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THE DEGENERATE CAUCHY PROBLEM IN BANACH SPACES

1. Introduction

We consider the well-posedness of the abstract Cauchy problem for the second order complete equation in regular and degenerate cases:

$$u''(t) = Au'(t) + Bu(t), \quad t \geq 0,\ u(0) = x,\ u'(0) = y, \tag{1}$$

$$Qu''(t) = Au'(t) + Bu(t), \quad t \geq 0,\ u(0) = x,\ u'(0) = y,\ \ker Q \neq \{0\}. \tag{2}$$

The complete equation (1) attracts particular interest because of its specificity as compared to the equation with $A = 0$ and the first order equation:

- a solution of the complete equation depends on the interconnection between the operators A, B;
- generally speaking, a solution of the well-posed Cauchy problem may be not exponentially bounded [1].

The Cauchy problem (1) was investigated for different classes of A, B. In [2]–[4] we have constructed the theory of M, N-functions for commuting operators A, B thus generalizing the theory of C, S-functions [5]. On the basis of this theory, we obtained necessary and sufficient conditions for the well-posedness of (1) in terms of the operator $R(\lambda^2) := (\lambda^2 - \lambda A - B)^{-1}$ called the resolvent of the operators A, B. In [6] an application of the M, N-theory without commutativity of A, B caused additional conditions on A, B. These conditions similar to those in [3] ensure the equivalence between the well-posedness of (1) and the well-posedness of the first order Cauchy problem in the product space:

$$v'(t) = Tv(t), \quad t \geq 0,\ v(0) = v_0, \tag{3}$$

$$T = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}, \quad v(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \quad v_0 = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The connection between (1) and (3) was used for the investigation of (1) in [7], [8] and others, the technique of propogators was used in [1], [9]. It seemed that the recently created integrated semigroup theory [10]–[12] would allow to solve

the well-posedness problem for (1) on the basis of the integrated semigroup technique for (3) or another system to which (1) might be reduced. However, a result of such type was obtained in [13] only for the case when the operator A is bounded. For such an operator, it was shown that the well-posedness of (1) is equivalent to the existence of an integrated semigroup with the generator T. On this basis, necessary and sufficient conditions were obtained. We shall show that, in the general case, the above equivalence may be proved for biclosed operators A, B.

In the paper, with the help of M, N-functions for the commuting operators A, B, we prove that the MFPHY-type condition

$$\exists K > 0, \omega \geq 0 \ \|R^{(k)}(\lambda^2)\|, \ \|R_1^{(k)}(\lambda^2)\| \leq K\, k!/(\Re\lambda - \omega)^{k+1}, \tag{4}$$
$$k = 0, 1, 2, \ldots, \ \Re\lambda > \omega, \ R_1(\lambda^2) := (\lambda - A)R(\lambda^2),$$

is necessary and sufficient for the well-posedness of (1). In this case, the operators A, B are biclosed. With the help of the integrated semigroup theory for the biclosed operators A, B, we prove that the following MFPHY-type condition

$$\exists K > 0, \omega \geq 0 \ \|R^{(k)}(\lambda^2)\|, \ \|R_2^{(k)}(\lambda^2)\| \leq K\, k!/(\Re\lambda - \omega)^{k+1}, \tag{5}$$
$$k = 0, 1, 2, \ldots, \ \Re\lambda > \omega, \quad R_2(\lambda^2) := \overline{R(\lambda^2))(\lambda - A)},$$

is necessary and sufficient for the well-posedness of (1). If the operator A is bounded, (5) coincides with the condition obtained in [13]: $\overline{R(\lambda^2))(\lambda - A)} = R(\lambda^2))(\lambda - A)$. If the operators A, B commute, the condition (5) coincides with (4).

Moreover, using the theory of degenerate integrated semigroups introduced in [14],[15], we pioneered in obtaining MFPHY-type necessary and sufficient conditions for the well-posedness of the degenerate Cauchy problem (2).

Different sufficient conditions for the first and second order degenerate Cauchy problems have been obtained in [16]–[20].

2. The M, N-function method in the regular case

We consider the Cauchy problem (1) where A, B are closed operators in a Banach space E.

Definition 2.1. *The Cauchy problem is called ω-well-posed on E_1, E_2 if, for any $x \in E_1$, $y \in E_2$, (1) has a unique solution which is exponentially bounded:*

$$\exists K > 0, \omega \geq 0 \ \|u(t)\| \leq K \exp(\omega t)\Big(\|x\| + \|y\|\Big).$$

Definition 2.2. *A one-parameter family of bounded commuting operators $M(t), N(t)$ is called an ω-strongly continuous family of M, N-functions (generated by the operators A, B) if the following conditions hold:*

(M1) $M(t+h) = M(t)M(h) + BN(t)N(h)$,
$N(t+h) = M(t)N(h) + M(h)N(t) + AN(t)N(h)$,
$N(t)Ax = AN(t)x,\ x \in D(A),\ N(t)Bx = BN(t)x,\ x \in D(B),\ t, h \geq 0$;

(M2) $N(0) = 0,\ M(0) = I,\ N'(0) = I,\ M'(0) = 0$;

(M3) $M(t), N(t)$ *are strongly continuous for* $t \geq 0$;

(M4) $\exists K > 0, \omega \geq 0\ \|M(t)\|, \|N(t)\| \leq K \exp(\omega t)$.

The operators A, B are called the generators of the family of M, N-functions.

From definition 2.2 it follows that the operators

$$M''(0)x := \lim_{h \to 0} \frac{M(2h) - 2M(h) + I}{h^2}\, x,$$

$$N''(0)x := \lim_{h \to 0} \frac{N(2h) - 2N(h)}{h^2}\, x$$

defined for $x \in E$ such that the limit exists coincide with the closed operators B, A, respectively.

Definition 2.3. *The operators A, B are said to be ω-closed if $\lambda A + B$ is a closed operator for $\Re\lambda > \omega$; A, B are biclosed if, for any $x_n \in D(A),\ x_n \to x$, and any $y_n \in D(B),\ y_n \to y$, such that $Ax_n + By_n \to z$, it follows $x \in D(A)$, $y \in D(B)$ and $Ax + By = z$.*

It is not difficult to see that if A, B are biclosed, then A, B are ω-closed and both the operators are closed. The following two examples demonstrate that if A, B are closed operators, they may be not ω-closed, and if A, B are ω-closed, they may be not biclosed.

1. $A := d^2/d^2s + d/ds, B := -c\, d^2/d^2s : C[a,b] \to C[a,b]$.

The operators A, B are closed, but they are not ω-closed for $\omega < c$. Indeed, let x_n be such that $D(A) = D(B) \ni x_n(s) \to x(s),\ x_n'(s) \to y(s)$, but $x_n''(s)$ does not converge in $C[a,b]$, then $(cA + B)x_n \to cy$, but $x \notin D(A) \cap D(B)$.

2. $A := d/ds, B := d^2/d^2s : C[a,b] \to C[a,b]$.

The operators A, B are ω-closed for any ω. The sequences $x_n(s) \in D(A)$ and

$y_n(s) = -\int_0^s x_n(t)dt$ such that $x_n \to x$, but x_n' is not convergent in $C[a,b]$, demonstrate that A, B are not biclosed.

Any pair consisting of a closed operator and a bounded operator or of closed operators with ranges in mutually orthogonal subspaces gives an example of biclosed operators.

Denote $d_1 = D(AB)$, $d_2 = D(A^2) \cap D(B)$. Then we have the following necessary and sufficient conditions for the well-posedness of the Cauchy problem (1), cf. [4],[2]:

Theorem 2.1. *Let A, B be closed commuting linear operators with domains such that*

$$\overline{d}_1 = \overline{d}_2 = E. \tag{d}$$

Then the following statements are equivalent:

(i) *the Cauchy problem* (1) *is ω-well-posed on d_1, d_2;*

(ii) *the operators A, B are generators of an ω-strongly continuous family of M, N-functions (in this case the solution has the form*

$$u(t) = M(t)x + N(t)y, \quad x \in d_1,\ y \in d_2);$$

(iii) *For $\Re\lambda > \omega$, there exists $R(\lambda^2)$, the resolvent of the operators A, B, and the condition* (4) *for $R(\lambda^2)$ holds.*

In order to observe the interplay between A and B, we give a pattern of the proof.

(i) $\Rightarrow$ (ii) Let us define the bounded operators $M(t), N(t)$ as the solution operators for the initial values $(x, 0), (0, y)$, respectively. Since A, B commute and the solution is unique, $M(t), N(h)$ $(t, h \geq 0)$ commute and M, N commute with A, B on $D(A), D(B)$, respectively. This implies (M1). (M2) follows from the choice of the initial values. Since the solution is continuous and exponentially bounded, (M3) and (M4) take place as well.

(ii) $\Rightarrow$ (iii) As the interdependence properties of A, B are under consideration, we pay a special attention to the proof of the relations:

$$R(\lambda^2)x = \int_0^\infty \exp(-\lambda t)N(t)x dt,\ x \in E, \tag{6}$$

$$R_1(\lambda^2)x = (\lambda - A)R(\lambda^2)x = \int_0^\infty \exp(-\lambda t)M(t)x dt,\ x \in E. \tag{7}$$

These relations and the exponential estimates for M, N-functions imply (4). From the commutativity of M, N-functions and the definition of generators we obtain the following equalities for derivatives:

(a) $\forall u \in D(B)\ M'(t)u = N(t)Bu = BN(t)u,$
$\forall u \in D(A)\ N'(t)u = M(t)u + N(t)Au = M(t)u + AN(t)u, \quad t \geq 0;$

(b) $\forall u \in D(AB)\ M''(t)u = BM(t)u + ABN(t)u = AM'(t)u + BM(t)u,$
$\forall u \in D(A^2) \cap D(B)\ N''(t)u = AN'(t)u + BN(t)u.$

From (a),(b) it follows that $u(t) = M(t)x + N(t)y,\ x \in d_1,\ y \in d_2$, is a solution of the Cauchy problem (1). Since M, N-functions are exponentially bounded, the solution is stable. Its uniqueness is proved on the basis of the estimates (4) for $R(\lambda^2)$, which in turn follow from the relations (6),(7). Let us prove (6) (the relation (7) may be proved by similar arguments). We apply the operator $\lambda^2 - \lambda A - B$ to $\int_0^\infty \exp(-\lambda t)N(t)xdt$ and integrate by parts. Using (a),(b) and the following (weaker than ω-closedness and hence biclosedness) property of the generators of the M, N-family:

$$\forall x_n \in D(A) \cap D(B),\ x_n \to x \in D(B)\ \&\ Px_n \to y \in D(P) \Rightarrow$$

$$\Rightarrow x \in D(P)\ \&\ Px = y \quad (P := \lambda A + B = \lambda N''(0) + M''(0),\ \Re\lambda > \omega),$$

we obtain

$$(\lambda^2 - \lambda A - B)\int_0^\infty \exp(-\lambda t)N(t)xdt = x,\ x \in X,$$

$$\int_0^\infty \exp(-\lambda t)N(t)(\lambda^2 - \lambda A - B)xdt = x,\ x \in D(A) \cap D(B),$$

and hence (6). The introduced property of P may be proved similarly to the closedness property for the generator of a strongly continuous semigroup [4],[8].

(iii) $\Rightarrow$ (i) is proved similarly to the proof for incomplete equations [5],[2].

3. The integrated semigroup method in the regular case

We consider (3) with the operator T generating an integrated semigroup in $X := E \times E$, $\|(x, y)\| = \max\{\|x\|_E, \|y\|_E\}$.

Definition 3.1. *Let $n \in \mathbb{N}$. A one-parameter family of linear bounded operators $\{V(t),\ t \geq 0\}$ is called n-times integrated (exponentially bounded and nondegenerate) semigroup if the following conditions are fulfilled:*

(V1) $\int_0^s [(s-r)^{n-1}V(t+r) - (t+s-r)^{n-1}V(r)]dr = (n-1)!V(t)V(s)$, $s,t \geq 0$;

(V2) $V(t)$ *is strongly continuous for* $t \geq 0$;

(V3) $\exists K > 0, \omega \in \mathbb{R}: \ \|V(t)\| \leq K\exp(\omega t), \ t \geq 0$;

(V4) $\forall t \geq 0, \ V(t)x = 0 \ \Rightarrow \ x = 0$.

The operator $Tx := \lambda x - \mathcal{R}(\lambda)^{-1}x, \ D(T) = range\mathcal{R}(\lambda)$, *where*

$$\mathcal{R}(\lambda) = \int_0^\infty \lambda^n \exp(-\lambda t)V(t)dt,$$

is called the generator of $V(t)$.

Definition 3.2. *The Cauchy problem* (3) *is said to be* (n,ω)*-well-posed on* $\mathcal{X} \subset D(T^{n+1})$ *if, for any* $v_0 \in \mathcal{X}$,
(a) *there exists a unique solution* $v(t) \in C\{[0,\infty),[D(T)]\} \bigcap C^1\{[0,\infty),X\}$;
(b) $\exists K > 0, \omega \in \mathbb{R} \ \|v(t)\| \leq K\exp(\omega t)\|v_0\|_n, \ \|v_0\|_n := \|v_0\| + \|Tv_0\| + \cdots + \|T^n v_0\|$.

If $\mathcal{X} = D(T^{n+1})$, *the problem* (3) *is called* (n,ω)*-well-posed.*

For any Banach spaces X, Y, denote by $\mathcal{L}(X,Y)$ the set of all linear operators from X to Y (if $X = Y$ we write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$), by $\mathcal{B}(X,Y)$ the set of all bounded linear operators from X to Y. Denote by $\rho(T)$ the resolvent set of T, by $\rho(A,B)$ the set of λ such that $R(\lambda^2) \in \mathcal{B}(E)$. If $\rho(T) \neq \emptyset$, the norm $\|x\|_n$ in Definition 3.2 is equivalent to the norm

$$\|x\|_n := \inf_{y:\ \mathcal{R}_A(\lambda)y = x} \|y\|, \quad \lambda \in \rho(T).$$

Theorem 3.1. [10], [11], [15]. *Let* $T \in \mathcal{L}(X)$ *be a densely defined operator with* $\rho(T) \neq \emptyset$. *Then the following statements are equivalent:*

(T1) *the operator* T *is the generator of* n*-times integrated semigroup* $V(t)$;

(T2) *the Cauchy problem* (3) *is* (n,ω)*-well-posed;*

(T3) *for* $\lambda > \omega$, *there exists the resolvent* $\mathcal{R}(\lambda)$ *of the operator* T *and*

$$\exists K > 0: \forall \lambda > \omega, \ \|[\mathcal{R}(\lambda)/\lambda^n]^{(k)}\| \leq Kk!/(\lambda-\omega)^{k+1}, \ k = 0,1,2,\ldots \ . \quad (8)$$

Theorem 3.2. *Let $A, B \in \mathcal{L}(E)$ be biclosed operators satisfying the density property* (d)*, and $\rho(A, B) \neq \emptyset$. Then the following statements are equivalent:*

(i) *the operator* $T = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ *is the generator of a* 1*-time integrated semigroup in* $X = E \times E$;

(ii) *the condition* (5) *is fulfilled;*

(iii) *the Cauchy problem* (1) *is ω-well-posed on d_1, d_2.*

Proof. (i) $\Rightarrow$ (ii) From the conditions of the theorem it follows that the operator T is closed (T is closed if and only if A, B are biclosed) and $\overline{D(T)} = X$. Since T is the generator of a 1-time integrated semigroup, the condition (8) with $n = 1$ is fulfilled for the operator

$$\mathcal{R}(\lambda) = (\lambda - T)^{-1} = \begin{pmatrix} \overline{R(\lambda^2)(\lambda - A)} & R(\lambda^2) \\ \overline{R(\lambda^2)B} & \lambda R(\lambda^2) \end{pmatrix}.$$

This condition is equivalent to the MFPHY-type conditions (5) for $R(\lambda^2)$.

(ii) $\Rightarrow$ (iii) The condition (5) is equivalent to (8) with $n = 1$, hence T is the generator of the 1-time integrated semigroup $V(t)$, and the Cauchy problem (3) is $(1, \omega)$-well-posed. That is, for all $v_0 \in D(T^2) = D(AB) \times D(A^2) \cap D(B)$, there exists a unique solution $v(t) = V'(t)v_0$ such that

$$\|v(t)\| = \|(u(t), u'(t))\| \leq K \exp(\omega t)\|v_0\|_1.$$

Then $w(t) = \mathcal{R}(\lambda)v(t)$ is the solution of (3) with the initial value $w(0) = \mathcal{R}(\lambda)v_0$ and with the stability property

$$\|w(t)\| \leq K \exp(\omega t)\|\mathcal{R}(\lambda)v_0\|_1 \leq K \exp(\omega t)\|v_0\|.$$

Applying $\mathcal{R}(\lambda)$ to the equation (3) and integrating the result, we obtain the equality

$$\int_0^t v(s)ds = -w(t) + w(0) + \lambda \int_0^t w(s)ds. \tag{9}$$

Hence

$$\| \int_0^t v(s)ds\| \leq 2K(1 + |\lambda|/\omega) \exp(\omega t)\|v_0\|.$$

Since $\int_0^t v(s)ds$ is stable with respect to v_0 in X if and only if $u(t)$ is stable with respect to $x, y \in E$, we have, for all $x \in d_1, y \in d_2$, $u(t)$ is the unique

solution of (1) which is stable in E. (This solution of (1) is connected with the mild solution of (3) [21]).

(iii) $\Rightarrow$ (i) If $\rho(A,B) \neq \emptyset$, then $\rho(T) \neq \emptyset$ and the well-posedness of the Cauchy problem (1) on d_1, d_2 is equivalent to the $(1,\omega)$-well-posedness of (3): $v(t)$ is stable with respect to v_0 by norm $\|\cdot\|_1$ if and only if $u(t)$ is stable with respect to $x, y \in E$. Hence by Theorem 3.1, T is the generator of the 1-time integrated semigroup.

Thus, we see that the well-posedness of (1) implies rather strong connections between operators A, B. The application of M, N-function and integrated semigroup methods makes it possible to clarify the interrelation between conditions on A, B and the well-posedness of the Cauchy problem (1): by Theorem 2.1 if A, B are commuting operators satisfying (d) and (4), then (1) is ω-well-posed on d_1, d_2 and A, B are biclosed. By Theorem 3.2, if A, B are biclosed operators satisfying (d) and (5), then (1) is ω-well-posed on d_1, d_2. The conditions (4) and (5) coincide if and only if A, B commute.

4. The well-posedness of the degenerate Cauchy problem

Let E, F be Banach spaces. We consider the Cauchy problem (2) with $A, B \in \mathcal{L}(E,F)$, $Q \in \mathcal{B}(E,F)$, and A, B being biclosed. Similarly to the regular case, it is not difficult to prove that the existence of a unique solution

$$u(t) \in C\{[0,\infty), D(B)\} \bigcap C^1\{[0,\infty), D(A)\} \bigcap C^2\{[0,\infty), E\}$$

is equivalent to the existence of the unique solution

$$v(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} \in C\{[0,\infty), D(T)\} \bigcap C^1\{[0,\infty), X\}$$

of the Cauchy problem

$$Sv'(t) = Tv(t), \quad t \geq 0, \ v(0) = v_0, \tag{10}$$

where

$$S = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}, \quad T = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}, \quad v_0 = \begin{pmatrix} x \\ y \end{pmatrix},$$

$T, S \in \mathcal{L}(X,Y)$, T is a closed operator, S is bounded, $X = E \times E$, $Y = F \times F$.

In [14],[15] the following definition of a degenerate integrated semigroup and several theorems concerning with the well-posedness of (10) are given.

Definition 4.1. *A one-parameter family of bounded linear operators* $\{V(t), t \geq 0\}$ *is said to be a degenerate* n*-times integrated semigroup if the relations* (V1)–(V3) *hold and* $\ker V(t) \neq \{0\}$. *The operators* S, T *are said to be the generators of the degenerate* n*-times integrated semigroup* $V(t)$ *if for* $\Re\lambda > \omega$

$$\mathcal{R}_d(\lambda)x := (\lambda S - T)^{-1}Sx = \int_0^\infty \lambda^n \exp(-\lambda t)V(t)x dt, \quad x \in X.$$

The operator $\mathcal{R}_d(\lambda)$ satisfies the resolvent identity and has no inverse, hence $\mathcal{R}_d(\lambda)$ is a pseudoresolvent.

Theorem 4.1. (Cf. [15].) *Let* S, T *be the generators of a degenerate* 1*-time integrated semigroup* $V(t)$ *satisfying*

$$\limsup_{\delta\to 0} \sup_{h\leq\delta} h^{-1}\|V(t+h) - V(t)\| \leq K \exp(\omega t), \quad t \geq 0. \tag{11}$$

Then

$$\mathcal{R}_d(\lambda)V(t) = V(t)\mathcal{R}_d(\lambda), \quad t \geq 0, \ \Re\lambda > \omega, \tag{12}$$

$$tSx = SV(t)x - T\int_0^t V(s)x ds, \quad x \in X_1 \oplus \ker S, \ X_1 := \overline{\mathcal{R}_d(\lambda)X}, \tag{13}$$

$$S\frac{d}{dt}V'(t)x = TV'(t)x, \quad x \in \mathcal{R}_d(\lambda)X_1, \tag{14}$$

and $V'(t)$ *is a degenerate strongly continuous semigroup on* $X_1 \oplus \ker S$.

Theorem 4.2. (Cf. [15].) *Let* $S, T \in \mathcal{L}(X, Y)$, T *be closed,* S *and* $(\lambda S - T)^{-1}S$ *(for some* λ*) be bounded. Then the following statements are equivalent:*

(ST1) S, T *are the generators of a degenerate strongly continuous semigroup;*

(ST2) *the* MFPHY*-type condition* (8) *with* $n = 0$ *is fulfilled for* $\mathcal{R}_d(\lambda)$ *and* $X = X_1 \oplus \ker S$;

(ST3) *the Cauchy problem* (10) *is uniformly well-posed on the maximal well-posedness class* $\mathcal{D}_1 := \mathcal{R}_d(\lambda)X = \{x \in D(T) : \ Tx \in S(X)\}$.

Remark 4.1. *For the well-posed Cauchy problem* (10)*, the decomposition*

$$X = X_1 \oplus \ker S$$

in (ST2) *is an analog of the condition* $X = \overline{D(T)} = \overline{\mathcal{R}(\lambda)X}$ *for the regular case* (3). *While studying the* (n, ω)*-well-posedness of the Cauchy problem* (10) *in the following theorems, we shall consider the decomposition*

$$X = X_{n+1} \oplus \ker \mathcal{R}_d^{n+1}(\lambda), \ X_{n+1} := \overline{D_{n+1}}, \ D_{n+1} := \mathcal{R}_d^{n+1}(\lambda)X. \tag{15}$$

Here D_{n+1} *does not depend on* λ *as well as* D_1. (15) *is the extension of the decomposition* $X = X_1 \oplus \ker S = X_1 \oplus \ker \mathcal{R}_d(\lambda)$ *for* $n \geq 1$.

Theorem 4.3. (Cf. [15].) *Let $S, T \in \mathcal{L}(X, Y)$, T be closed, S be bounded and* (8) *for $\mathcal{R}_d(\lambda)$ be fulfilled. Then S, T are generators of an $(n+1)$-times integrated semigroup and the Cauchy problem* (10) *is (n, ω)-well-posed on*

$$\mathcal{G}_{n+1} := \mathcal{R}_d^{n+1}(\lambda)\overline{\mathcal{R}_d(\lambda)X_1},$$

that is, for any $v_0 \in \mathcal{G}_{n+1}$, there exists a unique solution $v(t)$ such that

$$\|v(t)\| \le K\exp(\omega t)\|v_0\|_n, \quad \|v\|_n := \inf_{y:\ \mathcal{R}_d^n(\lambda)y=v} \|y\|.$$

Using these results we arrive at

Theorem 4.4. *Suppose $Q, A, B \in \mathcal{L}(E, F)$, Q is bounded, A, B are biclosed operators, and for*

$$R_d(\lambda^2) := (\lambda^2 Q - \lambda A - B)^{-1}, \ R_{d,2}(\lambda^2) := \overline{R_d(\lambda^2)(\lambda Q - A)}$$

the condition (5) *holds. Then for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{G}_2$, there exists a unique solution of* (2) *stable in E.*

Proof. If the operators Q, A, B are such that, for $R_d(\lambda^2)$, $R_{d,2}(\lambda^2)$, the condition (5) is fulfilled, then for the operator

$$\mathcal{R}_d(\lambda) = (\lambda S - T)^{-1}S = \begin{pmatrix} \overline{R_d(\lambda^2)(\lambda Q - A)} & R_d(\lambda^2) \\ \overline{R_d(\lambda^2)B} & \lambda R_d(\lambda^2) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix},$$

the condition (8) with $n = 1$ is fulfilled. Hence by Theorem 4.3 the degenerate 2-times integrated semigroup $V(t)$ exists and for all $v_0 = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{G}_2$, $v(t) = V''(t)v_0$ is the unique stable solution of (10): $\|v(t)\| \le K\exp(\omega t)\|v_0\|_1$. Then $w(t) = \mathcal{R}_d(\lambda)v(t)$ also is the solution of (10):

$$Sw'(t) = S(\mathcal{R}_d(\lambda)v(t))' = S\mathcal{R}_d(\lambda)v'(t) = S(\lambda S - T)^{-1}Sv'(t) =$$

$$S(\lambda S - T)^{-1}Tv(t) = T(\lambda S - T)^{-1}Sv(t) = T\mathcal{R}_d(\lambda)v(t) = Tw(t).$$

This solution has the initial value $w(0) = \mathcal{R}_d(\lambda)v_0$ and the following stability property:

$$\|w(t)\| \le K\exp(\omega t)\|\mathcal{R}_d(\lambda)v_0\|_1 \le K\exp(\omega t)\|v_0\|.$$

To secure the same stability for $\displaystyle\int_0^t v(s)ds = \begin{pmatrix} \int_0^t u(s)ds \\ u(t) - v_0 \end{pmatrix}$, we apply $(\lambda S - T)^{-1}$ to the equation (10) and obtain the following equalities:

$$(\lambda S - T)^{-1}Sv'(t) = w'(t) = (\lambda S - T)^{-1}(T \pm \lambda S)v(t) = -v(t) + \lambda\mathcal{R}_d(\lambda)v(t).$$

This implies (9) and stability of $\int_0^t v(s)ds$ in X. Hence $u(t)$ is stable in E and the Cauchy problem (2) is well-posed for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{G}_2$.

Now we prove the theorem which extends Theorems 2.1, 3.2, 4.3 and provides necessary and sufficient conditions for the $(n-1,\omega)$-well-posedness of (2) on the well-posedness class of x, y such that $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}_{n+1} = \mathcal{R}_d^{n+1}(\lambda)X$. It was proved in [15] that $\mathcal{R}_d(\lambda)X$ does not depend on λ.

Theorem 4.5. *Tet $Q, A, B \in \mathcal{L}(E,F)$, Q and $(\lambda S - T)^{-1}$ for some λ be bounded, A, B be biclosed operators, and the decomposition* (15) *take place. Then the following statements are equivalent:*

(i) *the Cauchy problem* (10) *is (n,ω)-well-posed on $\mathcal{D}_{n+1}$;*

(ii) *the* MFPHY-*type condition* (8) *for $\mathcal{R}_d(\lambda)$ holds;*

(iii) *the* MFPHY-*type condition*

$$\exists K > 0, \omega \geq 0: \ \|[R_d(\lambda^2)/\lambda^{n-1}]^{(k)}\|, \|[R_{d,2}(\lambda^2)/\lambda^{n-1}]^{(k)}\| \leq Kk!/(\Re\lambda - \omega)^{k+1},$$

$$k = 0, 1, 2, \ldots, \ \Re\lambda > \omega$$

holds;

(iv) *the Cauchy problem* (2) *is $(n-1,\omega)$-well-posed on the set of initial values such that $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}_{n+1}$.*

Proof will be given by the scheme (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (iv).

Let $v(t)$ be the unique solution of (10) with $v_0 \in \mathcal{D}_{n+1}$. In view of the stability property

$$\|v(t)\| \leq K \exp(\omega t)\|v_0\|_n,$$

it may be extended to $[\mathcal{D}_{n+1}]_n$. Taking into account the decomposition (15), we define the operator $U_0(t)$ on $[\mathcal{D}_{n+1}]_n \oplus \ker \mathcal{R}_d(\lambda)$ in the following way:

$$U_0(t)v_0 := v(t), \ v_0 \in [\mathcal{D}_{n+1}]_n, \quad U_0(t)v_0 := 0, \ v_0 \in \ker S = \ker \mathcal{R}_d(\lambda).$$

For $U_0(t)$ on $\mathcal{D}_{n+1}$, we have

$$\mathcal{R}_d(\lambda)U_0'(t)v_0 = (\lambda S - T)^{-1}SU_0'(t)v_0 = (\lambda S - T)^{-1}TU_0(t)v_0 = \\ -U_0(t)v_0 + \lambda\mathcal{R}_d(\lambda)U_0(t)v_0, \tag{16}$$

$$Sd/dt(\mathcal{R}_d(\lambda)U_0(t)v_0) = S(\lambda S - T)^{-1}TU_0(t)v_0 = T\mathcal{R}_d(\lambda)U_0(t)v_0.$$

Hence, $w(t) = \mathcal{R}_d(\lambda)U_0(t)v_0 = U_0(t)\mathcal{R}_d(\lambda)v_0, v_0 \in \mathcal{D}_{n+1}$, is the solution of (10) with the estimates

$$\|w(t)\| \le K\exp(\omega t)\|\mathcal{R}_d(\lambda)v_0\|_n \le K\exp(\omega t)\|v_0\|_{n-1},$$

and $U_0(t)\mathcal{R}_d(\lambda)$ may be extended to $[\mathcal{D}_n]_{n-1} \cup \ker \mathcal{R}_d^2(\lambda)$. Integrating (16) by t, we obtain

$$\int_0^t U_0(s)v_0 ds = \mathcal{R}_d(\lambda)v_0 - \mathcal{R}_d(\lambda)U_0(t)v_0 + \lambda\int_0^t \mathcal{R}_d(\lambda)U_0(s)v_0 ds =$$

$$\mathcal{R}_d(\lambda)v_0 - U_0(t)\mathcal{R}_d(\lambda)v_0 + \lambda\int_0^t U_0(s)\mathcal{R}_d(\lambda)v_0 ds.$$

The right hand side of the equality is defined on $[\mathcal{D}_n]_{n-1} \cup \ker \mathcal{R}_d^2(\lambda)$, and we introduce on $[\mathcal{D}_n]_{n-1} \cup \ker \mathcal{R}_d^2(\lambda)$ the operator-function $U_1(t)$:

$$U_1(t)v_0 := \mathcal{R}_d(\lambda)v_0 - U_0(t)\mathcal{R}_d(\lambda)v_0 + \lambda\int_0^t U_0(s)\mathcal{R}_d(\lambda)v_0 ds.$$

For $U_1(t)$ we have

$$\int_0^t U_1(s)v_0 ds = t\mathcal{R}_d(\lambda)v_0 - U_1(t)\mathcal{R}_d(\lambda)v_0 + \lambda\int_0^t U_1(s)\mathcal{R}_d(\lambda)v_0 ds. \qquad (17)$$

In the same manner we define $U_2(t)$ on $[\mathcal{D}_{n-1}]_{n-2} \cup \ker \mathcal{R}_d^3(\lambda)$ equal to the right hand side of (17), and $U_k(t)$ on $[\mathcal{D}_{n-k+1}]_{n-k} \cup \ker \mathcal{R}_d^{k+1}(\lambda)$ in the following way:

$$U_k(t)v_0 := \frac{t^{k-1}}{(k-1)!}\mathcal{R}_d(\lambda)v_0 - U_{k-1}(t)\mathcal{R}_d(\lambda)v_0 + \lambda\int_0^t U_{k-1}(s)\mathcal{R}_d(\lambda)v_0 ds. \qquad (18)$$

$$\|U_k(t)v_0\| \le K\exp(\omega t)\|v_0\|_{n-k} \ (\|v_0\|_0 = \|v_0\|), \ k = 1, 2, \ldots, n. \qquad (19)$$

Thus, $U_k(t)$ are defined on $[\mathcal{D}_{n-k+1}]_{n-k} \cup \ker \mathcal{R}_d^{k+1}(\lambda)$, in particular, $U_n(t)$ is defined on $\overline{\mathcal{D}_1} \cup \ker \mathcal{R}_d^{n+1}(\lambda)$, hence on $X_{n+1} \oplus \ker \mathcal{R}_d^{n+1}(\lambda) = X$.

Denote $\int_0^t U_n(s)ds$ by $V(t)$. We show that $V(t)$ is an $(n+1)$-times integrated semigroup with the generators S, T, and (8) takes place. For any $v_0 \in X$, $V(t)v_0$ is continuous by t, that is, (V2) holds; (V3) follows from (19). Let us verify (V1). As (V1) for $V(t)$ takes place if and only if $\int_0^\infty \lambda^{n+1}\exp(-\lambda t)V(t)dt$ satisfies the resolvent identity, it suffices to prove the equality

$$\mathcal{R}_d(\lambda)v_0 = \int_0^\infty \lambda^n \exp(-\lambda t)U_n(t)v_0 dt = \int_0^\infty \lambda^{n+1}\exp(-\lambda t)V(t)v_0 dt \qquad (20)$$

for $\Re\lambda > \omega$. Multiplying (18) through by $\lambda^k \exp(-\lambda t)$ for $k = n$ and integrating with respect to t from 0 to ∞, we have (20) for λ from an open set where $\mathcal{R}_d(\lambda)$ exists by the condition. Using the resolvent identity for the function

$$\int_0^\infty \lambda^n \exp(-\lambda t) U_n(t) v_0 dt, \quad \Re\lambda > \omega, \ v_0 \in X,$$

the analytical expansion of $\mathcal{R}_d(\lambda)$ to the halfplane $\Re\lambda > \omega$, we obtain (20) for $\Re\lambda > \omega$. (20) implies (8) for $\mathcal{R}_d(\lambda)$.

Thus, (ii) and the equivalent condition (iii) are valid.

If (iii) takes place, then by Theorem 4.3 the Cauchy problem (10) is (n, ω)-well-posed on

$$\mathcal{G}_{n+1} = \mathcal{R}_d^{n+1}(\lambda)\overline{\mathcal{R}_d(\lambda)X_1} \quad (X_1 = \overline{\mathcal{D}_1}).$$

Let us show that (15) and the (n, ω)-well-posedness on $\mathcal{G}_{n+1}$ imply the (n, ω)-well-posedness on $\mathcal{D}_{n+1}$. From (15) it follows $\mathcal{R}_d^{n+1}(\lambda)\overline{\mathcal{R}_d(\lambda)X} = \mathcal{R}_d^{n+1}(\lambda)X$ and

$$\mathcal{G}_{n+1} = \mathcal{R}_d^{n+1}(\lambda)\overline{\mathcal{R}_d(\lambda)\overline{\mathcal{D}_1}} \supseteq \mathcal{R}_d^{n+1}(\lambda)\mathcal{R}_d(\lambda)\overline{\mathcal{D}_1} =$$

$$\mathcal{R}_d(\lambda)\mathcal{R}_d^{n+1}(\lambda)\overline{\mathcal{D}}_1 = \mathcal{R}_d(\lambda)\mathcal{R}_d^{n+1}(\lambda)X = \mathcal{R}_d^{n+1}(\lambda)\mathcal{R}_d(\lambda)X = \mathcal{R}_d^{n+1}(\lambda)\mathcal{D}_1.$$

Hence

$$\mathcal{G}_{n+1} \supseteq \mathcal{R}_d^{n+1}(\lambda)\overline{\mathcal{D}_1} = \mathcal{R}_d^{n+1}(\lambda)X = \mathcal{D}_{n+1}.$$

Since $\mathcal{G}_{n+1} \subseteq \mathcal{D}_{n+1}$, we have $\mathcal{G}_{n+1} = \mathcal{D}_{n+1}$.

Thus, the Cauchy problem (10) is (n, ω)-well-posed on $\mathcal{D}_{n+1}$. That is, for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}_{n+1}$, there exists a unique solution $v = \begin{pmatrix} u \\ u' \end{pmatrix}$ stable by the norm $\|\cdot\|_n$ (which is equivalent to the $(S^{-1}T)^n$-graph norm if S^{-1} exists). Hence for the denoted x, y the unique solution $u(t)$ of (2) exists and depends continuously on $\begin{pmatrix} x \\ y \end{pmatrix}$ by the norm $\|\cdot\|_{n-1}$. Such problem (2) is called $(n-1, \omega)$-well-posed.

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