}$ of two semigroup varieties $\mathbf{X}$ and $\mathbf{Y}$ is the least variety containing both $\mathbf{X}$ and $\mathbf{Y}$; in other words, $\mathbf{X} \vee \mathbf{Y}$ is the closure of the class-theoretic union $\mathbf{X} \cup \mathbf{Y}$ under the operators of taking homomorphic images, subsemigroups, and direct products. As mentioned in the introduction, our main result is the following

Theorem. $\overline{\mathbf{A_2}} = \overline{\mathbf{B_2}} \vee \mathbf{B_2}$.

Proof. Let $\mathbf{V} = \overline{\mathbf{B_2}} \vee \mathbf{B_2}$. Since $\overline{\mathbf{A_2}}$ is the largest proper subvariety in $\mathbf{A_2}$ and both $\overline{\mathbf{B_2}}$ and $\mathbf{B_2}$ are proper subvarieties in $\mathbf{A_2}$, we conclude that $\mathbf{V} \subseteq \overline{\mathbf{A_2}}$. Arguing by contradiction, assume that this inclusion is strict. Then there exists an identity that holds in the variety $\mathbf{V}$ but fails in the variety $\overline{\mathbf{A_2}}$. We choose an identity $u = v$ with this property with minimum possible value of $|\operatorname{alph}(u)|$. Observe that necessarily $\operatorname{alph}(u) = \operatorname{alph}(v)$ – otherwise the identity $u = v$ would fail in the two element semilattice and could not be satisfied by $\mathbf{V}$.

The core of the proof consists in verifying the following

Claim. *The words u and v are connected.*

Proof. Suppose that the word u is not connected. Then arguing as in the proof of Proposition 3 we may decompose u as a product of two non-empty words u' and u'' such that $\operatorname{alph}(u') \cap \operatorname{alph}(u'') = \varnothing$. Now we make use of the fact that the subsemigroup $A_0 = \{b, ba, ab, 0\}$ of A_2 belongs to the variety $\overline{\mathbf{B_2}}$, and therefore, it must satisfy the identity $u = v$. Let the homomorphism $\zeta : \Sigma^+ \to A_0$ be defined as follows:

$$x\zeta = \begin{cases} ba & \text{if } x \in \operatorname{alph}(u'), \\ ab & \text{otherwise.} \end{cases} \tag{9}$$

As in the proof of Proposition 3, one readily obtains that $u\zeta = b$. Hence also $v\zeta = b$. By (9) $v\zeta$ is a product of the idempotents ab and ba in some order. Since $ab \cdot ba = 0$, if such a product is not equal to 0, then no occurrence of ab precedes an occurrence of ba. This implies that in the word v no occurrence of a letter from $\operatorname{alph}(u'')$ precedes an occurrence of a letter from $\operatorname{alph}(u')$. Therefore v decomposes as $v \equiv v'v''$ where $\operatorname{alph}(v') = \operatorname{alph}(u')$, $\operatorname{alph}(v'') = \operatorname{alph}(u'')$.

We want to show that both the identities $u' = v'$ and $u'' = v''$ hold in the variety $\mathbf{V}$. First consider an arbitrary homomorphism $\varphi : \Sigma^+ \to B_2$. Suppose that $u'\varphi \neq v'\varphi$. Then one of the two elements is not equal to 0; without any loss we may assume that $s = u'\varphi \neq 0$. Let s' be the unique inverse of s in B_2. then $s's$ is a non-zero idempotent and, as one can easily verify, for any $r \in B_2$ the product $rs's$ is equal to either r or 0. In particular,

$$u'\varphi \cdot s's = s \neq v'\varphi \cdot s's \in \{v'\varphi, 0\}.$$

Now we define a "modification" ξ of the homomorphism φ by letting

$$x\xi = \begin{cases} x\varphi & \text{if } x \in \text{alph}(u'), \\ s's & \text{otherwise.} \end{cases}$$

Taking into account the equalities $\text{alph}(v') = \text{alph}(u')$, $\text{alph}(v'') = \text{alph}(u'')$ and $\text{alph}(u') \cap \text{alph}(u'') = \varnothing$, we obtain

$$u\xi = u'\xi \cdot u''\xi = u'\varphi \cdot s's \neq v'\varphi \cdot s's = v'\xi \cdot v''\xi = v\xi.$$

This contradicts the assumption that the identity $u = v$ holds in the variety $\mathbf{V}$ and hence in the semigroup B_2. Thus, we conclude that $u'\varphi = v'\varphi$ under every homomorphism $\varphi : \Sigma^+ \to B_2$, that is, B_2 satisfies the identity $u' = v'$. By the left-right symmetry of B_2, it also satisfies the identity $u'' = v''$.

Since the identity $u = v$ holds in the variety $\overline{\mathbf{B}}_\mathbf{2}$, then by Birkhoff's Completeness Theorem for Equational Logic (cf. [7, Theorem 14.19]) there is a *deduction* of $u = v$ from the identity basis (5) of $\overline{\mathbf{B}}_\mathbf{2}$. Recall that such a deduction is a sequence of words

$$u \equiv w_0 \Rightarrow w_1 \Rightarrow \cdots \Rightarrow w_k \equiv v \tag{10}$$

in which $w_i \Rightarrow w_{i+1}$ $(i = 0, 1, \ldots, k-1)$ denotes that either $w_{i+1} \equiv w_i$ or w_{i+1} is obtained from w_i by a single application of an identity $g_i = h_i$ from the system (5), that is, there exist $p_i, q_i \in \Sigma^*$ and an endomorphism $\gamma_i : \Sigma^+ \to \Sigma^+$ such that $w_i \equiv p_i(g_i\gamma_i)q_i$ and $w_{i+1} \equiv p_i(h_i\gamma_i)q_i$. Since for each $i = 0, 1, \ldots, k$ the identity $u = w_i$ holds in $\overline{\mathbf{B}}_\mathbf{2}$, and therefore, in the semigroup A_0, we can apply to each word w_i the argument from the first paragraph of the proof of our claim. This gives for each $i = 0, 1, \ldots, k$ a (necessarily unique) decomposition

$$w_i \equiv w_i' w_i'' \tag{11}$$

with $\text{alph}(w_i') = \text{alph}(u')$, $\text{alph}(w_i'') = \text{alph}(u'')$ and $\text{alph}(w_i') \cap \text{alph}(w_i'') = \varnothing$.

Given an index $i < k$, we want to analyze how the application of the identity $g_i = h_i$ to the word w_i interacts with the decomposition (11). Observe that all words involved in the identity system (5) are connected and by Corollary 4 so are their images under endomorphisms of the free semigroup Σ^+. In particular, the factor $g_i\gamma_i$ of w_i is connected whence it must occur in w_i either before the bridge corresponding to the decomposition (11) or after this bridge. In the first case, we have $w_i' \equiv p_i(g_i\gamma_i)q_i'$ and $q_i \equiv q_i' w_i''$, see the left hand side of Fig. 3. Then

$$w_{i+1} \equiv p_i(h_i\gamma_i)q_i \equiv p_i(h_i\gamma_i)q_i' \cdot w_i''. \tag{12}$$

For every identity $g_i = h_i$ in the system (5) $\text{alph}(h_i) = \text{alph}(g_i)$ whence also $\text{alph}(p_i(h_i\gamma_i)q_i') = \text{alph}(p_i(g_i\gamma_i)q_i') = \text{alph}(w_i')$. We see that the right hand side

Fig. 3: Two possible ways for applying the identity $g_i = h_i$ to the word w_i

of (12) gives a decomposition of the word w_{i+1} into a product of a word over $\mathrm{alph}(w'_i) = \mathrm{alph}(u')$ with a word over $\mathrm{alph}(w''_i) = \mathrm{alph}(u'')$. As mentioned, such a decomposition is unique but, on the other hand, w_{i+1} decomposes as $w_{i+1} \equiv \equiv w'_{i+1}w''_{i+1}$ where again $\mathrm{alph}(w'_{i+1}) = \mathrm{alph}(u')$, $\mathrm{alph}(w''_{i+1}) = \mathrm{alph}(u'')$. Thus, we must have $w'_{i+1} \equiv p_i(h_i\gamma_i)q'_i$ and $w''_{i+1} \equiv w''_i$. In the second case, when the factor $g_i\gamma_i$ of w_i occurs within w''_i, we have $w''_i \equiv p''_i(g_i\gamma_i)q_i$ and $p_i \equiv w'_ip''_i$ (this situation is illustrated by the right hand side of Fig. 3). Repeating the above argument, we then conclude that $w'_{i+1} \equiv w'_i$ and $w''_{i+1} \equiv p''_i(h_i\gamma_i)q_i$.

We see that whenever the deduction step $w_i \Rightarrow w_{i+1}$ is ensured by an application of one of the identities (5), then also $w'_i \Rightarrow w'_{i+1}$ and $w''_i \Rightarrow w''_{i+1}$. Of course, the same conclusion holds true if the deduction step is trivial, that is, $w_{i+1} \equiv w_i$. Thus, the deduction (10) gives rise to the two deductions

$$u' \equiv w'_0 \Rightarrow w'_1 \Rightarrow \cdots \Rightarrow w'_k \equiv v',$$
$$u'' \equiv w''_0 \Rightarrow w''_1 \Rightarrow \cdots \Rightarrow w''_k \equiv v'',$$

that show that both the identities $u' = v'$ and $u'' = v''$ follow from the identity basis (5) of $\overline{\mathbf{B_2}}$. Thus, the identities $u' = v'$ and $u'' = v''$ hold in the variety $\overline{\mathbf{B_2}}$. As we already have proved that they hold in the variety $\mathbf{B_2}$, they hold also in $\mathbf{V} = \overline{\mathbf{B_2}} \vee \mathbf{B_2}$.

Since $|\,\mathrm{alph}(u')|, |\,\mathrm{alph}(u'')| < |\,\mathrm{alph}(u)|$, our choice of the identity $u = v$ ensures that both the identities $u' = v'$ and $u'' = v''$ hold in the variety $\overline{\mathbf{A_2}}$. However, together they obviously imply the identity $u = v$ that cannot hold in $\overline{\mathbf{A_2}}$. This contradiction completes the proof of our claim.

We return to the proof of the main theorem. Consider $F_\Sigma(\overline{\mathbf{A_2}})$, the free semigroup of the variety $\overline{\mathbf{A_2}}$ over the alphabet Σ, and let $\chi : \Sigma^+ \to F_\Sigma(\overline{\mathbf{A_2}})$ be the canonical homomorphism. By the above claim and Proposition 3, $u\chi$ and $v\chi$ are distinct regular elements of $F_\Sigma(\overline{\mathbf{A_2}})$. We are in a position to apply Kublanovskii's lemma (Proposition 5) according to which there exists a completely 0-simple semigroup K and a surjective homomorphism $\varphi : F_\Sigma(\overline{\mathbf{A_2}}) \to K$ such that $(u\chi)\varphi \neq \neq (v\chi)\varphi$. We will arrive to a final contradiction by analyzing the possible structure of the sandwich matrix P of the semigroup K in its presentation as a Rees matrix semigroup (cf. [4, Chapter 3]).

Since the homomorphism φ is surjective, the semigroup K belongs to the variety $\overline{\mathbf{A_2}}$. This means, in particular, that the subgroups of K are trivial whence all entries of the matrix P are 1's and possibly 0's. Further, K cannot contain a subsemigroup isomorphic to A_2 because $A_2 \notin \overline{\mathbf{A_2}}$. The sandwich matrix in any presentation of A_2 as a Rees matrix semigroup over the trivial group is a 2×2-matrix with three entries equal to 1 and one entry equal to 0. Therefore none of 2×2-submatrices of the matrix P can have exactly one entry equal to 0. It is easy to realize that, permuting rows and columns of such a matrix, one can collect all non-zero entries in rectangular blocks placed along the main diagonal. We may thus assume that the matrix P is written in this block-diagonal form. Fig. 4 presents a typical example of such a block-diagonal matrix.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Fig. 4: A typical block-diagonal matrix

Now let B_ω be the countable Brandt semigroup over the trivial group and $R_{\omega\times\omega}$ the rectangular band with countably many rows and columns. It is well known (and easy to verify) that B_ω belongs to the variety $\mathbf{B_2}$ and it is obvious that $R_{\omega\times\omega}$ belongs to the variety $\overline{\mathbf{B_2}}$. In the direct product $B_\omega \times R_{\omega\times\omega}$ take the set I of all pairs of the form $(0,r)$ with $r \in R_{\omega\times\omega}$. Clearly, I is an ideal of $B_\omega \times R_{\omega\times\omega}$. It is well known (see, e. g., [18]) that the Rees quotient $B_\omega \times R_{\omega\times\omega}/I$ is a completely 0-simple semigroup which has a Rees matrix presentation with the sandwich matrix Q being the Kronecker product of the sandwich matrices of B_ω and $R_{\omega\times\omega}$. The sandwich matrix of B_ω is the identity $\omega \times \omega$-matrix and the sandwich matrix of $R_{\omega\times\omega}$ is the $\omega \times \omega$-matrix filled by 1's, whence Q can be thought of as the block-diagonal matrix with countably many blocks of 1's and with countably many rows and columns in each such block. Since the semigroup $F_\Sigma(\overline{\mathbf{A_2}})$ is countable, the semigroup K is countable or finite, and the matrix P has at most countably many rows and columns. Therefore we can select some rows and columns of the matrix Q such that the submatrix formed by the intersection of these rows and columns coincides with P. This proves that the semigroup K embeds into the semigroup $B_\omega \times R_{\omega\times\omega}/I$, and thus, K is a subsemigroup of a homomorphic image of a direct product of two semigroups from $\overline{\mathbf{B_2}} \cup \mathbf{B_2}$. Hence $K \in \mathbf{V} = \overline{\mathbf{B_2}} \vee \mathbf{B_2}$. Since the identity $u = v$ has been chosen to hold in $\mathbf{V}$, the images of the words u and v under the homomorphism $\chi\varphi : \Sigma^+ \to K$ must coincide. This contradicts the assumption that $(u\chi)\varphi \neq (v\chi)\varphi$ in K. The theorem is proved.

Acknowledgment. The author is very much grateful to E. W. H. Lee for a number of stimulating discussions.

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Статья поступила 05.01.2005 г.